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MULTIVARIABLE MIXED MEANS AND INEQUALITIES OF HARDY AND LEVIN–COCHRAN–LEE TYPE

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Abstract. We consider integral power means of arbitrary real order, taken over cells in $\mathbf{R}^n$, and their dual means. We establish related mixed-means inequalities and then apply obtained results to derive multivariable analogues and some new generalizations of Hardy and Levin-Cochran-Lee type inequalities. Moreover, we prove the constant factors involved in the right-hand sides of these relations to be the best possible, that is, they cannot be replaced with smaller constants.

1. Introduction

In 1928, G. H. Hardy proved one of the most important classical one-dimensional integral inequalities (cf. [7], or [14]):

THEOREM A. *Let $p,k \in \mathbf{R}$, $p>1$ and $k \neq 1$. Suppose f is a non-negative measurable function such that $x^{1-\frac{k}{p}}f \in L^p(0,\infty)$, and the function F is defined on $\langle 0,\infty\rangle$ by*

$$F(x)=\begin{cases}\displaystyle\int_0^x f(t)\,dt, & k>1,\\[2ex] \displaystyle\int_x^\infty f(t)\,dt, & k<1.\end{cases}$$

Then

$$\int_0^\infty x^{-k}F^p(x)\,dx<\left(\frac{p}{|k-1|}\right)^p\int_0^\infty x^{p-k}f^p(x)\,dx, \tag{1}$$

unless $f\equiv 0$. The constant $\left(\frac{p}{|k-1|}\right)^p$ is the best possible.

Besides the famous Hardy's inequality (1), we also need to consider a pair of one-dimensional weighted exponential integral inequalities, closely related to (1), but discovered more than fifty years later. They are given in

THEOREM B. *Let $\alpha,\gamma\in\mathbf{R}$, $\alpha\neq 0$, and f be a positive measurable function on $\langle 0,\infty\rangle$ such that $\int_0^\infty x^{\gamma-1}f(x)\,dx<\infty$. Then the inequalities*

$$\int_0^\infty x^{\gamma-1}\exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right]dx\leqslant e^{\frac{\gamma}{\alpha}}\int_0^\infty x^{\gamma-1}f(x)\,dx, \tag{2}$$

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for $\alpha > 0$*, and*

$$\int_0^\infty x^{\gamma-1} \exp\left[-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right]dx \leqslant e^{\frac{\gamma}{\alpha}}\int_0^\infty x^{\gamma-1} f(x)\,dx, \tag{3}$$

for $\alpha < 0$*, hold. The constant* $e^{\frac{\gamma}{\alpha}}$ *is the best possible for both inequalities.*

Inequality (2) is due to J. A. Cochran and C.-S. Lee, [2], while inequality (3) represents its dual result, proved by E. R. Love in [12] (see also [14] for both results). However, it is not widely known that (2) is only a special case of a more general inequality of V. Levin, published in 1938 in his unnoticed paper [11], written in Russian. Because of this reason, inequalities (2) and (3) will be called the Levin-Cochran-Lee inequalities.

Although classical, inequalities (1), (2) and (3) were during the last decade generalized in many different ways by numerous mathematicians. One possible generalization of Theorem A and Theorem B is to derive their various multivariable analogues. In this paper we will generalize these theorems to n-cells, that is, we will obtain inequalities with integrals taken over axis-parallel rectangular blocks in $\mathbf{R}^n$.

Before presenting our idea, it is necessary to introduce some notation. For two vectors $\mathbf{c} = (c_1,\dots,c_n), \mathbf{d} = (d_1,\dots,d_n) \in \mathbf{R}^n$ we define the vector $\mathbf{c}\cdot\mathbf{d}$ by

$$\mathbf{c}\cdot\mathbf{d} = (c_1d_1,\dots,c_nd_n).$$

We also write $\mathbf{c} \ll \mathbf{d}$ (or, equivalently, $\mathbf{d} \gg \mathbf{c}$) if componentwise $c_i < d_i$, $i = 1,\dots,n$. In particular, if we denote $\mathbf{0} = (0,\dots,0), \mathbf{1} = (1,\dots,1) \in \mathbf{R}^n$, then cases $c_i > 0$ and $d_i > 1$, $i = 1,\dots,n$, can be written as $\mathbf{c} \gg \mathbf{0}$ and $\mathbf{d} \gg \mathbf{1}$ respectively.

Further, for vectors $\mathbf{c} = (c_1,\dots,c_n) \gg \mathbf{0}$ and $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbf{R}^n$, let $\mathbf{c}^{\boldsymbol{\alpha}} = \prod_{i=1}^n c_i^{\alpha_i}$. Of course, $\mathbf{c}^{\mathbf{1}} = \prod_{i=1}^n c_i$ and $\mathbf{c}^{-\mathbf{1}} = \prod_{i=1}^n \frac{1}{c_i}$. On the other hand, if $\mathbf{c} \in \mathbf{R}^n$ is such that $c_i \neq 0$, $i = 1,\dots,n$, we denote $\frac{\mathbf{1}}{\mathbf{c}} = \left(\frac{1}{c_1},\dots,\frac{1}{c_n}\right)$.

We also introduce a notation for some special sets. For arbitrary $\mathbf{c},\mathbf{d} \in \mathbf{R}^n$, $\mathbf{c} \ll \mathbf{d}$, a subset $R_{\mathbf{c},\mathbf{d}} \subset \mathbf{R}^n$ let be defined by

$$R_{\mathbf{c},\mathbf{d}} = \{(x_1,\dots,x_n) \in \mathbf{R}^n : c_i < x_i < d_i, i = 1,\dots,n\}.$$

Similarly, if $\mathbf{d} \in \mathbf{R}^n$, let

$$R_{\mathbf{d},\infty} = \{(x_1,\dots,x_n) \in \mathbf{R}^n : d_i < x_i < \infty, i = 1,\dots,n\},$$

and finally, let

$$R = R_{\mathbf{0},\infty} = \{(x_1,\dots,x_n) \in \mathbf{R}^n : 0 < x_i < \infty, i = 1,\dots,n\}.$$

Since throughout this paper all integrals will be taken over these sets, by $\int_{R_{\mathbf{c},\mathbf{d}}} u(\mathbf{z})\,d\mathbf{z}$ and $\int_R u(\mathbf{z})\,d\mathbf{z}$ we denote n-fold integrals

$$\int_{c_1}^{d_1}\cdots\int_{c_n}^{d_n} u(z_1,\dots,z_n)\,dz_n\dots dz_1 \quad\text{and}\quad \int_0^\infty\cdots\int_0^\infty u(z_1,\dots,z_n)\,dz_n\dots dz_1.$$

The first multivariable version of Hardy's inequality was given by B. G. Pachpatte in [19]. Using Fubini's theorem and Jensen's inequality, he proved the next theorem.

THEOREM C. *Let $p \in \mathbf{R}$, $p > 1$, and let $f \in L^p(R)$ be a non-negative function. If the function F is defined on R by*

$$F(\mathbf{x}) = \int_{R_{\mathbf{0},\mathbf{x}}} f(\mathbf{y})\, d\mathbf{y},$$

then

$$\int_R \mathbf{x}^{-p\mathbf{1}} F^p(\mathbf{x})\, d\mathbf{x} < \left(\frac{p}{p-1}\right)^{np} \int_R f^p(\mathbf{x})\, d\mathbf{x},$$

unless $f \equiv 0$. The constant $\left(\frac{p}{p-1}\right)^{np}$ is the best possible.

Motivated by Pachpatte's result, in this paper we generalize Theorem C to obtain n-variable Hardy's inequality which is a natural generalization of Theorem A. We also derive a multivariable version of Theorem B of the same type and establish n-variable Levin-Cochran-Lee type inequalities. Moreover, we prove that the constants obtained on the right-hand sides of these inequalities are the best possible, that is, none of them can be replaced with a smaller constant factor.

Furthermore, since the outer integrals on the both sides of derived multivariable relations are taken over R, here we also investigate inequalities of the same type, with the outer integrals taken over $R_{\mathbf{0},\mathbf{b}}$ or $R_{\mathbf{b},\infty}$. We show that in that case Hardy and Levin-Cochran-Lee type inequalities can be improved by providing sharp smaller upper bounds for their left-hand sides.

A technique that will be used in the proofs consists of introducing two types of multivariable integral means of arbitrary real order, over $R_{\mathbf{c},\mathbf{d}}$ and $R_{\mathbf{d},\infty}$, with power weights, and proving the corresponding mixed (r,s)-means inequalities. All desired results will be elegantly obtained as limit cases of related mixed-means inequalities.

The idea of using mixed means in deriving Hardy and Levin-Cochran-Lee type inequalities has already been applied to one-dimensional integrals, finite and infinite series (cf. [3] and [5]), and to integrals over balls in $\mathbf{R}^n$ centered at the origin (cf. [4] and [6]).

