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ON SOME NEW SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY ORLICZ FUNCTIONS

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Abstract. The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of a Orlicz function, invariant mean and lacunary convergence. We also examine some topological properties of these spaces and establish some elementary connections between lacunary $[w]_\sigma$ -convergence and lacunary $[w]_\sigma$ -convergence with respect to an Orlicz functions which satisfy a Δ_2 -condition.

1. Introduction

Let l_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively.

A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide, (see, Banach [1]). Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [8] proved that

$$\hat{c} = \{x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) \text{ exists, uniformly in } n\},$$

where $t_{mn}(x) = \dfrac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1}$.

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [10] and also independently by Freedman et al. [4] as follows:

$$[\hat{c}] = \{x \in l_\infty : \lim_{m\to\infty} t_{mn}(|x - le|) = 0, \text{ uniformly in } n, \text{ for some } l\}$$

where $e = (1, 1, \ldots)$.

Schaefer [18] defined the σ-convergence as follows:

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ-mean if and only if,

(i) $\varphi(x) \geqslant 0$ when the sequences $x = (x_n)$ has $x_n \geqslant 0$ for all n,
(ii) $\varphi(e) = 1$, and
(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in l_\infty$.

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In case σ is the translation mapping $n \to n+1$, a σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x \in l_\infty : \lim_k t_{km}(x) = l, \text{ uniformly in } m\},$$

$l = \sigma\text{-}\lim x$, where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^k(m)}}{k+1}.$$

Here $\sigma^k(m)$ denotes the k^{th} iterate of the mapping σ at m.

A σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if for all $n \geqslant 0, j \geqslant 1$ $\sigma^j(n) \neq n$ (see, Mursaleen [11]).

Just as the concept of almost convergence led naturally to the concept of strong almost convergence, σ-convergence leads naturally to the concept of strong σ-convergence. A sequence $x = (x_k)$ is said to be strongly σ-convergent if there exists a number l such that

$$(|x_k - l|) \in V_\sigma \tag{1}$$

with the limit zero, (see, Mursaleen [12]).

We write $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1) holds, we write $[V_\sigma] - \lim x = l$. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ-convergence generalizes the concept of strong almost convergence. Note that

$$c \subset [V_\sigma] \subset V_\sigma \subset l_\infty.$$

Using the concept of invariant means, the following sequence spaces have been recently introduced and examined by Mursaleen et al. [13] is a generalization of the results of Das and Sahoo [3].

$$w_\sigma = \left\{ x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} t_{km}(x - l) = 0, \text{ for some } l \in C, \text{ uniformly in } m \right\},$$

$$[w]_\sigma = \left\{ x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x - l)| = 0, \text{ for some } l \in C, \text{ uniformly in } m \right\},$$

$$[w_\sigma] = \left\{ x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} t_{km}(|x - l|) = 0, \text{ for some } l \in C, \text{ uniformly in } m \right\}.$$

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots,$ where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [4] as

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l \right\}.$$

Quite recently the concept of lacunary strong σ-convergence was introduced by Savas [16] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra [2].

If $[V_\sigma^\theta]$ denotes the set of all lacunary strongly σ-convergent sequences then Savas [16] defined

$$[V_\sigma^\theta] = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_\sigma k_{(n)} - l| = 0, \text{ for some } l, \text{ uniformly in } n \right\}.$$

Recall [5,7] that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of M is replaced by subadditivity, then this function is called a modulus function (see, Ruckle [15]).

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the following sequence spaces. Let s be the space of all real or complex sequence $x = (x_k)$,

$$l_M = \left\{ x : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is called an Orlicz sequence space. l_M is a Banach space with the norm,

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leqslant 1 \right\}.$$

Also, recently Parashar and Choudhary [14] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[c, 1, p]$, $[c, 1, p]_0$ and $[c, 1, p]_\infty$. It may be noted here that the space of strongly summable sequences were discussed by Maddox [9].

