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INEQUALITIES ON POLYNOMIAL ROOTS

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Abstract. The paper presents a survey of inequalities involving roots of univariate polynomials with complex coefficients. These allow improvements in the methods of Bernoulli and Graeffe. Inequalities involving the length of a polynomial are also deduced.

1. Introduction

Let $P(X) = a_d X^d + \cdots + a_1 X + a_0$ be a polynomial with complex coefficients of degree $d \geqslant 2$ and let $z_1, z_2, \ldots, z_d \in \mathbb{C}$ be its roots. The determination of the roots z_j arises frequently in applications. Since the exact computation of the zeros in function of the coeffents of the polynomial is not possible for general polynomials, for all practical purposes it is useful to handle efficient methods for estimation. With these devices there are related various inequalities satisfied by moduli of polynomial zeros. Bounds for roots were obtained, among other, by Cauchy, Kuniyeda, Fujiwara, Landau and Montel. Upper and lower bounds can be derived using polynomial sizes defined in function of the coefficients, such that the norm, the length, the heigth and the measure. (See [12], [11], [8], [14]).

The methods of D. Bernoulli and of Dandelin–Graeffe give estimates for the largest absolute values of polynomial roots (see, for example, [7]). The method of Bernoulli involves inequalities on linear recurrent sequences, derived through an approximation theorem of Dirichlet [2].

Finally we discuss inequalities on the length of a polynomial divisor. They allow us to obtain other upper bounds for polynomial roots.

2. Bounds for polynomial roots

If $z \in \mathbb{C}$ is a root of the polynomial $P \in \mathbb{C}[X] \setminus \mathbb{C}$ one searches for positive numbers s_0, r_0 such that

$$s_0 \leqslant |z| \leqslant r_0 .$$

The first significant result was obtained by Cauchy [3]:

THEOREM 2.1. (A.–L. Cauchy, 1829) *All the roots of the nonconstant complex polynomial $P(X) = a_0 + a_1 X + \cdots + a_d X^d$ are contained in the disk $|z| \leqslant \xi$, where ξ is the unique positive solution of the equation*

$$|a_d| X^d = |a_0| + |a_1| X + \cdots + |a_{d-1}| X^{d-1} . \tag{1}$$

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Among other estimates for r_0 we mention the results of Fujiwara and Kuniyeda:

THEOREM 2.2. (M. Fujiwara, 1926) *If* $\lambda_1, \ldots, \lambda_d \in (0, \infty)$ *and*

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_d} = 1,$$

then all the roots of the polynomial P *are contained in the disk* $|z| \leqslant \xi$, *where*

$$\xi = \max_{1 \leqslant k \leqslant d} \left(\lambda_k \left| \frac{a_{d-k}}{a_d} \right| \right)^{\frac{1}{k}}.$$

THEOREM 2.3. (M. Kuniyeda, 1916) *If and* $p, q > 0$ *are such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then all the roots of the polynomial* P *are contained in the disk* $|z| \leqslant \xi$, *where*

$$\xi = \left(1 + \left(\sum_{j=0}^{d-1} \left| \frac{a_j}{a_d} \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

There exist many other useful results, see, for example, [8] and [14].

3. Dominant roots

A root $\alpha \in \mathbb{C}$ of the polynomial $P \in \mathbb{C}[X]$ is called *dominant* if $|\alpha| > |\beta|$ for any other root β .

The computation of dominant roots was considered by Newton (1707) in his *Arithmetica Universalis* [17], no. 133–137. His idea was deleloped by Daniel Bernoulli (1728, [1]), who used linear recurrent sequences for approaching the dominant roots. The method of Bernoulli was improved by Jacobi (1834, [10]), and another approach was proposed by Dandelin (1826) and Graeffe (1833), see [9].

3.1. Bernoulli

Daniel Bernoulli (1728, [1]) invented a method for estimating the dominant roots, based on the study of appropriate linear recurrent sequences.