Although the mixed-means inequalities will be used here only as a technical help in proving the main theorems, they are interesting in their own right. Their different discrete and integral variants were investigated very recently (cf. [1], [3], [4], [8], [9], [10], [13], [15], [16], [17], or [20]).

The analysis used in the proofs is mostly based on classical real analysis, on the well-known Minkowski's and Jensen's inequality for integrals, and on the properties of integral means (cf. [7]).

2. Multivariable integral means and related inequalities for mixed means

We start by introducing multivariable integral power means over n-cells and their natural dual means. Let $\boldsymbol{\alpha}, \mathbf{c}, \mathbf{d} \in R$ be such that $\mathbf{c} \ll \mathbf{d}$, and let f be a non-negative measurable function on R. For $r \in \mathbf{R}$, $r \neq 0$, as in [7], we define the integral mean of order r, with the power weight, of f, $M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha})$, by

$$M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) = \left[\boldsymbol{\alpha}^{\mathbf{1}}(\mathbf{d}-\mathbf{c})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{c},\mathbf{d}}}(\mathbf{x}-\mathbf{c})^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{x})\,d\mathbf{x}\right]^{\frac{1}{r}}, \tag{4}$$

with the convention that $M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) = 0$ if $r < 0$ and f vanishes on a subset of $R_{\mathbf{c},\mathbf{d}}$ of positive measure. Especially, if f is a positive function, by

$$\begin{aligned} G(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) &= M^{[0]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) \\ &= \exp\left(\boldsymbol{\alpha}^{\mathbf{1}}(\mathbf{d}-\mathbf{c})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{c},\mathbf{d}}}(\mathbf{x}-\mathbf{c})^{\boldsymbol{\alpha}-\mathbf{1}}\log f(\mathbf{x})\,d\mathbf{x}\right) \end{aligned} \tag{5}$$

we denote its geometric mean over $R_{\mathbf{c},\mathbf{d}}$.
These means have the further properties (cf. [7]):

$$M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) \leqslant M^{[s]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}), \quad \text{for } r,s \in \mathbf{R},\ r < s, \tag{6}$$

and

$$\lim_{r\to 0-} M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) = \lim_{r\to 0+} M^{[r]}(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}) = G(f;\mathbf{c},\mathbf{d},\boldsymbol{\alpha}). \tag{7}$$

The means defined by (4) and (5) naturally generate their dual means. Let $\mathbf{d},\boldsymbol{\alpha} \in \mathbf{R}^n$ be such that $\mathbf{d} \gg \mathbf{0}$ and $\boldsymbol{\alpha} \ll \mathbf{0}$. If f is a non-negative measurable function on R, for $r \in \mathbf{R}$, $r \neq 0$, the relation

$$M_*^{[r]}(f;\mathbf{d},\boldsymbol{\alpha}) = \left[(-\boldsymbol{\alpha})^{\mathbf{1}}\mathbf{d}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{d},\infty}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{x})\,d\mathbf{x}\right]^{\frac{1}{r}} \tag{8}$$

defines the dual mean of order r of f. We set $M_*^{[r]}(f;\mathbf{d},\boldsymbol{\alpha}) = 0$ if $r < 0$ and $f(\mathbf{x}) = 0$ on a subset of $R_{\mathbf{d},\infty}$ of positive measure. Finally, for a positive function f, the relation

$$G_*(f;\mathbf{d},\boldsymbol{\alpha}) = M_*^{[0]}(f;\mathbf{d},\boldsymbol{\alpha}) = \exp\left((-\boldsymbol{\alpha})^{\mathbf{1}}\mathbf{d}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{d},\infty}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log f(\mathbf{x})\,d\mathbf{x}\right) \tag{9}$$

defines the dual geometric mean of f.

The main properties of dual means (8) and (9) are similar to those of means (4) and (5):

$$M_*^{[r]}(f;\mathbf{d},\boldsymbol{\alpha}) \leqslant M_*^{[s]}(f;\mathbf{d},\boldsymbol{\alpha}), \quad \text{for } r,s \in \mathbf{R},\ r < s, \tag{10}$$

and

$$\lim_{r\to 0-} M_*^{[r]}(f;\mathbf{d},\boldsymbol{\alpha}) = \lim_{r\to 0+} M_*^{[r]}(f;\mathbf{d},\boldsymbol{\alpha}) = G_*(f;\mathbf{d},\boldsymbol{\alpha}). \tag{11}$$

Now we state the basic result of this paper.

THEOREM 1. *Let f be a non-negative measurable function on R and $r,s\in\mathbf{R}$, $\mathbf{a},\mathbf{b}\in R$ be such that $r,s\neq 0$, $r<s$, and $\mathbf{a}\ll\mathbf{b}$. If $\boldsymbol{\alpha},\boldsymbol{\gamma}\in\mathbf{R}^n$ are arbitrary, then*

$$\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}\left[(\mathbf{x}-\mathbf{a})^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{a},\mathbf{x}}}(\mathbf{t}-\mathbf{a})^{\boldsymbol{\Upsilon}-\mathbf{1}}f^{s}(\mathbf{t})\,d\mathbf{t}\right]^{\frac{r}{s}}d\mathbf{x}\right\}^{\frac{1}{r}}$$
$$\geqslant\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[(\mathbf{x}-\mathbf{a})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\mathbf{x}}}(\mathbf{t}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{t})\,d\mathbf{t}\right]^{\frac{s}{r}}d\mathbf{x}\right\}^{\frac{1}{s}}. \tag{12}$$

Equality holds if and only if f is of the form

$$f(\mathbf{a}+\mathbf{u}\cdot(\mathbf{x}-\mathbf{a}))=f_1(\mathbf{u})\cdot f_2(\mathbf{x}),\quad \mathbf{u}\in R_{\mathbf{0},\mathbf{1}},\ \mathbf{x}\in R_{\mathbf{a},\mathbf{b}}. \tag{13}$$

Proof. Using the change $\mathbf{t}=\mathbf{a}+\mathbf{u}\cdot(\mathbf{x}-\mathbf{a})$ of the independent variable in the inner integral, the left-hand side of (12) becomes

$$\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}\left[\int_{R_{\mathbf{0},\mathbf{1}}}\mathbf{u}^{\boldsymbol{\Upsilon}-\mathbf{1}}f^{s}(\mathbf{a}+\mathbf{u}\cdot(\mathbf{x}-\mathbf{a}))\,d\mathbf{u}\right]^{\frac{r}{s}}d\mathbf{x}\right\}^{\frac{1}{r}}, \tag{14}$$

since the Jacobian of the transformation is equal to $|\frac{\partial t_i}{\partial u_j}|=(\mathbf{x}-\mathbf{a})^{\mathbf{1}}$. Further, by applying Minkowski's integral inequality, we have that (14) is greater than or equal to

$$\left\{\int_{R_{\mathbf{0},\mathbf{1}}}\mathbf{u}^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{a}+\mathbf{u}\cdot(\mathbf{x}-\mathbf{a}))\,d\mathbf{x}\right]^{\frac{s}{r}}d\mathbf{u}\right\}^{\frac{1}{s}}. \tag{15}$$

Now, put $\mathbf{a}+\mathbf{u}\cdot(\mathbf{x}-\mathbf{a})=\mathbf{t}$ back and denote $\tilde{\mathbf{u}}=\mathbf{a}+\mathbf{u}\cdot(\mathbf{b}-\mathbf{a})$. Since $|\frac{\partial x_i}{\partial t_j}|=\mathbf{u}^{-\mathbf{1}}$, the term (15) is equal to

$$\left\{\int_{R_{\mathbf{0},\mathbf{1}}}\mathbf{u}^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\alpha}}\mathbf{u}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\tilde{\mathbf{u}}}}(\mathbf{t}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{t})\,d\mathbf{t}\right]^{\frac{s}{r}}d\mathbf{u}\right\}^{\frac{1}{s}}$$
$$=\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[(\mathbf{x}-\mathbf{a})^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{a},\mathbf{x}}}(\mathbf{t}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}}f^{r}(\mathbf{t})\,d\mathbf{t}\right]^{\frac{s}{r}}d\mathbf{x}\right\}^{\frac{1}{s}},$$

that is, the right-hand side of inequality (12). The last equality is due to the substitution $\mathbf{x}=\mathbf{a}+\mathbf{u}\cdot(\mathbf{b}-\mathbf{a})$.

It is obvious that the equality in (12) holds if and only if it occurs in Minkowski's inequality, that is, if and only if the function f fulfills the condition (13). □

The dual inequality to (12) is stated in the following theorem.