The main object of this paper is to introduce and study the following sequence spaces which are defined by combining the concepts of an Orlicz function, invariant mean and lacunary convergence. We examine some topological properties of these spaces and establish some elementary connections between lacunary $[w]_\sigma$-convergence and lacunary $[w]_\sigma$-convergence with respect to an Orlicz function which satisfies Δ_2-condition.

Now we introduce the following sequence spaces:

DEFINITION 1. Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. Now we define the following sequence spaces.

$$[w^\theta, M, p]_\sigma = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\frac{|t_{km}(x-l)|}{\rho}\right)\right]^{p_k} = 0, \right.$$
$$\left. \text{for some } l \text{ and } \rho > 0, \text{ uniformly in } m \right\},$$

$$[w^\theta, M, p]_\sigma^0 = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M\Big(\frac{|t_{km}(x)|}{\rho} \Big) \Big]^{p_k} = 0,$$

$$\text{for some } \rho > 0, \text{ uniformly in } m \Big\},$$

$$[w^\theta, M, p]_\sigma^\infty = \Big\{ x = (x_k) : \sup_{r,m} \frac{1}{h_r} \sum_{k \in I_r} \Big[M\Big(\frac{|t_{km}(x)|}{\rho} \Big) \Big]^{p_k} < \infty,$$

$$\text{for some } \rho > 0 \Big\}.$$

Some sequence spaces are obtained by specializing $\theta, M,$ and p. For example, if we take $p_k = 1$ for all k, then we write the spaces $[w^\theta, M]_\sigma, [w^\theta, M]_\sigma^0$ and $[w^\theta, M]_\sigma^\infty$ in place of the spaces $[w^\theta, M, p]_\sigma, [w^\theta, M, p]_\sigma^0$ and $[w^\theta, M, p]_\sigma^\infty$.

If $x \in [w^\theta, M]_\sigma$, we say that x is lacunary $[w]_\sigma$-convergent with respect to the Orlicz function M.

When $\sigma(n) = n+1$, the spaces $[w^\theta, M, p]_\sigma, [w^\theta, M, p]_\sigma^0$ and $[w_1^\theta, M, p]_\sigma^\infty$ reduce to the spaces $[\hat{w}_\theta, M, p], [\hat{w}_\theta, M, p]_0$ and $[\hat{w}_\theta, M, p]_\infty$ respectively, which are defined as

$$[\hat{w}_\theta, M, p] = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{m \in I_r} \Big[M\Big(\frac{|t_{mn}(x-l)|}{\rho} \Big) \Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } l \text{ and } \rho > 0 \Big\},$$

$$[\hat{w}_\theta, M, p] = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{m \in I_r} \Big[M\Big(\frac{|t_{mn}(x)|}{\rho} \Big) \Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } \rho > 0 \Big\},$$

$$[\hat{w}_\theta, M, p]_\infty = \Big\{ x = (x_k) : \sup_{r,n} \frac{1}{h_r} \sum_{m \in I_r} \Big[M\Big(\frac{|t_{mn}(x)|}{\rho} \Big) \Big]^{p_k} < \infty,$$

$$\text{for some } \rho > 0 \Big\}.$$

If $M(x) = x$, $\theta = (2^r)$ and $p_k = 1$ for all k, then $[w^\theta, M, p]_\sigma = [w]_\sigma$ (see, Mursaleen et al. [13]) and $[\hat{w}_\theta, M, p] = [\hat{w}]$ (see, Das and Sahoo[3]). When $M(x) = x$, and $p_k = 1$ for all k, then $[w^\theta, M, p]_\sigma = [w^\theta]_\sigma$, $[w^\theta, M, p]_\sigma^o = [w^\theta]_\sigma^o$. If $\theta = (2^r)$ then $[w^\theta, M, p]_\sigma = [w, M, p]_\sigma$, $[w^\theta, M, p]_\sigma^0 = [w, M, p]_\sigma^0$ and $[w^\theta, M, p]_\sigma^\infty = [w, M, p]_\sigma^\infty$.