Linear recurrent sequences

If $P(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d$ a *linear recurrent sequence* (lrs) associated to P is a sequence $(x_n)_{n \in \mathbb{N}}$ that satisfies the relations

$$a_0x_n + a_1x_{n-1} + \cdots + a_dx_{n-d} = 0 \quad \text{for all } n \in \mathbb{N}, n \geqslant d.$$

Newton's relations

THEOREM 3.1. (I. Newton) *Let* $P \in \mathbb{C}(X] \setminus \mathbb{C},\ P(0) \neq 0$,

$$P(X) \ = \ a_0X^d + a_1X^{d-1} + \cdots + a_d \ = \ a_0(X-\alpha_1)\cdots(X-\alpha_d)\,.$$

Let (x_n) *be the linear recurrent associated to the polynomial* P *and with starting values* $x_0,\ x_1,\ \dots\ x_{d-1}$ *given by*

$$x_0 \ = \ -\frac{a_1}{a_0}, \qquad x_{k-1} \ = \ -\frac{ka_k + a_{k-1}a_0 + \cdots + a_1a_{k-2}}{a_0} \quad for \quad k = 2,3,\dots,d\,.$$

Then

$$x_n \ = \ \alpha_1^{n+1} + \alpha_2^{n+1} + \cdots + \alpha_d^{n+1} \quad for\ all \quad n \in \mathbb{N}\,.$$

Proof. Consider Newton's sums

$$P_k \ = \ \alpha_1^k + \cdots + \alpha_d^k$$

and the reciprocal polynomial

$$Q(X) \ = \ X^d\,P(X^{-1}) \ = \ a_dX^d + a_{d-1}X^{d-1} + \cdots + a_0\,.$$

We have

$$Q(X) \ = \ \beta\Big(X - \frac{1}{\alpha_1}\Big)\cdots(X - \frac{1}{\alpha_d})\,,$$

with $\beta = (-1)^d a_0\alpha_1\cdots\alpha_d \in \mathbb{C}\setminus\{0\}$.
Let $f(z) = -Q'(z)/Q(z)$ and its Taylor series at $z=0$

$$f(z) \ = \ \sum_{n=0}^{\infty} x_nz^n\,.$$

On the other hand

$$f(z) \ = \ -\sum_{j=1}^{d}\frac{1}{z-\alpha_j^{-1}} = \sum_{j=1}^{d}\frac{\alpha_j}{1-\alpha_jz}$$

$$= \ \sum_{j=1}^{d}\left(\sum_{n=0}^{\infty}\alpha_j^{n+1}z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{j=1}^{d}\alpha_j^{n+1}\right)z^n\,.$$

It follows that

$$x_n \ = \ \sum_{j=1}^{d}\alpha_j^{n+1} \ = \ P_{n+1}\,.$$

Now we compare the coeffients of z^n in both sides of the relation $Q(z)f(z) = -Q'(z)$, i. e.

$$\sum_{n=0}^{\infty}(a_0x_n + a_1x_{n-1} + \cdots + a_dx_{n-d})z^n \ = \ -\sum_{j=0}^{d-1}(j+1)a_{j+1}z^j$$

and we obtain

$$\begin{cases} a_0x_k + a_1x_{k-1} + \cdots + a_dx_{k-d} & = \ 0 \ \text{ for } \ k \geqslant d\,, \\ a_0x_k + a_1x_{k-1} + \cdots + a_dx_{n-d} + (k+1)a_{k+1} & = \ 0 \ \text{ for } \ k = 0,1,\ldots,d-1\,. \end{cases},$$

which are equivalent to Newton's relations. It follows that the linear recurrent sequence $(x_n)_n$ associated with P and having the initial values

$$x_0 = -a_1/a_0\,, \quad x_{k-1} = -(ka_k + a_{k-1}x_0 + \cdots + a_1x_{k-2})/a_0 \quad \text{for} \quad k = 2,3,\ldots,d\,,$$

is $x_n = \alpha_1^{n+1} + \cdots + \alpha_d^{n+1}$. □

For another proof see [16].

Theorem 3.1 has applications to the problem of estimating the dominant roots (see [16]). For obtaining upper bounds it is necessary to consider other linear recurrent sequences. In [16] such an upper bound is obtained using the linear recurrent sequence $(v_n)_n$ associated to P and with initial values

$$v_0 = v_1 = \cdots = v_{d-2} = 0,\ v_{d-1} = 1\,.$$

Finally these inequalities allow to obtain the absolute value of a dominant root as a limit.

3.1.1. The approximation theorem of Dirichlet

Other bounds for dominant roots can be deduced by Dirichlet's approximation theorem.