THEOREM 2. *Let* $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbf{R}^n$, $\mathbf{b} \in R$, *and let* f *be a non-negative measurable function on* R. *If* $r, s \in \mathbf{R}$ *are such that* $r, s \neq 0$ *and* $r < s$, *then*

$$\left\{\mathbf{b}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{b},\infty}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} \left[\mathbf{x}^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{x},\infty}} \mathbf{t}^{\boldsymbol{\gamma}-\mathbf{1}} f^s(\mathbf{t})\, d\mathbf{t}\right]^{\frac{r}{s}} d\mathbf{x}\right\}^{\frac{1}{r}}$$
$$\geqslant \left\{\mathbf{b}^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{b},\infty}} \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} \left[\mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{x},\infty}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} f^r(\mathbf{t})\, d\mathbf{t}\right]^{\frac{s}{r}} d\mathbf{x}\right\}^{\frac{1}{s}}. \tag{16}$$

Proof. Define the function g by $g(\mathbf{x}) = f\left(\frac{\mathbf{1}}{\mathbf{x}}\right)$, $\mathbf{x} \in R$. If we, instead to f, $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, apply Theorem 1 to g, $\mathbf{0}$, $\frac{\mathbf{1}}{\mathbf{b}}$, $-\boldsymbol{\alpha}$, $-\boldsymbol{\gamma}$ respectively, (12) can be written as

$$\left\{\mathbf{b}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\frac{\mathbf{1}}{\mathbf{b}}}} \mathbf{u}^{-\boldsymbol{\alpha}-\mathbf{1}} \left[\mathbf{u}^{\boldsymbol{\gamma}} \int_{R_{\mathbf{0},\mathbf{u}}} \mathbf{v}^{-\boldsymbol{\gamma}-\mathbf{1}} g^s(\mathbf{v})\, d\mathbf{v}\right]^{\frac{r}{s}} d\mathbf{u}\right\}^{\frac{1}{r}}$$
$$\geqslant \left\{\mathbf{b}^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{0},\frac{\mathbf{1}}{\mathbf{b}}}} \mathbf{u}^{-\boldsymbol{\gamma}-\mathbf{1}} \left[\mathbf{u}^{\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{u}}} \mathbf{v}^{-\boldsymbol{\alpha}-\mathbf{1}} g^r(\mathbf{v})\, d\mathbf{v}\right]^{\frac{s}{r}} d\mathbf{u}\right\}^{\frac{1}{s}}. \tag{17}$$

Inequality (16) now easily follows by using the substitution $\mathbf{t} = \frac{\mathbf{1}}{\mathbf{v}}$ in the inner integrals, and then $\mathbf{x} = \frac{\mathbf{1}}{\mathbf{u}}$ in the outer integrals on the both sides of (17). $\square$

Note that the parameter $\boldsymbol{\alpha} \in \mathbf{R}^n$ in Theorem 1 and Theorem 2 was arbitrary. Furthermore, the previous results avoided cases that include geometric and dual geometric mean, that is, the cases where $r = 0$ or $s = 0$. These cases will be treated in the next two theorems, but this time with some constraints on $\boldsymbol{\alpha}$.

THEOREM 3. *Let* $f : R \to \mathbf{R}$ *be a positive measurable function. If* $\mathbf{a}, \mathbf{b}, \boldsymbol{\alpha} \in R$ *and* $\boldsymbol{\gamma} \in \mathbf{R}^n$ *are such that* $\mathbf{a} \ll \mathbf{b}$, *then the inequality*

$$\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{a},\mathbf{b}}} (\mathbf{x}-\mathbf{a})^{\boldsymbol{\gamma}-\mathbf{1}} \left[\exp\left(\boldsymbol{\alpha}^{\mathbf{1}} (\mathbf{x}-\mathbf{a})^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{a},\mathbf{x}}} (\mathbf{t}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\, d\mathbf{t}\right)\right]^{s} d\mathbf{x}\right\}^{\frac{1}{s}}$$
$$\leqslant \exp\left\{\boldsymbol{\alpha}^{\mathbf{1}} (\mathbf{b}-\mathbf{a})^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{a},\mathbf{b}}} (\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}-\mathbf{1}} \log\left[(\mathbf{x}-\mathbf{a})^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{a},\mathbf{x}}} (\mathbf{t}-\mathbf{a})^{\boldsymbol{\gamma}-\mathbf{1}} f^s(\mathbf{t})\, d\mathbf{t}\right]^{\frac{1}{s}} d\mathbf{x}\right\} \tag{18}$$

holds for all $s \in \mathbf{R}$, $s > 0$.

Proof. For arbitrary $0 < r < s$, apply Theorem 1 to the function $\boldsymbol{\alpha}^{\frac{1}{r}\mathbf{1}} f$ instead of f, and denote

$$h(\mathbf{x}) = \left[(\mathbf{x}-\mathbf{a})^{-\boldsymbol{\gamma}} \int_{R_{\mathbf{a},\mathbf{x}}} (\mathbf{t}-\mathbf{a})^{\boldsymbol{\gamma}-\mathbf{1}} f^s(\mathbf{t})\, d\mathbf{t}\right]^{\frac{1}{s}}, \quad \mathbf{x} \in R.$$

Using (4), inequality (12) becomes

$$\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[M^{[r]}(f;\mathbf{a},\mathbf{x},\boldsymbol{\alpha})\right]^{s}d\mathbf{x}\right\}^{\frac{1}{s}} \leqslant M^{[r]}(h;\mathbf{a},\mathbf{b},\boldsymbol{\alpha}). \tag{19}$$

The definition and the properties (6) and (7) of means M imply that the sequence $\left(M^{[r]}(f;\mathbf{a},\mathbf{x},\boldsymbol{\alpha})\right)$ is non-negative and monotonically converges to the geometric mean $G(f;\mathbf{a},\mathbf{x},\boldsymbol{\alpha})$, as r decreases to 0. Hence, by Lebesgue's monotone convergence theorem, the limit of the left-hand side of (19), as $r\searrow 0$, is equal to

$$\left\{(\mathbf{b}-\mathbf{a})^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{a},\mathbf{b}}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[G(f;\mathbf{a},\mathbf{x},\boldsymbol{\alpha})\right]^{s}d\mathbf{x}\right\}^{\frac{1}{s}}.$$

Since (7) also implies that $\lim_{r\searrow 0}M^{[r]}(h;\mathbf{a},\mathbf{b},\boldsymbol{\alpha}) = G(h;\mathbf{a},\mathbf{b},\boldsymbol{\alpha})$, the inequality (18) holds by taking $\lim_{r\searrow 0}$ of (19). □

The same analysis can be used to prove the dual result of Theorem 3.

THEOREM 4. *Let* $\mathbf{b},\boldsymbol{\alpha},\boldsymbol{\gamma}\in\mathbf{R}^n$ *be such that* $\mathbf{b}\gg\mathbf{0}$, $\boldsymbol{\alpha}\ll\mathbf{0}$, *and let* $s\in\mathbf{R}$, $s>0$. *If* f *is a positive measurable function on* $\mathbf{R}$, *then*

$$\begin{aligned}&\left\{\mathbf{b}^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{b},\infty}}\mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}}\left[\exp\left((-\boldsymbol{\alpha})^{\mathbf{1}}\mathbf{x}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{x},\infty}}\mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}}\log f(\mathbf{t})\,d\mathbf{t}\right)\right]^{s}d\mathbf{x}\right\}^{\frac{1}{s}}\\ &\quad\leqslant \exp\left\{(-\boldsymbol{\alpha})^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{b},\infty}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log\left[\mathbf{x}^{-\boldsymbol{\Upsilon}}\int_{R_{\mathbf{x},\infty}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f^{s}(\mathbf{t})\,d\mathbf{t}\right]^{\frac{1}{s}}d\mathbf{x}\right\}.\end{aligned} \tag{20}$$

Proof. Inequality (20) follows from the definition and properties of means M_*, by using the same tools as in the proof of the previous theorem, if Theorem 1 is rewritten for $0<r<s$, $\boldsymbol{\alpha}\ll\mathbf{0}$, and the function $(-\boldsymbol{\alpha})^{\frac{1}{r}\mathbf{1}}f$ instead of f. □

The main result of this section are the multivariable mixed (r,s)-means inequalities for means M and their dual means M_*.

THEOREM 5. *Let* $\boldsymbol{\alpha},\boldsymbol{\gamma}\in R$ *and* $\mathbf{a},\mathbf{b}\in\mathbf{R}^n$ *be such that* $\mathbf{a}\ll\mathbf{b}$. *If* $f:R\to\mathbf{R}$ *is a positive measurable function, then the inequality*

$$M^{[s]}\left(M^{[r]}(f;\mathbf{a},\mathbf{x},\boldsymbol{\alpha});\mathbf{a},\mathbf{b},\boldsymbol{\gamma}\right)\leqslant M^{[r]}\left(M^{[s]}(f;\mathbf{a},\mathbf{x},\boldsymbol{\gamma});\mathbf{a},\mathbf{b},\boldsymbol{\alpha}\right) \tag{21}$$

holds for all $r,s\in\mathbf{R}$, $r<s$.

Proof. Directly from Theorem 1, by replacing the function f with $\boldsymbol{\alpha}^{\frac{1}{r}\mathbf{1}}\boldsymbol{\gamma}^{\frac{1}{s}\mathbf{1}}f$, and using the same analysis as in the proof of Theorem 3. □

We conclude this section with mixed-means inequalities for dual means M_*.