2. Main Results

We have

THEOREM 2.1. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[w^\theta,M,p]_\sigma$, $[w^\theta,M,p]_\sigma^o$ and $[w^\theta,M,p]_\sigma^\infty$ are linear spaces over the set of complex numbers.*

Proof. We shall prove the result only for $[w^\theta,M,p]_\sigma^o$. The others can be treated similarly. Let $x,y\in[w^\theta,M,p]_\sigma^o$ and $\alpha,\beta\in C$. In order to prove the result we need to find some $\rho_3>0$ such that $\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\alpha x+\beta y)|}{\rho_3}\right)\right]^{p_k}=0$, uniformly in m.

Since $x,y\in[w^\theta,M,p]_\sigma^o$, there exists positive ρ_1 and ρ_2 such that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho_1}\right)\right]^{p_k}=0$$

$$\text{and } \lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho_2}\right)\right]^{p_k}=0, \text{ uniformly in } m.$$

Define, $\rho_3=\max\,(2|\alpha|\rho_1,\ 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\alpha x+\beta y)|}{\rho_3}\right)\right]^{p_k}\\
&\quad\leqslant \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\alpha x)|}{\rho_3}+\frac{|t_{km}(\beta y)|}{\rho_3}\right)\right]^{p_k}\\
&\quad\leqslant \frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|t_{km}(x)|}{\rho_1}\right)+M\left(\frac{|t_{km}(y)|}{\rho_2}\right)\right]^{p_k}\\
&\quad< \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho_1}\right)+M\left(\frac{|t_{km}(y)|}{\rho_2}\right)\right]^{p_k}\\
&\quad\leqslant C\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho_1}\right)\right]^{p_k}+C\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(y)|}{\rho_2}\right)\right]^{p_k}
\end{aligned}$$

$\to 0$ as $r\to\infty$, uniformly in m, where $C=\max(1,2^{H-1})$, $H=\sup p_k$, so that $\alpha x+\beta y\in[w^\theta,M,p]_\sigma^o$.

This completes the proof.

THEOREM 2.2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[w^\theta,M,p]_\sigma^o$ is a topological linear space, totally paranormed by*

$$g(x)=\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right)^{1/H}\leqslant 1,\ r=1,2,\ ,\ m=1,2,\dots\right\}$$

where $H=\max\,(1,\sup p_k)$.

Proof. It is easy to see that $g(x) = g(-x)$. By using theorem 2.1, for $\alpha = \beta = 1$, we get $g(x+y) \leqslant g(x) + g(y)$. Since $M(0) = 0$, we get $\inf\{\rho^{pr/H}\} = 0$ for $x = 0$. Conversely, suppose $g(x) = 0$, then

$$\inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leqslant 1\right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_3 (0 < \rho_3 < \varepsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho_3}\right)\right]^{p_k}\right)^{1/H} \leqslant 1.$$

Thus

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\varepsilon}\right)\right]^{p_k}\right)^{1/H} \leqslant \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)}{\rho_3}\right)\right]^{p_k}\right)^{1/H} \leqslant 1,$$

for each r and m.

Suppose that $x_{\sigma^i(j)} \neq 0$ for each i and j. This implies that $t_{ij}(x) \neq 0$, for each i and j. Let $\varepsilon \to 0$. Then

$$\left(\frac{|t_{ij}(x)|}{\varepsilon}\right) \to \infty.$$

It follows that

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|t_{ij}(x)|}{\varepsilon}\right)\right]^{p_k}\right)^{1/H} \to \infty$$

which is a contradiction.

Therefore, $t_{ij}(x) = 0$ for each i and j, and thus $x_{\sigma^i(j)} = 0$ for all i and j. Finally, we prove that scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\lambda x)|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leqslant 1,\ r = 1,2,\ ,\ m = 1,2,\dots\right\}$$

Then

$$g(\lambda x) = \inf\left\{(|\lambda|u)^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{u}\right)\right]^{p_k}\right)^{1/H} \leqslant 1,\ r = 1,2,\ ,\ m = 1,2,\dots\right\}$$

where $u = \rho/|\lambda|$. Since $|\lambda|^{p_r} \leqslant \max\,(1,\ |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leqslant (\max(1, |\lambda|^{\sup p_r}))^{1/H} \inf\left\{u^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{u}\right)\right]^{p_k}\right)^{1/H} \leqslant 1, r = 1,2,\ ,\ m = 1,2,\dots\right\},$$

which converges to zero as x converges to zero in $[w^\theta, M, p]^o_\sigma$.