DEFINITION. The *norm* of the real number θ is

$$||\theta|| \ = \ \min\{|\theta - n|\,;\, n \in \mathbb{Z}\}\,.$$

REMARK. Note that $||\theta||$ is the distance between θ and the nearest integer, with the convention $||n + \frac{1}{2}|| = \frac{1}{2}$. We have

$$||\theta|| \ = \ \min\big\{\{\theta\}, 1 - \{\theta\}\big\}$$

and

$$\begin{aligned} ||\theta_1 + \theta_2|| \ &\leqslant \ ||\theta_1|| + ||\theta_2|| \quad \text{for all} \quad \theta_1, \theta_2 \in \mathbb{R}\,, \\ ||n\theta|| \ &\leqslant \ |n|\,||\theta|| \quad \text{for all} \quad n \in \mathbb{N},\ \theta \in \mathbb{R}\,. \end{aligned}$$

THEOREM 3.2. (Dirichlet) *Let* $\theta_1,\ldots,\theta_k \in \mathbb{R}$, $Q \geqslant 1$. *There exists* $q \in \mathbb{Z}$ *such that* $0 \leqslant q < (Q+1)^k$ *and* $||q\theta_j|| \leqslant \frac{1}{Q}$ *for all* $j = 1,2,\ldots,k$.

Proof. We first assume that Q is an integer. Then we decompose the cube $[0,1]^k$ into Q^k "small" cubes and look to the points

$$\Theta_r \ = \ (\{r\theta_1\},\ldots,\{r\theta_1\}), \quad r \in \mathbb{Z}, 0 \leqslant r \leqslant Q^k\,,$$

where $\{q\theta_j\}$ is the fractional part of $q\theta_j$.

There exist two points Θ_r, $\Theta_{r'}$ in the same cube. We have

$$\frac{u_j}{Q} \leqslant \{r\theta_j\}, \{r'\theta_j\} \leqslant \frac{u_j+1}{Q}, \quad \text{with} \quad 0 \leqslant u_j < Q.$$

We put $q = |r' - r|$. Therefore $q \neq 0$, $1 \leqslant q < Q^k$.
On the other hand we have $\{r\theta_j\} = r\theta_j - s$, with $s = \lfloor r\theta_j \rfloor \in \mathbb{Z}$ and, similarly, $\{r'\theta_j\} = r'\theta_j - s'$, with $s' \in \mathbb{Z}$. Therefore

$$||q\theta_j|| = \min_{n\in\mathbb{Z}}\{|q\theta_j - n|\} \leqslant |(r-r')\theta_j - (s-s')| = |\{r\theta_j\} - \{r'\theta_j\}| \leqslant \frac{1}{Q}.$$

so

$$||q\theta_j|| \leqslant \frac{1}{Q} \quad \text{for all} \quad j = 1, 2, \ldots, r.$$

Note that if $Q \notin \mathbb{Z}$ the theorem is verified with $\lfloor Q \rfloor + 1$ instead of Q. Let $q \in \mathbb{Z}$ be such that $0 < q < \left(\lfloor Q \rfloor + 1\right)^k$ and $||q\theta_j|| \leqslant \left(\lfloor Q \rfloor + 1\right)^{-1}$ for all j. Therefore $1 \leqslant q \leqslant (Q+1)^k$ and

$$||q\theta_j|| \leqslant \frac{1}{\lfloor Q \rfloor + 1} \leqslant \frac{1}{Q} \quad \text{for all} \quad j. \quad \square$$

REMARK. We observe that for $k = 1$, then $1 \leqslant q \leqslant Q$.
If $k \geqslant 2$ and $Q \in \mathbb{Z}$ one can find $1 \leqslant q < Q^k$.
The $\leqslant$ in the theorem cannot be improved to strict inequality. In fact, if we take $k = 1$ and $\theta_1 = Q^{-1}$, then $||\theta_1 q|| \geqslant Q^{-1}$ for all $0 < q < Q$.