THEOREM 6. *Let* $\mathbf{b} \in R$ *and* $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbf{R}^n$ *be such that* $\boldsymbol{\alpha}, \boldsymbol{\gamma} \ll \mathbf{0}$. *If* $f : R \to \mathbf{R}$ *is a positive measurable function, then the inequality*

$$M_*^{[s]}\left(M_*^{[r]}(f;\mathbf{x},\boldsymbol{\alpha});\mathbf{b},\boldsymbol{\gamma}\right) \leqslant M_*^{[r]}\left(M_*^{[s]}(f;\mathbf{x},\boldsymbol{\gamma});\mathbf{b},\boldsymbol{\alpha}\right) \tag{22}$$

holds for all $r, s \in \mathbf{R}$, $r < s$.

Proof. If $r, s \neq 0$, inequality (22) follows immediately from Theorem 2, applied to the function $(-\boldsymbol{\alpha})^{\frac{1}{r}\mathbf{1}}(-\boldsymbol{\gamma})^{\frac{1}{s}\mathbf{1}}f$. In cases where $0 = r < s$ or $r < s = 0$, (22) is obtained by using (10), (11) and Lebesgue's monotone convergence theorem, as in the proof of Theorem 4. □

If $r, s \neq 0$, it is easy to see that inequalities (21) and (22) hold also for non-negative functions.

3. Multivariable Hardy's inequality

Mixed means and related results, established in the previous section, can be used as an approach to different integral inequalities. In this section we will apply relations (12) and (16) to prove a natural multivariable analogue of Theorem A, and also to obtain some of its new sharp generalizations.
First, we need to prove the following two technical lemmas.

LEMMA 1. *If* $(b_n)_{n\in\mathbf{N}}$ *is a strictly increasing sequence of non-negative real numbers such that* $\lim_{n\to\infty} b_n = 1$, *then for each* $\varepsilon \in \langle 0, 1 - b_1\rangle$ *there exists* $N_0 \in \mathbf{N}$ *such that the inequality*

$$\sum_{n=1}^{N} \frac{b_n}{n} > (1-\varepsilon)\sum_{n=1}^{N}\frac{1}{n}$$

holds for all $N \in \mathbf{N}$, $N \geqslant N_0$.

Proof. Let $\varepsilon \in \langle 0, 1-b_1\rangle$ be arbitrary. Since $b_1 < 1-\varepsilon$ and $b_n \nearrow 1$, there exists a number $n_0 = \max\{n \in \mathbf{N} : b_n \leqslant 1-\varepsilon\}$. Hence, $1 - b_1 - \varepsilon > \cdots > 1 - b_{n_0} - \varepsilon \geqslant 0$ and $\sum_{n=1}^{n_0}(1 - b_n - \varepsilon)\frac{1}{n} > 0$. Moreover, if $\delta = \frac{1+\varepsilon-b_{n_0+1}}{2}$, then $\varepsilon - \delta > 0$, and

$$b_1 < \cdots < b_{n_0} \leqslant 1-\varepsilon < 1-\delta < b_{n_0+1} < b_{n_0+2} < \cdots < 1. \tag{23}$$

Finally, since the series $\sum_{n=n_0+1}^{\infty}\frac{1}{n}$ is divergent, there exists $N_0 \in \mathbf{N}$ such that

$$\sum_{n=n_0+1}^{N}\frac{1}{n} > \frac{1}{\varepsilon-\delta}\sum_{n=1}^{n_0}(1-b_n-\varepsilon)\frac{1}{n}, \quad N \geqslant N_0,$$

or, equivalently,

$$(1-\varepsilon)\sum_{n=1}^{N}\frac{1}{n} < \sum_{n=1}^{n_0}\frac{b_n}{n} + \sum_{n=n_0+1}^{N}(1-\delta)\frac{1}{n}, \quad N \geqslant N_0.$$

Considering (23), the right-hand side of the last inequality is less than

$$\sum_{n=1}^{n_0} \frac{b_n}{n} + \sum_{n=n_0+1}^{N} \frac{b_n}{n} = \sum_{n=1}^{N} \frac{b_n}{n},$$

so the proof is completed. □

LEMMA 2. *Let $k,p \in \mathbf{R}$ be such that $k,p > 1$, and let $N \in \mathbf{N}$ be arbitrary. If the function f_N is defined on $\langle 0,\infty\rangle$ by $f_N(x) = x^{\frac{k-1}{p}-1}\chi_{[1,N+1]}(x)$, then*

(i) $\displaystyle\int_0^\infty x^{p-k} f_N^p(x)\,dx < \sum_{n=1}^{N} \frac{1}{n}$;

(ii) $\displaystyle\int_0^\infty x^{-k}\left[\int_0^x f_N(t)\,dt\right]^p dx > \left(\frac{p}{k-1}\right)^p \sum_{n=1}^{N} \left(\frac{n}{n+1}\right)^k \left(1-n^{\frac{1-k}{p}}\right)^p \frac{1}{n}$.

Proof. Observe that $\sum_{n=1}^{N} \frac{1}{n}$ is an upper Darboux sum for the function $x \mapsto \frac{1}{x}$ on $[1,N+1]$. Therefore,

$$\int_0^\infty x^{p-k} f_N^p(x)\,dx = \int_1^{N+1} \frac{1}{x}\,dx < \sum_{n=1}^{N} \frac{1}{n},$$

so (i) is proved. Inequality (ii) is obtained by a straightforward computation:

$$\begin{aligned}
\int_0^\infty x^{-k}\left[\int_0^x f_N(t)\,dt\right]^p dx &> \int_1^{N+1} x^{-k}\left[\int_0^x f_N(t)\,dt\right]^p dx \\
&= \int_1^{N+1} x^{-k}\left[\int_1^x t^{\frac{k-1}{p}-1}\,dt\right]^p dx \geqslant \sum_{n=1}^{N} \int_n^{n+1} x^{-k}\left[\int_1^n t^{\frac{k-1}{p}-1}\,dt\right]^p dx \\
&= \left(\frac{p}{k-1}\right)^p \sum_{n=1}^{N} \left(n^{\frac{k-1}{p}}-1\right)^p \int_n^{n+1} x^{-k}\,dx \\
&\geqslant \left(\frac{p}{k-1}\right)^p \sum_{n=1}^{N} \left(n^{\frac{k-1}{p}}-1\right)^p \frac{1}{(n+1)^k} \\
&= \left(\frac{p}{k-1}\right)^p \sum_{n=1}^{N} \left(\frac{n}{n+1}\right)^k \left(1-n^{\frac{1-k}{p}}\right)^p \frac{1}{n}. \qquad \square
\end{aligned}$$

In the next theorem we prove the multivariable Hardy's inequality.

THEOREM 7. *Suppose $p \in \mathbf{R}$, $p > 1$, and $\mathbf{k} \in \mathbf{R}^n$, $\mathbf{k} \gg \mathbf{1}$ or $\mathbf{k} \ll \mathbf{1}$. If f is a non-negative measurable function such that $\mathbf{x}^{\mathbf{1}-\frac{1}{p}\mathbf{k}} f \in L^p(R)$, and the function F is defined on R by*

$$F(\mathbf{x}) = \begin{cases} \displaystyle\int_{R_{\mathbf{0},\mathbf{x}}} f(\mathbf{t})\,d\mathbf{t}, & \mathbf{k} \gg \mathbf{1}, \\ \displaystyle\int_{R_{\mathbf{x},\infty}} f(\mathbf{t})\,d\mathbf{t}, & \mathbf{k} \ll \mathbf{1}, \end{cases} \tag{24}$$

then

$$\int_R \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\,d\mathbf{x} \leqslant \left[p^n \cdot \left|(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right|\right]^p \int_R \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\,d\mathbf{x}. \tag{25}$$

The constant $\left[p^n \cdot \left|(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right|\right]^p$ *is the best possible.*

Proof. Consider the case $\mathbf{k} \gg \mathbf{1}$ first. If $\mathbf{a} = \mathbf{0}$, arbitrary $\mathbf{b} \in R$, $r = 1$, $s = p > 1$, $\boldsymbol{\alpha} = \mathbf{1}$ and $\boldsymbol{\gamma} = (p+1)\mathbf{1} - \mathbf{k}$ are chosen as parameters in Theorem 1, inequality (12) can be written in the form

$$\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\,d\mathbf{x} \leqslant \mathbf{b}^{\mathbf{1}-\mathbf{k}} \left\{ \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}} \left[\int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\,d\mathbf{t} \right]^{\frac{1}{p}} d\mathbf{x} \right\}^p. \tag{26}$$

Since

$$I_{\mathbf{b}} = \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\,d\mathbf{t} \geqslant \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\,d\mathbf{t}, \quad \mathbf{x} \in R_{\mathbf{0},\mathbf{b}},$$

the right-hand side of (26) is not greater than

$$\begin{aligned} \mathbf{b}^{\mathbf{1}-\mathbf{k}} \left[\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}} d\mathbf{x} \right]^p \cdot I_{\mathbf{b}} &= \mathbf{b}^{\mathbf{1}-\mathbf{k}} \cdot \left\{ \mathbf{b}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})} \cdot \left[\frac{1}{p}(\mathbf{k}-\mathbf{1}) \right]^{-\mathbf{1}} \right\}^p \cdot I_{\mathbf{b}} \\ &= \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \cdot I_{\mathbf{b}}\,. \end{aligned}$$

Hence,

$$\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\,d\mathbf{x} \leqslant \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\,d\mathbf{t}, \tag{27}$$

so (25) holds by taking $\lim_{\mathbf{b}\to\infty}$ (that is, $b_i \to \infty$, $i = 1, \dots, n$).