Now we assume that $\lambda_n \to 0$ and x is fixed in $[w^\theta, M, p]^o_\sigma$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} < (\varepsilon/2)^H \text{ for some } \rho > 0,\ r > N$$

and all m. This implies that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \varepsilon/2 \text{ for some } \rho > 0,\ r > N$$

and all m.

Let $0 < |\lambda| < 1$. Using the convexity of M, for $r > N$ and all m, we get

$$\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|t_{km}(\lambda x)|}{\rho}\right)\right]^{p_k} < \frac{1}{h_r}\sum_{k\in I_r}\left[|\lambda|\, M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} < (\varepsilon/2)^H.$$

Since M is continuous everywhere in $[0,\infty)$, then for $r \leqslant N$,

$$f(s) = \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(sx)|}{\rho}\right)\right]^{p_k} \text{ is continuous at 0.}$$

So there is $1 > \delta > 0$ such that $|f(s)| < (\varepsilon/2)^H$ for $0 < s < \delta$. Let K be such that $|\lambda_i| < \delta$ for $i > K$. Then for $i > K$, $r \leqslant N$ and all m,

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\lambda_i x)|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \varepsilon/2.$$

Thus

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\lambda_i x)|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \varepsilon$$

for $i > K$ and all r and m, so that $g(\lambda x) \to 0(\lambda \to 0)$.

DEFINITION 2. An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leqslant KM(u),\ u \geqslant 0$.

It is also easy to see that always $K > 2$. The Δ_2-condition equivalent to the satisfaction of inequality $M(lu) \leqslant K(lu)M(u)$ for all values of u and for all $l > 1$. (see [6]).

THEOREM 2.3. *For any Orlicz function M which satisfies the Δ_2-condition, we have* $[w^\theta]_\sigma \subseteq [w^\theta, M]$.

To prove this theorem we need the following lemma.

LEMMA 2.4. *Let M be an Orlicz function which satisfies the Δ_2-condition and let $0<\delta<1$. Then for each $x \geqslant \delta$ we have $M(x) < Kx\delta^{-1}M(2)$ for some constant $K>0$.*

Proof. It follows by a straightforward computation using the Δ_2-condition.

Proof of Theorem 2.3. Let $x \in [w^\theta]_\sigma$. Then we have

$$A_r = \frac{1}{h_r}\sum_{k\in I_r} |t_{km}(x-l)| \to 0$$

as $r\to\infty$, uniformly in m, for some l.

Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\leqslant t\leqslant\delta$. We can write

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} M(|t_{km}(x-l)|) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |t_{km}(x-l)|\leqslant\delta}} M(|t_{km}(x-l)|) + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |t_{km}(x-l)|>\delta}} M(|t_{km}(x-l)|)\\ &< h_r^{-1}(h_r\varepsilon) + h_r^{-1}K\delta^{-1}M(2)h_rA_r,\end{aligned}$$

by lemma 2.4. Letting $r\to\infty$, it follows that $x\in[w^\theta, M]_\sigma$.

This completes the proof of Theorem 2.3.

By using the method of the proof of Theorem 2.3. it is not hard to see that $[w^\theta]^0_\sigma \subset [w^\theta, M]^0_\sigma$ and $[w^\theta]^\infty_\sigma \subset [w^\theta, M]^\infty_\sigma$.

In the following two theorems we prove the inclusion $[w,M,p]_\sigma \subset [w^\theta, M, p]_\sigma$ and $[w^\theta, M, p]_\sigma \subset [w,M,p]_\sigma$ under certain restrictions on $\theta=(k_r)$.