Lower and upper bounds for dominant roots

LEMMA 3.3. *If $(x_n)_n$ is the lrs defined in Theorem 3.1, we have*

$$\left|\frac{x_n}{d}\right|^{1/(n+1)} \leqslant |\alpha_1|.$$

Proof. By Newton's relations we obtain

$$|x_n| \leqslant \sum_{i=1}^{d} |\alpha_i|^{n+1} \leqslant d|\alpha_1|^{n+1},$$

which gives $d^{-1}|x_n| \leqslant |\alpha_1|^{n+1}$, therefore $|x_n/d|^{1/(n+1)} \leqslant |\alpha_1|$. $\square$

COROLLARY 3.4. *Let*

$$X_n = \max\{|x_1|^{1/2}, |x_2|^{1/3}, \ldots, |x_n|^{1/(n+1)}\}$$

and

$$Y_n = \max\{|x_1/d|^{1/2}, |x_2/d|^{1/3}, \ldots, |x_n/d|^{1/(n+1)}\}.$$

We have

$$X_n \cdot d^{-1/(n+1)} \leqslant |\alpha_1| \quad \textit{and} \quad Y_n \leqslant |\alpha_1|.$$

Another lower bound is described below.

PROPOSITION 3.5. *If* $(u_n)_n$ *is a linear recurrent sequence associated to the polynomial* P*, there exists a constant* $C = C(u_n) > 0$ *that depends only on* $(u_n)_n$ *such that*

$$|u_n|^{1/n} \cdot \left(Cn^{d-1}\right)^{-1/n} \leqslant |\alpha_1| .$$

Proof. We may write $u_n = \sum_{j=1}^{s} P_j(n)\alpha_{i_j}$, where $\alpha_{i_1}, \ldots, \alpha_{i_s}$ are the distinct roots of the polynomial P and P_j is a polynomial of degree at most $m_j - 1$, with m_j the multiplicity of α_{i_j}. Note that $\deg(P_j) \leqslant d - 1$.

Choosing C to be the sum of the absolute values of the coefficients of the polynomials $P_1, P_2, \ldots, P_s$, we get

$$|u_n| \leqslant C \cdot n^{d-1} \cdot |\alpha_1|^n ,$$

which gives the inequality. □

For obtaining upper bounds for dominant roots one needs a linear recurrent sequence $(u_n)_n$ associated to the polynomial P that satisfies the condition

$$\begin{vmatrix} u_0 & u_1 & \ldots & u_{d-1} \\ u_1 & u_2 & \ldots & u_d \\ \vdots & \vdots & \vdots & \vdots \\ u_{d-1} & u_d & \ldots & u_{2d-2} \end{vmatrix} \neq 0 . \qquad (*)$$

PROPOSITION 3.6. *If* $(u_n)_n$ *is a linear recurrent sequence that satisfy the condition* $(*)$*, there exists a constant* $C = C(u_n) > 0$ *that depends only on* $(u_n)_n$ *such that*

$$|\alpha_1| \leqslant C^{1/n} \cdot \max\{|u_n|, |u_{n+1}|, \ldots, |u_{n+d-1}|\}^{1/n} \quad \textit{for all} \quad n \in \mathbb{N} .$$

Proof. Let $(s_n)_n$ be a linear recurrent sequence associated with P. By Lemma 3 of [16] there exists a constant $C > 0$ such that

$$|s_n| \leqslant C \cdot \max\{|u_n|, |u_{n+1}|, \ldots, |u_{n+d-1}|\} \quad \text{for all} \quad n \in \mathbb{N} .$$

Because

$$a_0\alpha_1^n + a_1\alpha_1^{n-1} + \cdots + a_d\alpha_1^{n-d} = 0 ,$$

the sequence $(\alpha_1^n)_n$ is a lrs and we find C. □

COROLLARY 3.7. *Let* $(u_n)_n$ *be a linear recurrent sequence that satisfy the condition* $(*)$. *There exist a constant* $C = C(u_n) > 0$, $K = K(u_n)$ *that depend only on* $(u_n)_n$ *such that*

$$|u_n|^{1/n} \cdot \left(Kn^{d-1}\right)^{-1/n} \leqslant |\alpha_1| \leqslant C^{1/n} \cdot \max\{|u_n|, |u_{n+1}|, \ldots, |u_{n+d-1}|\}^{1/n} \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. The right hand–side inequality follows by Proposition 3.6, and the other inequality by Proposition 3.5. □