Now we prove that the constant $\left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p$ is the best possible for (25). Let $\varepsilon \in \langle 0, 1\rangle$ be arbitrary and $(f_N)_{N\in\mathbf{N}}$ be the sequence of functions on R, defined by $f_N(\mathbf{x}) = \prod_{i=1}^n f_{i,N}(x_i)$, where $f_{i,N}(x) = x^{\frac{k_i-1}{p}-1} \chi_{[1,N+1]}(x)$, $x > 0$. Using Fubini's theorem and Lemma 2, (ii), the left-hand side of inequality (25), rewritten for the function f_N, is equal to

$$\begin{aligned} \mathscr{L} &= \int_R \mathbf{x}^{-\mathbf{k}} \left[\int_{R_{\mathbf{0},\mathbf{x}}} f_N(\mathbf{t})\,d\mathbf{t} \right]^p d\mathbf{x} = \prod_{i=1}^n \int_0^\infty x_i^{-k_i} \left[\int_0^{x_i} f_{i,N}(t_i)\,dt_i \right]^p dx_i \\ &\geqslant \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \prod_{i=1}^n \left[\sum_{l=1}^N \left(\frac{l}{l+1} \right)^{k_i} \left(1 - l^{\frac{1-k_i}{p}} \right)^p \frac{1}{l} \right]. \end{aligned} \tag{28}$$

Obviously, the sequence $\left(b_l^{(i)}\right)_{l\in\mathbf{N}}$, where

$$b_l^{(i)} = \left(\frac{l}{l+1} \right)^{k_i} \left(1 - l^{\frac{1-k_i}{p}} \right)^p,$$

fulfills the conditions of Lemma 1. Therefore, for any $i = 1,\dots n$ there exists $N_i = N_i(\varepsilon) \in \mathbf{N}$ such that

$$\sum_{l=1}^{N}\left(\frac{l}{l+1}\right)^{k_i}\left(1-l^{\frac{1-k_i}{p}}\right)^p\frac{1}{l} > (1-\varepsilon)\sum_{l=1}^{N}\frac{1}{l},\ \ N \geqslant N_i.$$

Finally, if $N = N(\varepsilon) \geqslant \max\{N_1,\dots,N_n\}$, then using Lemma 2, (i), and Fubini's theorem, the second row in (28) is greater than

$$\begin{aligned}&(1-\varepsilon)^n\left[p^n\cdot(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p\left(\sum_{l=1}^{N}\frac{1}{l}\right)^n\\ &\quad > (1-\varepsilon)^n\left[p^n\cdot(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p\prod_{i=1}^{n}\left[\int_0^\infty x_i^{p-k_i}f_{i,N}^p(x_i)\,dx_i\right]\\ &\quad = (1-\varepsilon)^n\left[p^n\cdot(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p\int_R \mathbf{x}^{p\mathbf{1}-\mathbf{k}}f_N^p(\mathbf{x})\,d\mathbf{x}.\end{aligned}\tag{29}$$

By $\mathscr{R}$ denote the right-hand side of (25), that is, let

$$\mathscr{R} = \left[p^n\cdot(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p\int_R \mathbf{x}^{p\mathbf{1}-\mathbf{k}}f_N^p(\mathbf{x})\,d\mathbf{x}\,.$$

Combining (25), (28) and (29), we obtain

$$(1-\varepsilon)^n\mathscr{R} < \mathscr{L} \leqslant \mathscr{R},$$

that proves the desired result.

For the case $\mathbf{k} \ll \mathbf{1}$, inequality (25) is derived in the similar way. This time, by starting from (16), with $r = 1$, $s = p > 1$, $\boldsymbol{\alpha} = \mathbf{1}$, $\boldsymbol{\gamma} = (p+1)\mathbf{1}-\mathbf{k}$ and arbitrary $\mathbf{b} \in R$ as parameters, we obtain the relation

$$\int_{R_{\mathbf{b},\infty}}\mathbf{x}^{-\mathbf{k}}F^p(\mathbf{x})\,d\mathbf{x} \leqslant \mathbf{b}^{\mathbf{1}-\mathbf{k}}\left\{\int_{R_{\mathbf{b},\infty}}\mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}}\left[\int_{R_{\mathbf{x},\infty}}\mathbf{t}^{p\mathbf{1}-\mathbf{k}}f^p(\mathbf{t})\,d\mathbf{t}\right]^{\frac{1}{p}}d\mathbf{x}\right\}^p.\tag{30}$$

By making an analogous estimate to the one in the previous case, we proceed to

$$\int_{R_{\mathbf{b},\infty}}\mathbf{x}^{-\mathbf{k}}F^p(\mathbf{x})\,d\mathbf{x} \leqslant \left[p^n\cdot(\mathbf{1}-\mathbf{k})^{-\mathbf{1}}\right]^p\int_{R_{\mathbf{b},\infty}}\mathbf{t}^{p\mathbf{1}-\mathbf{k}}f^p(\mathbf{t})\,d\mathbf{t},\tag{31}$$

so (25) holds by taking $\lim_{\mathbf{b}\to\mathbf{0}}$.

Observe that the pair of inequalities in (25) (for $\mathbf{k} \gg \mathbf{1}$ and $\mathbf{k} \ll \mathbf{1}$) are mutually equivalent, since writing one of them for $2\cdot\mathbf{1}-\mathbf{k}$ and $\mathbf{x}^{-2\cdot\mathbf{1}}f\left(\frac{\mathbf{1}}{\mathbf{x}}\right)$, instead of $\mathbf{k}$ and f, and then substituting $\mathbf{z} = \frac{\mathbf{1}}{\mathbf{t}}$ and $\mathbf{y} = \frac{\mathbf{1}}{\mathbf{x}}$ on its left-hand side, and $\mathbf{y} = \frac{\mathbf{1}}{\mathbf{x}}$ on its right-hand side, we obtain the other inequality of the pair. Therefore, $\left[p^n\cdot(\mathbf{1}-\mathbf{k})^{-\mathbf{1}}\right]^p$ is the best possible constant for the case $\mathbf{k} \ll \mathbf{1}$. $\square$

It is easy to see that generality of (25) will not be increased if the parameter $\boldsymbol{\alpha}$ is left to be arbitrary, since the function f can always be replaced with $\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}f$.

Now, let us take a closer look to inequalities (27) and (31), obtained in the proof of Theorem 7. Both of them are of the same type as Hardy's inequality (25), but with one difference: the outer integrals on their both sides, instead of being over R, are taken over its subsets $R_{\mathbf{0},\mathbf{b}}$ or $R_{\mathbf{b},\infty}$. By a careful analysis of the proof of Theorem 7, it is not hard to improve these results by providing smaller upper bounds for their left-hand sides. Moreover, it will be shown that obtained new inequalities are sharp, that is, the constant factor $\left[p^n \cdot \left|(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right|\right]^p$ is still the best possible.
These improvements are given in the following theorem.

THEOREM 8. *Let $p \in \mathbf{R}$, $\mathbf{b} \in R$, and $\mathbf{k} \in \mathbf{R}^n$ be such that $p > 1$, and $\mathbf{k} \gg \mathbf{1}$ or $\mathbf{k} \ll \mathbf{1}$. Suppose f is a non-negative measurable function, the function F is defined by* (24)*, and the function w is defined on R by*

$$w(\mathbf{x}; p, \mathbf{k}, \mathbf{b}) = \prod_{i=1}^{n} \left[1 - \left(\frac{x_i}{b_i} \right)^{\frac{k_i - 1}{p}} \right].$$

(i) *If $\mathbf{k} \gg \mathbf{1}$ and $0 < \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\, d\mathbf{x} < \infty$, then*

$$\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\, d\mathbf{x} < \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \int_{R_{\mathbf{0},\mathbf{b}}} w(\mathbf{x}, p, \mathbf{k}, \mathbf{b})\, \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\, d\mathbf{x}. \tag{32}$$

(ii) *If $\mathbf{k} \ll \mathbf{1}$ and $0 < \int_{R_{\mathbf{b},\infty}} \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\, d\mathbf{x} < \infty$, then*

$$\int_{R_{\mathbf{b},\infty}} \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\, d\mathbf{x} < \left[p^n \cdot (\mathbf{1}-\mathbf{k})^{-\mathbf{1}}\right]^p \int_{R_{\mathbf{b},\infty}} w(\mathbf{x}, p, \mathbf{k}, \mathbf{b})\, \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\, d\mathbf{x}. \tag{33}$$

The constant $\left[p^n \cdot \left|(\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right|\right]^p$ is the best possible for both inequalities.