THEOREM 2.5. *Let $\theta=(k_r)$ be a lacunary sequence with $\liminf_r q_r>1$. Then for any Orlicz function M, $[w,M,p]_\sigma \subset [w^\theta,M,p]_\sigma$, where*

$$[w,M,p]_\sigma = \left\{ x=(x_k) : \lim_n \frac{1}{n+1}\sum_{k=0}^{n}\left[M\left(\frac{|t_{km}(x-l)|}{\rho}\right)\right]^{p_k} = 0 \right.$$

$$\left. \textit{uniformly in } m, \textit{ for some } l \textit{ and } \rho>0 \right\}.$$

(In case $l=0$, we write $[w,M,p]_\sigma = [w,M,p]^0_\sigma$).

Proof. It is sufficient to show that $[w,M,p]^0_\sigma \subset [w^\theta,M,p]^0_\sigma$; the general inclusion follows by linearity. Suppose $\liminf_r q_r>1$, then there exists $\delta>0$ such that

$q_r = (k_r/k_{r-1}) \geqslant 1+\delta$ for all $r \geqslant 1$. Then for $x \in [w,M,p]^0_\sigma$, we write

$$\begin{aligned}
A_r = \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} &= \frac{1}{h_r}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} \\
&\quad - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}\left[M\left(\frac{|t_{km}(x-l)|}{\rho}\right)\right]^{p_k} \\
&= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right) \\
&\quad - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right).
\end{aligned}$$

Since, $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leqslant \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1}}{h_r} \leqslant \frac{1}{\delta}.$$

$$\text{The terms } k_r^{-1}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} \text{ and } k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}$$

both converge to zero uniformly in m, and it follows that A_r converges to 0 as $r \to \infty$, uniformly in m, that is, $x \in [w^\theta,M,p]^o_\sigma$. This completes the proof.

THEOREM 2.6.. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then for any Orlicz function M, $[w^\theta,M,p]_\sigma \subset [w,M,p]_\sigma$.*

Proof. If $\limsup_r q_r < \infty$, there exists $B > 0$ such that $q_r < B$ for all $r \geqslant 1$. Let $x \in [w^\theta,M,p]^o_\sigma$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geqslant R$ and all m,

$$A_j = \frac{1}{h_j}\sum_{k\in I_j}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1,2,\ldots$.

Now let n be any integer with $k_{r-1} < n \leqslant k_r$, where $r > R$.

Then

$$
\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} \leqslant k_{r-1}^{-1}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\\
&= k_{r-1}^{-1}\Bigg\{\sum_{k\in I_1}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} + \sum_{k\in I_2}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\\
&\quad + \ldots + \sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\Bigg\}\\
&= \frac{k_1}{k_{r-1}}k_1^{-1}\sum_{k\in I_1}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} + \frac{k_2-k_1}{k_{r-1}}(k_2-k_1)^{-1}\sum_{k\in I_2}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\\
&\quad + \ldots + \frac{k_R-k_{R_{-1}}}{k_{r-1}}(k_R-k_{R_{-1}})^{-1}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\\
&\quad + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}}(k_r-k_{r-1})^{-1}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\\
&= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}A_R + \frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1} + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r\\
&\leqslant \left(\sup_{j\geqslant 1}A_j\right)\frac{k_R}{k_{r-1}} + \left(\sup_{j\geqslant R}A_j\right)\frac{k_r-k_R}{k_{r-1}}\\
&< K\frac{k_R}{k_{r-1}} + \varepsilon B.
\end{aligned}
$$

Since $k_{r-1}\to\infty$ as $n\to\infty$, it follows that

$$
\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k} \to 0,
$$

uniformly in m and consequently $x\in[w^\theta,M,p]_\sigma^o$.

This completes the proof.

THEOREM 2.7. *Let* $\theta=(k_r)$ *be a lacunary sequence with* $1<\liminf_r q_r\leqslant\limsup_r q_r<\infty$. *Then for any Orlicz function* M, $[w,M,p]_\sigma=[w^\theta,M,p]_\sigma$.

Proof. Theorem 2.7 follows theorems 2.5 and 2.6.

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