The approach of Jacobi

G. Jacobi (1834, [10]) proposed a modification of Bernoulli's method.

Let $\alpha_1, \ldots, \alpha_s$ be the dominant roots of the polynomial P of degree $d \geqslant 2$. Instead of Newton's sums

$$S_n = \alpha_1^n + \cdots + \alpha_d^n,$$

Jacobi considered the truncated sums

$$T_n = \alpha_1^n + \cdots + \alpha_s^n$$

and the polynomial

$$T = (X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_s) = X^s + A_1X^{s-1} + \cdots + A_s.$$

Eliminating $A_1, \ldots, A_s$ from $T=0$ and Newton's relations

$$T_{s+k} + T_{s+k-1}A_1 + \cdots + T_kA_s = 0 \quad \text{for} \quad k = n, n+1, \ldots, n+s-1$$

leads to the sth degree polynomial equation

$$Q_{n,s} = \begin{vmatrix} X^s & X^{s-1} & \cdots & 1 \\ T_{n+s} & T_{n+s-1} & \cdots & T_n \\ \vdots & \vdots & \ldots & \vdots \\ T_{n+2s-1} & T_{n+2s-2} & \cdots & T_{n+s-1} \end{vmatrix} = 0.$$

Jacobi then states that, because they have the same order of magnitude, the truncated sums can be replaced by Newton's sums in the polynomial equation $Q_{n,s} = 0$, obtaining another polynomial $R_{n,s}$. He finally concludes that the roots of $R_{n,s}$ have the absolute values approximately equal to those of the dominant roots of P.

Jacobi gives no proof of this rule but it can be proved that his method is valid for polynomials having only simple dominant roots. A detailed discussion of Jacobi's rule is given by Mignotte–Ştefănescu [16]. We present briefly the case of three dominant roots, namely $s=3$.

REMARK. Jacobi's rule does not work for polynomials with multiple dominant roots.

Let us consider, for example,

$$P = (X-a)^3\,(X-b)^2 \quad \text{with} \quad |a| > |b|.$$

We obtain

$$S_n = 3a^n + 2b^n$$

and

$$T_n = 3a^n.$$

Note that the quadratic equation formed with the truncated sums of Jacobi is the null equation. Using Newton's sums we obtain

$$A_0 \;=\; -6a^n b^n (a-b)^2, \quad A_1 \;=\; 6a^n b^n (a+b)(a-b)^2, \quad A_2 \;=\; -6a^{n+1} b^{n+1} (a-b)^2\,,$$

so

$$R_{n,2}(X) \;=\; -6a^n b^n (a-b)^2 \cdot \left(X^2-(a+b)X+ab\right)$$

whose roots are a and b. But for infinitely many a and b the absolute values $|a|$ and $|b|$ are not approximately equal.

But a is a dominant triple root of P, in contradiction with Jacobi's rule.

However we have the following result.

THEOREM 3.8. *Let P be a polynomial with three dominant roots α_1, α_2, α_3. Then*

$$Q_n = Q_{n,3} \;=\; (\alpha_1\alpha_2\alpha_3)^n \cdot \prod_{j<i}(\alpha_i-\alpha_j)^2 \cdot (X-\alpha_1)(X-\alpha_2)(X-\alpha_s)$$

and $Q_n \neq 0$ if and only if α_1, α_2 and α_3 are simple roots.

Proof. Denote by $\alpha_1, \alpha_2, \ldots, \alpha_d$ the roots of P. We have

$$|\alpha_1| \;=\; |\alpha_2| \;=\; |\alpha_3| \;>\; |\alpha_j| \quad \text{for} \quad j \geqslant 4\,.$$

Then Jacobi's polynomial Q_n is of degree three and α_1, α_2 and α_3 are its zeros. It follows that

$$Q_n \;=\; A_n\,(X-\alpha_1)(X-\alpha_2)(X-\alpha_3)$$

and the leading coefficient is

$$A_n = \begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+3} & T_{n+2} & T_{n+1} \\ T_{n+4} & T_{n+3} & T_{n+2} \end{vmatrix}.$$

We observe that

$$\begin{pmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+3} & T_{n+2} & T_{n+1} \\ T_{n+4} & T_{n+3} & T_{n+2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1^n & \alpha_1^{n+1} & \alpha_1^{n+2} \\ \alpha_2^n & \alpha_2^{n+1} & \alpha_2^{n+2} \\ \alpha_3^n & \alpha_3^{n+1} & \alpha_3^{n+2} \end{pmatrix}.$$