Proof. First, we prove (32). As in the proof of Theorem 7, we start from the relation (26). Denote

$$A_{\mathbf{b}} = \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}}\, d\mathbf{x} = p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}} \cdot \mathbf{b}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})}.$$

The right-hand side of (26) is then equal to

$$\begin{aligned}
&\left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \left\{ \frac{1}{A_{\mathbf{b}}} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}} \left[\int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\, d\mathbf{t} \right]^{\frac{1}{p}} d\mathbf{x} \right\}^p \\
&\quad < \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \cdot \frac{1}{A_{\mathbf{b}}} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\, d\mathbf{t}\, d\mathbf{x} \\
&\quad = \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^{p-1} \mathbf{b}^{\frac{1}{p}(\mathbf{1}-\mathbf{k})} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\, d\mathbf{t}\, d\mathbf{x} \\
&\quad = \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^{p-1} \mathbf{b}^{\frac{1}{p}(\mathbf{1}-\mathbf{k})} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t}) \int_{R_{\mathbf{t},\mathbf{b}}} \mathbf{x}^{\frac{1}{p}(\mathbf{k}-\mathbf{1})-\mathbf{1}}\, d\mathbf{x}\, d\mathbf{t} \\
&\quad = \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p \int_{R_{\mathbf{0},\mathbf{b}}} w(\mathbf{t}; p, \mathbf{k}, \mathbf{b})\, \mathbf{t}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{t})\, d\mathbf{t},
\end{aligned} \tag{34}$$

so (32) is proved. Note that $0 < w(\mathbf{t}; p, \mathbf{k}, \mathbf{b}) < 1$, $\mathbf{t} \in R_{\mathbf{0},\mathbf{b}}$. The inequality in the second row of (34) is obtained by using Jensen's inequality, while the equality in the fourth row of (34) is a consequence of Fubini's theorem. Owing to the conditions on f from the statement of the theorem, the inequality sign in (34) is strict.

Now we prove that the constant $C = \left[p^n \cdot (\mathbf{k}-\mathbf{1})^{-\mathbf{1}}\right]^p$ is the best possible for inequality (32). If it is not true, then there exists a smaller constant D, $0 < D < C$, such that

$$\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{-\mathbf{k}} F^p(\mathbf{x})\, d\mathbf{x} < D \int_{R_{\mathbf{0},\mathbf{b}}} w(\mathbf{x}, p, \mathbf{k}, \mathbf{b})\, \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f^p(\mathbf{x})\, d\mathbf{x}. \tag{35}$$

Since $\lim_{\varepsilon\to 0} \left\{p^n \cdot [\mathbf{k} + (\varepsilon-1)\mathbf{1}]^{-\mathbf{1}}\right\}^p = C$, there exists a small number $\varepsilon > 0$ such that $0 < D < \left\{p^n \cdot [\mathbf{k} + (\varepsilon-1)\mathbf{1}]^{-\mathbf{1}}\right\}^p < C$. Define the function f_ε by $f_\varepsilon(\mathbf{x}) = \mathbf{x}^{\frac{1}{p}[\mathbf{k}+(\varepsilon-1)\mathbf{1}]-\mathbf{1}}$, $\mathbf{x} \in R_{\mathbf{0},\mathbf{b}}$. Then we have

$$\begin{aligned}\int_{R_{\mathbf{0},\mathbf{b}}} w(\mathbf{x}, p, \mathbf{k}, \mathbf{b})\, \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f_\varepsilon^p(\mathbf{x})\, d\mathbf{x} &\leqslant \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f_\varepsilon^p(\mathbf{x})\, d\mathbf{x} \\ &= \prod_{i=1}^n \int_0^{b_i} x_i^{\varepsilon-1} dx_i = \frac{\mathbf{b}^{\varepsilon\mathbf{1}}}{\varepsilon^n},\end{aligned}$$

and therefore

$$\begin{aligned}\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{-\mathbf{k}} \left(\int_{R_{\mathbf{0},\mathbf{x}}} f_\varepsilon(\mathbf{t})\, d\mathbf{t}\right)^p d\mathbf{x} &= \left\{p^n \cdot [\mathbf{k} + (\varepsilon-1)\mathbf{1}]^{-\mathbf{1}}\right\}^p \prod_{i=1}^n \int_0^{b_i} x_i^{\varepsilon-1} dx_i \\ &= \left\{p^n \cdot [\mathbf{k} + (\varepsilon-1)\mathbf{1}]^{-\mathbf{1}}\right\}^p \frac{\mathbf{b}^{\varepsilon\mathbf{1}}}{\varepsilon^n} > D \cdot \frac{\mathbf{b}^{\varepsilon\mathbf{1}}}{\varepsilon^n} \\ &\geqslant D \int_{R_{\mathbf{0},\mathbf{b}}} w(\mathbf{x}, p, \mathbf{k}, \mathbf{b})\, \mathbf{x}^{p\mathbf{1}-\mathbf{k}} f_\varepsilon^p(\mathbf{x})\, d\mathbf{x}.\end{aligned}$$

This contradicts (35), so C is the best possible constant for (32).

The proof of (33) is similar, but this time we start from (30), and then use Jensen's inequality and Fubini's theorem. To prove that $\left[p^n \cdot (\mathbf{1}-\mathbf{k})^{-\mathbf{1}}\right]^p$ is the best possible constant factor for (33), it suffices to consider the function g_ε, defined by $g_\varepsilon(\mathbf{x}) = \mathbf{x}^{\frac{1}{p}[\mathbf{k}-(\varepsilon+1)\mathbf{1}]-\mathbf{1}}$, $\mathbf{x} \in R_{\mathbf{b},\infty}$. □

4. Levin-Cochran-Lee inequalities for multivariable functions

Theorem 3 and Theorem 4 can be used to derive n-variable Levin-Cochran-Lee type inequalities, a natural multivariable analogue of Theorem B. In this section we also establish some new generalizations of these inequalities, of the same type as the improvements (32) and (33) of (25), derived in the previous section. Moreover, we discuss the best possible constants for obtained inequalities.
First, we need to prove the next lemma.

LEMMA 3. *Let* $\alpha, \gamma, \varepsilon \in \mathbf{R}$, $\alpha, \varepsilon > 0$. *If the function* f_ε *is defined on* $\langle 0, \infty \rangle$ *by*

$$f_\varepsilon(x) = \begin{cases} \alpha e^{-\frac{\gamma}{\alpha}} x^{\alpha\varepsilon - \gamma}, & 0 < x < 1, \\ \alpha e^{-\frac{\gamma}{\alpha}} x^{-\alpha\varepsilon - \gamma}, & x \geqslant 1, \end{cases}$$

then:

(i) $\displaystyle\int_0^\infty x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1} \log f_\varepsilon(t)\, dt\right] dx \geqslant \frac{2}{\varepsilon} \cdot e^{-\varepsilon}$;

(ii) $\displaystyle\int_0^\infty t^{\gamma-1} f_\varepsilon(t)\, dt = \frac{2}{\varepsilon} \cdot e^{-\frac{\gamma}{\alpha}}$.

Proof. Elementary calculus and the estimate $e^{-\frac{2\varepsilon}{x^\alpha}} > e^{-2\varepsilon}$, $x > 1$, give the following:

$$\begin{aligned} &\int_0^\infty x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1} \log f_\varepsilon(t)\, dt\right] dx \\ &= \int_0^1 x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1} \log\left(\alpha e^{-\frac{\gamma}{\alpha}} x^{\alpha\varepsilon-\gamma}\right) dt\right] dx \\ &\quad + \int_1^\infty x^{\gamma-1} \exp\left\{\frac{\alpha}{x^\alpha}\left[\int_0^1 t^{\alpha-1} \log\left(\alpha e^{-\frac{\gamma}{\alpha}} x^{\alpha\varepsilon-\gamma}\right) dt \right.\right. \\ &\quad \left.\left. + \int_1^x t^{\alpha-1} \log\left(\alpha e^{-\frac{\gamma}{\alpha}} x^{-\alpha\varepsilon-\gamma}\right) dt\right]\right\} dx \\ &= \alpha\left(e^{-\varepsilon}\int_0^1 x^{\alpha\varepsilon-1} dx + e^{\varepsilon}\int_1^\infty x^{-\alpha\varepsilon-1} e^{-\frac{2\varepsilon}{x^\alpha}} dx\right) \\ &\geqslant \alpha e^{-\varepsilon}\left(\int_0^1 x^{\alpha\varepsilon-1} dx + \int_1^\infty x^{-\alpha\varepsilon-1} dx\right) = \frac{2}{\varepsilon} \cdot e^{-\varepsilon}, \end{aligned}$$

that is, (i), and

$$\int_0^\infty t^{\gamma-1} f_\varepsilon(t)\, dt = \alpha e^{-\frac{\gamma}{\alpha}}\left(\int_0^1 x^{\alpha\varepsilon-1} dx + \int_1^\infty x^{-\alpha\varepsilon-1} dx\right) = \frac{2}{\varepsilon} \cdot e^{-\frac{\gamma}{\alpha}},$$

that is, (ii). □

Now we prove the multivariable Levin-Cochran-Lee type inequalities.