Therefore

$$A_n \;=\; (\alpha_1\alpha_2\alpha_3)^n \cdot \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}^2 \;=\; (\alpha_1\alpha_2\alpha_3)^n \cdot (\alpha_3-\alpha_1)^2(\alpha_3-\alpha_2)^2(\alpha_2-\alpha_3)^2\,.$$

It also follows that Jacobi's polynomial Q_n is nonzero if and only if α_1, α_2 and α_3 are distinct. $\square$

COROLLARY 3.9. *The rule of Jacobi holds if the polynomial P has only three simple dominant roots.*

Proof. In this case $Q_n \neq 0$ and its roots are the dominant roots of P. The corollary then follows by the theorem of continuity (Weber, [19]) of the roots of a polynomial with respect to the coefficients. $\square$

3.2. The method of Graeffe

The method of Graeffe was introduced independently by Dandelin (1826), Graeffe (1833, 1837) and Lobatchevskiĭ (1834), see [9] for a historical presentation. In the restricted version there are used the polynomials $F_n(X) = \mathrm{Res}_Y\big(P(Y), Y^{2^n} - X\big)$.

For $n = 2$ consider

$$P(x) \;=\; a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_d),$$

$$(-1)^d P(-x) = a_0(x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_d),$$

and

$$F_2(x) := (-1)^d F(-x)F(x) = a_0^2(x^2-\alpha_1^2)(x^2-\alpha_2^2)\cdots(x^2-\alpha_d^2)\,.$$

If

$$|\alpha_1| \gg |\alpha_2| \gg \cdots \gg |\alpha_d|$$

then

$$|\alpha_j| \sim \sqrt{\frac{|a_j^{(2)}|}{|a_{j-1}^{(2)}|}}\,.$$

If we continue the process we find

$$F_n(x) := a_0^{2^{n-1}}(x-\alpha_1^{2^{n-1}})(x-\alpha_2^{2^{n-1}})\cdots(x-\alpha_d^{2^{n-1}}) = \sum a_j^{(n)} x^{d-j}.$$

For example, taking $F(x) = 2x^3 - 7x^2 - 13x + 16$, we have

$$\begin{aligned}
F_{(2)}(x) &= 4x^3 + 101x^2 + 393x + 256\\
F_{(3)}(x) &= 16x^3 + 7057x^2 + 102\,700x + 65\,540\\
F_{(4)}(x) &= 256x^3 + 4651\cdot 10^4 x^2 + 9622\cdot 10^6 x + 4295\cdot 10^6
\end{aligned}$$

and the estimates

$$\begin{aligned}
|\alpha_1| &= \left(\frac{46\,510\,000}{256}\right)^{1/8} &\sim 4.544\\
|\alpha_2| &= \left(\frac{9\,622\,000\,000}{46\,510\,000}\right)^{1/8} &\sim 1.947\\
|\alpha_3| &= \left(\frac{4\,295\,000\,000}{9\,622\,000\,000}\right)^{1/8} &\sim 0.904
\end{aligned}$$

which are very closed to the true values of the roots.

In the general case, the method of Graeffe is based on the study of the sequence $(P_n)_n$ of polynomials associated to P, where

$$P_n(X) \;=\; \mathrm{Res}_Y\big(P(Y), Y^n - X\big)\,.$$

Let

$$P_n(X) \;=\; \sum_{i=0}^{d} a_i^{(n)} X^{d-i} \quad \text{for all} \quad n \in \mathbb{N}\,,$$

with the convention $P_1(X) = P(X) = \sum_{i=0}^{d} a_i X^{d-i}$.

Both approaches use relations between the roots and the coefficients of the polynomial P. The simplest case is that of a unique dominant root α_1 whose absolute value is “larger enough" than the modules of the other roots. In this case $|\alpha_1| \sim |a_1/a_0|$.

Note that if

$$|\alpha_1| \gg |\alpha_2| \gg \cdots \gg |\alpha_d|, \qquad \text{we have}$$

$$|\alpha_1| \sim |\alpha_1| \cdot \left|1 + \frac{\alpha_2}{\alpha_1} + \cdots + \frac{\alpha_d}{\alpha_1}\right| = \left|\frac{a_1}{a_0}\right|,$$

$$|\alpha_1\alpha_2| \sim |\alpha_1\alpha_2| \cdot \left|1 + \frac{\alpha_1\alpha_3}{\alpha_1\alpha_2} + \cdots + \frac{\alpha_{d-1}\alpha_d}{\alpha_1\alpha_2}\right| = \left|\frac{a_2}{a_0}\right|$$

$$\vdots$$

$$\text{so} \qquad |\alpha_j| \sim \left|\frac{a_j}{a_{j-1}}\right|.$$

We observe that if $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_d|$, then α_1, $\alpha_2, \ldots,$ α_d are real numbers.