THEOREM 9. *Let* $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbf{R}^n$ *and* f *be a positive measurable function on* R *such that* $\int_R \mathbf{t}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{t})\, d\mathbf{t} < \infty$. *Then the inequalities*

$$\int_R \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} \exp\left[\boldsymbol{\alpha}^{\mathbf{1}} \mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\, d\mathbf{t}\right] d\mathbf{x} \leqslant \exp\left(\sum_{i=1}^n \frac{\gamma_i}{\alpha_i}\right) \int_R \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{x})\, d\mathbf{x}, \quad (36)$$

for $\boldsymbol{\alpha} \gg \mathbf{0}$, *and*

$$\int_R \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} \exp\left[(-\boldsymbol{\alpha})^{\mathbf{1}} \mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{x},\infty}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\, d\mathbf{t}\right] d\mathbf{x} \leqslant \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{\alpha_i}\right) \int_R \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} f(\mathbf{x})\, d\mathbf{x}, \tag{37}$$

for $\boldsymbol{\alpha} \ll \mathbf{0}$, *hold. The constant* $\exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{\alpha_i}\right)$ *is the best possible for both inequalities.*

Proof. First, we prove (36). By putting $s = 1$, $\mathbf{a} = \mathbf{0}$ and arbitrary $\mathbf{b} \in R$ in Theorem 3 as parameters, inequality (18) becomes

$$\begin{aligned} &\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} \exp\left[\boldsymbol{\alpha}^{\mathbf{1}} \mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\, d\mathbf{t}\right] d\mathbf{x} \\ &\leqslant \exp\left\{\log \mathbf{b}^{\boldsymbol{\Upsilon}} + \boldsymbol{\alpha}^{\mathbf{1}} \mathbf{b}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} \log\left[\mathbf{x}^{-\boldsymbol{\Upsilon}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}} f(\mathbf{t})\, d\mathbf{t}\right] d\mathbf{x}\right\}. \end{aligned} \tag{38}$$

Considering relations

$$J_{\mathbf{b}} = \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}} f(\mathbf{t})\, d\mathbf{t} \geqslant \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}} f(\mathbf{t})\, d\mathbf{t}, \quad \mathbf{x} \in R_{\mathbf{0},\mathbf{b}},$$

and

$$\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} \log x_j\, d\mathbf{x} = \left(\log b_j - \frac{1}{\alpha_j}\right) \boldsymbol{\alpha}^{-\mathbf{1}} \mathbf{b}^{\boldsymbol{\alpha}}, \quad j = 1, \ldots, n, \tag{39}$$

the second row of (38) is not greater than

$$\begin{aligned} &\exp\left\{\log \mathbf{b}^{\boldsymbol{\Upsilon}} + \boldsymbol{\alpha}^{\mathbf{1}} \mathbf{b}^{-\boldsymbol{\alpha}} \left[\log J_{\mathbf{b}} \cdot \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x} - \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} \log \mathbf{x}^{\boldsymbol{\Upsilon}}\, d\mathbf{x}\right]\right\} \\ &= \exp\left\{\sum_{i=1}^{n} \gamma_i \log b_i + \log J_{\mathbf{b}} - \boldsymbol{\alpha}^{\mathbf{1}} \mathbf{b}^{-\boldsymbol{\alpha}} \sum_{j=1}^{n} \gamma_j \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}} \log x_j\, d\mathbf{x}\right\} \\ &= J_{\mathbf{b}} \cdot \exp\left(\sum_{j=1}^{n} \frac{\gamma_j}{\alpha_j}\right). \end{aligned}$$

Inequality (36) now follows by taking $\lim_{\mathbf{b}\to\boldsymbol{\infty}}$. The other inequality, (37), is derived from Theorem 4 by the same technique, choosing $s = 1$ in (20) as parameter and taking $\lim_{\mathbf{b}\to\mathbf{0}}$.

To find out that $\exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{\alpha_i}\right)$ is the best possible constant for both inequalities, for any $\varepsilon > 0$ we define the function f_ε on R by $f_\varepsilon(\mathbf{x}) = \prod_{i=1}^{n} f_{i,\varepsilon}(x_i)$, where

$$f_{i,\varepsilon}(x) = \begin{cases} \alpha_i e^{-\frac{\gamma_i}{\alpha_i}} x^{\alpha_i\varepsilon-\gamma_i}, & 0 < x < 1, \\ \alpha_i e^{-\frac{\gamma_i}{\alpha_i}} x^{-\alpha_i\varepsilon-\gamma_i}, & x \geqslant 1. \end{cases}$$

By Lemma 3 and Fubini's theorem, the left-hand side of inequality (36), rewritten for f_ε, is equal to

$$\begin{aligned}\mathscr{L}_\varepsilon &= \int_R \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} \exp\left[\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f_\varepsilon(\mathbf{t})\,d\mathbf{t}\right] d\mathbf{x} \\ &= \prod_{i=1}^n \int_0^\infty x_i^{\gamma_i-1} \exp\left[\frac{\alpha_i}{x_i^{\alpha_i}} \int_0^{x_i} t_i^{\alpha_i-1} \log f_{i,\varepsilon}(t_i)\,dt_i\right] dx_i \geqslant \left(\frac{2}{\varepsilon}e^{-\varepsilon}\right)^n,\end{aligned}$$

while on the right-hand side we have

$$\mathscr{R}_\varepsilon = \exp\left(\sum_{i=1}^n \frac{\gamma_i}{\alpha_i}\right) \int_R \mathbf{t}^{\boldsymbol{\gamma}-\mathbf{1}} f_\varepsilon(\mathbf{t})\,d\mathbf{t} = \prod_{i=1}^n \left[e^{\frac{\gamma_i}{\alpha_i}} \int_0^\infty x_i^{\gamma_i-1} f_{i,\varepsilon}(x_i)\,dx_i\right] = \left(\frac{2}{\varepsilon}\right)^n.$$

Thus,

$$1 \leqslant \frac{\mathscr{R}_\varepsilon}{\mathscr{L}_\varepsilon} \leqslant e^{n\varepsilon} \to 1, \quad \text{as } \varepsilon \to 0,$$

so the constant $\exp\left(\sum_{j=1}^n \frac{\gamma_j}{\alpha_j}\right)$ is the best possible for (36). On the other hand, the proof that the same constant is the best possible for (37) is similar, if the functions

$$f_{i,\varepsilon}(x) = \begin{cases} -\alpha_i e^{-\frac{\gamma_i}{\alpha_i}} x^{-\alpha_i\varepsilon-\gamma_i}, & 0 < x < 1, \\ -\alpha_i e^{-\frac{\gamma_i}{\alpha_i}} x^{\alpha_i\varepsilon-\gamma_i}, & x \geqslant 1 \end{cases}$$

are considered. □

We conclude this paper with the following result.

THEOREM 10. *Let* $\mathbf{b} \in R$ *and* $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbf{R}^n$ *be such that* $\boldsymbol{\alpha} \gg \mathbf{0}$ *or* $\boldsymbol{\alpha} \ll \mathbf{0}$. *Suppose* f *is a positive measurable function on* R *and the function* h *is defined by*

$$h(\mathbf{x};\mathbf{b},\boldsymbol{\alpha}) = \prod_{i=1}^n \left[1 - \left(\frac{x_i}{b_i}\right)^{\alpha_i}\right], \quad \mathbf{x} \in R.$$

(i) *If* $\boldsymbol{\alpha} \gg \mathbf{0}$ *and* $0 < \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{x})\,d\mathbf{x} < \infty$, *then*

$$\begin{aligned}&\int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} \exp\left[\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\,d\mathbf{t}\right] d\mathbf{x} \\ &\qquad < \exp\left(\sum_{i=1}^n \frac{\gamma_i}{\alpha_i}\right) \int_{R_{\mathbf{0},\mathbf{b}}} h(\mathbf{x};\mathbf{b},\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{40}$$

(ii) *If* $\boldsymbol{\alpha} \ll \mathbf{0}$ *and* $0 < \int_{R_{\mathbf{b},\infty}} \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{x})\,d\mathbf{x} < \infty$, *then*

$$\begin{aligned}&\int_{R_{\mathbf{b},\infty}} \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} \exp\left[(-\boldsymbol{\alpha})^{\mathbf{1}}\mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{x},\infty}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log f(\mathbf{t})\,d\mathbf{t}\right] d\mathbf{x} \\ &\qquad < \exp\left(\sum_{i=1}^n \frac{\gamma_i}{\alpha_i}\right) \int_{R_{\mathbf{b},\infty}} h(\mathbf{x};\mathbf{b},\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\gamma}-\mathbf{1}} f(\mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{41}$$