Bounds by Graeffe

Instead of estimating a dominant root of the transformed polynomial P_n, we estimate a dominant product $\alpha_1 \cdots \alpha_k$ of roots of the given polynomial P by considering $(\alpha_1 \cdots \alpha_k)^n$ as a product of roots of P_n (see [15]). Then the approximation theorem of Dirichlet 3.2 is again invoked.

The next result was implicitely used in [15]:

LEMMA 3.10. *Let β_1, $\beta_2, \ldots,$ β_d be distinct complex numbers such that*

$$\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_d\,.$$

Then

$$\limsup_{n \leftarrow \infty} \left|\beta_1^n + \beta_2^n + \cdots + \beta_d^n\right|^{1/n} \leqslant |\beta_1|\,.$$

Proof. We have

$$\left|\beta_1^n + \beta_2^n + \cdots + \beta_d^n\right| \leqslant d|\beta_1|^n ,$$

hence

$$\left|\beta_1^n + \beta_2^n + \cdots + \beta_d^n\right|^{1/n} \leqslant d^{1/n} \cdot |\beta_1| ,$$

which proves the inequality. □

This allows to obtain

COROLLARY 3.11. (Mignotte-Ştefănescu) *We have*

$$\limsup_{n \leftarrow \infty} \left|\beta_1^n + \beta_2^n + \cdots + \beta_d^n\right|^{1/n} = |\beta_1| .$$

Proof. We use Lemma 3.10 and Dirichlet's theorem, by which

$$\left|\beta_1^n + \beta_2^n + \cdots + \beta_d^n\right|^{1/n} \geqslant \left(\frac{r}{2\sqrt{2}}\right)^{1/n} \cdot |\beta_1| \quad \text{for infinitely many } n$$

with $r \leqslant d$.

This proves the converse inequality. □

The following result is then true:

THEOREM 3.12. (Mignotte–Ştefănescu) *If* $\Sigma_n = \beta_1^n + \beta_2^n + \cdots + \beta_d^n$, *we have*

$$\lim_{n \leftarrow \infty} (\max\{|\Sigma_n|, |\Sigma_{n+1}|, \ldots, |\Sigma_{n+d-1}|\}) = |\beta_1| .$$

Note that Theorem 3.12 allows the computation of dominant roots by both the methods of Bernoulli and Graeffe (see [15] and [16]).

Remarks:

1. In practice the method of Graeffe is slower than that of Bernoulli.
2. There exist still open problems for estimating the dominant roots. For example, let us suppose that a complex polynomial P has exactly four dominant roots $\alpha_1, \ldots,$ α_4 such that α_1, α_2 are real and α_3, α_4 are complex conjugate. Neither Bernoulli's nor Graeffe's methods gives convenient results.

4. Bounds for the length

If $P(X) = \sum_{i=0}^d a_i X^i \in \mathbb{C}[X]$, the length of P is $\mathrm{L}(P) = \sum_{i=0}^d |a_i|$. If P divides Q in $\mathbb{C}[X]$, we obtain estimates of $\mathrm{L}(P)$ as functions of $\mathrm{L}(Q)$. The first step is obtaining inequalities between lengths in the particular case $\deg(Q/P) = 1$. Then in the general case it is possiblle to derive inequalities between $\mathrm{L}(P)$ and $\mathrm{L}(Q)$ involving the size of the roots of Q. The similar problem for the height $\mathrm{H}(P) = \max_{i=0}^d |a_i|$ was solved by Mignotte [13].

Suppose that $P \in \mathbb{C}[X]$ and let $Q(X) = (X - \alpha)P(X)$, with $\alpha \in \mathbb{C} \setminus \{0\}$. Assume that $d = \deg(P) \geqslant 1$ and let

$$P(X) = \sum_{i=0}^{d} a_i X^i, \qquad Q(X) = \sum_{i=0}^{d+1} b_i X^i .$$

LEMMA 4.1. *If* $\alpha \in \mathbb{C} \setminus \{0\}$*, then*

$$|\alpha|\,\mathrm{L}(P) \leqslant \sum_{i=0}^{d}\Big(\sum_{j=0}^{d-i}|\alpha|^{-j}\Big)\cdot|b_i|\,.$$

Proof. We have

$$b_i = a_{i-1} - \alpha a_i \quad \text{for all } i = 0,\ldots,d+1\,,$$

with the conventions $a_{-1} = a_{d+1} = 0$. We obtain

$$\begin{aligned}
\alpha^i\,a_i &= \alpha^{i-1}\,a_{i-1} - \alpha^{i-1}\,b_i,\\
\alpha^{i-1}\,a_{i-1} &= \alpha^{i-2}\,a_{i-2} - \alpha^{i-2}\,b_{i-1},\\
&\;\;\vdots\\
\alpha^2\,a_2 &= \alpha\,a_1 - \alpha\,b_2,\\
\alpha\,a_1 &= a_0 - b_1,
\end{aligned}$$

therefore

$$\alpha^i\,a_i = a_0 - \sum_{j=1}^{i}\alpha^{j-1}b_j = -\alpha^{-1}b_0 - \sum_{j=1}^{i}\alpha^{j-1}b_j = -\sum_{j=0}^{i}\alpha^{j-1}b_j\,,$$

which gives

$$\begin{aligned}
|\alpha|\cdot|a_0| &= |b_0|,\\
|\alpha|\cdot|a_1| &\leqslant \frac{1}{|\alpha|}\cdot|b_0| + |b_1|,\\
|\alpha|\cdot|a_2| &\leqslant \frac{1}{|\alpha|^2}\cdot|b_0| + \frac{1}{|\alpha|}\cdot|b_1| + |b_2|,\\
&\;\;\vdots\\
|\alpha|\cdot|a_d| &\leqslant \frac{1}{|\alpha|^d}\cdot|b_0| + \frac{1}{|\alpha|^{d-1}}\cdot|b_1| + \cdots + |b_d|\,.
\end{aligned}$$

By summation we get the result. □

PROPOSITION 4.2. *If* $|\alpha| > 1$*, then*

$$(|\alpha|-1)\,\mathrm{L}(P) \leqslant \Big(1 - \frac{1}{|\alpha|^{d+1}}\Big)\,\mathrm{L}(Q)\,.$$

Proof. By Lemma 4.1, we have

$$|\alpha|\,\mathrm{L}(P) \leqslant \Big(\sum_{j=0}^{d}|\alpha|^{-j}\Big)\cdot\Big(\sum_{i=0}^{d}|b_i|\Big) \leqslant \frac{1-|\alpha|^{-d-1}}{1-|\alpha|^{-1}}\,\mathrm{L}(Q)\,,$$

which gives the inequality. □

PROPOSITION 4.3. *If* $0 < |\alpha| < 1$, *then*

$$(1-|\alpha|)\,\mathrm{L}(P) \leqslant (1-|\alpha|^{d+1})\,\mathrm{L}(Q)\,.$$

Proof. We apply Proposition 4.2 to the reciprocal polynomials P^*, Q^* associated to P, respectively Q. Since $Q^* = (X - \frac{1}{\alpha})(-\alpha P^*)$, $\mathrm{L}(P^*) = \mathrm{L}(P)$ and $\mathrm{L}(Q^*) = \mathrm{L}(Q)$, it follows that

$$\Big(\frac{1}{|\alpha|} - 1\Big)\cdot|\alpha|\,\mathrm{L}(P) \leqslant (1-|\alpha|^{d+1})\cdot\mathrm{L}(Q)\,,$$

therefore $(1-|\alpha|)\,\mathrm{L}(P) \leqslant (1-|\alpha|^{d+1})\,\mathrm{L}(Q)$. □

COROLLARY 4.4. *We have*

$$\big|1-|\alpha|\big|\cdot\mathrm{L}(P) \leqslant \mathrm{L}(Q) \quad \textit{for all} \quad \alpha\in\mathbb{C}\,.$$

Proof. For $\alpha = 0$ we have equality. If $|\alpha| > 1$ we apply Proposition 4.2, while for $0 < |\alpha| < 1$ we use Proposition 4.3. □

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