The constant $\exp\left(\sum_{i=1}^{n}\frac{\gamma_i}{\alpha_i}\right)$ *is the best possible for both inequalities.*

Proof. First, let $\boldsymbol{\alpha} \gg \mathbf{0}$. As in the proof of Theorem 9, we start from (38). Considering (39), the right-hand side of (38) is equal to

$$\begin{aligned}
&\exp\left\{\log \mathbf{b}^{\boldsymbol{\Upsilon}} + \boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log\left(\mathbf{x}^{-\boldsymbol{\Upsilon}}\right)d\mathbf{x}\right.\\
&\qquad\left.+\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log\left[\int_{R_{\mathbf{0},\mathbf{x}}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\,d\mathbf{t}\right]d\mathbf{x}\right\}\\
&=\exp\left\{\log \mathbf{b}^{\boldsymbol{\Upsilon}} - \boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\sum_{j=1}^{n}\gamma_j\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log x_j\,d\mathbf{x}\right.\\
&\qquad\left.+\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log\left[\int_{R_{\mathbf{0},\mathbf{x}}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\,d\mathbf{t}\right]d\mathbf{x}\right\}\\
&=\exp\left(\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)\exp\left\{\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\log\left[\int_{R_{\mathbf{0},\mathbf{x}}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\,d\mathbf{t}\right]d\mathbf{x}\right\}\\
&<\exp\left(\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\int_{R_{\mathbf{0},\mathbf{x}}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\,d\mathbf{t}\,d\mathbf{x}\\
&=\exp\left(\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{b}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\int_{R_{\mathbf{t},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}}\,d\mathbf{x}\,d\mathbf{t}\\
&=\exp\left(\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)\int_{R_{\mathbf{0},\mathbf{b}}}h(\mathbf{t};\mathbf{b},\boldsymbol{\alpha})\,\mathbf{t}^{\boldsymbol{\Upsilon}-\mathbf{1}}f(\mathbf{t})\,d\mathbf{t},
\end{aligned}\tag{42}$$

so (40) is proved. Inequality (42) was obtained by Jensen's inequality and Fubini's theorem. Note that under the conditions of the theorem the inequality sign in (42) is strict.

The dual inequality (41) can be derived from the proof of Theorem 9 by the same technique.

It is only left to prove that the constant factor $\exp\left(\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)$ is the best possible for inequalities (40) and (41). Consider $\boldsymbol{\alpha} \gg \mathbf{0}$ first. For any $\varepsilon > 0$ and the function f_ε defined on $R_{\mathbf{0},\mathbf{b}}$ by $f_\varepsilon(\mathbf{x}) = \boldsymbol{\alpha}^{\mathbf{1}}\exp\left(-\sum_{j=1}^{n}\frac{\gamma_j}{\alpha_j}\right)\mathbf{x}^{\varepsilon\boldsymbol{\alpha}-\boldsymbol{\Upsilon}}$, the left-hand side of (40) is equal to

$$\mathscr{L}_\varepsilon = \int_{R_{\mathbf{0},\mathbf{b}}}\mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}}\exp\left[\boldsymbol{\alpha}^{\mathbf{1}}\mathbf{x}^{-\boldsymbol{\alpha}}\int_{R_{\mathbf{0},\mathbf{x}}}\mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}}\log f_\varepsilon(\mathbf{t})\,d\mathbf{t}\right]d\mathbf{x}$$

$$
\begin{aligned}
&= \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} \exp\Bigg\{ \boldsymbol{\alpha}^{\mathbf{1}} \mathbf{x}^{-\boldsymbol{\alpha}} \log\left[\boldsymbol{\alpha}^{\mathbf{1}} \exp\left(-\sum_{j=1}^{n} \frac{\gamma_j}{\alpha_j}\right)\right] \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{t} \\
&\quad + \boldsymbol{\alpha}^{\mathbf{1}} \mathbf{x}^{-\boldsymbol{\alpha}} \int_{R_{\mathbf{0},\mathbf{x}}} \mathbf{t}^{\boldsymbol{\alpha}-\mathbf{1}} \log\left(\mathbf{t}^{\varepsilon\boldsymbol{\alpha}-\boldsymbol{\Upsilon}}\right) d\mathbf{t} \Bigg\} d\mathbf{x} \\
&= \boldsymbol{\alpha}^{\mathbf{1}} e^{-n\varepsilon} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\varepsilon\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x} = \frac{\mathbf{b}^{\varepsilon\boldsymbol{\alpha}}}{\varepsilon^n} \cdot e^{-n\varepsilon},
\end{aligned}
$$

while on the right-hand side of that relation we have

$$
\begin{aligned}
\mathscr{R}_\varepsilon &= \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{\alpha_i}\right) \int_{R_{\mathbf{0},\mathbf{b}}} h(\mathbf{x};\mathbf{b},\boldsymbol{\alpha})\, \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} f_\varepsilon(\mathbf{x})\, d\mathbf{x} \\
&\leqslant \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{\alpha_i}\right) \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\boldsymbol{\Upsilon}-\mathbf{1}} f_\varepsilon(\mathbf{x})\, d\mathbf{x} = \boldsymbol{\alpha}^{\mathbf{1}} \int_{R_{\mathbf{0},\mathbf{b}}} \mathbf{x}^{\varepsilon\boldsymbol{\alpha}-\mathbf{1}} d\mathbf{x} = \frac{\mathbf{b}^{\varepsilon\boldsymbol{\alpha}}}{\varepsilon^n}.
\end{aligned}
$$

Hence,

$$
1 \leqslant \frac{\mathscr{R}_\varepsilon}{\mathscr{L}_\varepsilon} \leqslant e^{n\varepsilon} \to 1, \quad [\,\text{as}\,]\;\; \varepsilon \to 0,
$$

so the constant from the statement of the theorem is the best possible.

To prove that the same constant is the best possible for (41), consider the function f_ε defined on $R_{\mathbf{b},\infty}$ by $f_\varepsilon(\mathbf{x}) = (-\boldsymbol{\alpha})^{\mathbf{1}} \exp\left(-\sum_{j=1}^{n} \frac{\gamma_j}{\alpha_j}\right) \mathbf{x}^{\varepsilon\boldsymbol{\alpha}-\boldsymbol{\Upsilon}}$. □

REFERENCES

[1] P. S. BULLEN, *Inequalities due to T. S. Nanjundiah*, from Milovanović, G. V. (ed.) *Recent progress in inequalities. Dedicated to Prof. Dragoslav S. Mitrinović*, Kluwer Academic Publishers, 1998, 203–211.

[2] J. A. COCHRAN AND C.-S. LEE, *Inequalities related to Hardy's and Heinig's*, Math. Proc. Cambridge Phil. Soc. **96** (1984), 1–7.

[3] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Mixed means and Hardy's inequality*, Math. Inequal. Appl. **1**, No. 4 (1998), 491–506.

[4] A. ČIŽMEŠIJA, J. PEČARIĆ AND I. PERIĆ, *Mixed means and inequalities of Hardy and Levin-Cochran-Lee type for multidimensional balls*, Proc. Amer. Math. Soc. **128**, No. 9 (2000), 2543–2552.

[5] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Some new generalisations of inequalities of Hardy and Levin-Cochran-Lee*, Bull. Austral. Math. Soc. **63** (2001), 105–113.

[6] A. ČIŽMEŠIJA AND J. PEČARIĆ, *New generalizations of inequalities of Hardy and Levin-Cochran-Lee type for multidimensional balls*, to appear in Math. Inequal. Appl.

[7] G. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, second edition, Cambridge University Press, Cambridge, 1967.

[8] F. HOLLAND, *On a mixed arithmetic-mean, geometric-mean inequality*, Mathematics Competitions **5** (1992), 60–64.

[9] K. KEDLAYA, *Proof of a Mixed Arithmetic-Mean, Geometric-Mean Inequality*, Amer. Math. Monthly, **101** (1994), 355–357.

[10] K. KEDLAYA, *A Weighted Mixed-Mean Inequality*, Amer. Math. Monthly, **106** (1999), 355–358.

[11] V. LEVIN, *O neravenstvah III: Neravenstva, vypolnjaemie geometričeskim srednim neotricatel'noi funkcii*, Math. Sbornik 4 **46** (1938), 325–331.

[12] E. R. LOVE, *Inequalities related to those of Hardy and of Cochran and Lee*, Math. Proc. Cambridge Phil. Soc. **99** (1986), 395–408.

[13] T. MATSUDA, *An Inductive Proof of a Mixed Arithmetic-Geometric Mean Inequality*, Amer. Math. Monthly **102** (1995), 634–637.

[14] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991.

[15] B. MOND AND J. PEČARIĆ, *A Mixed means Inequality*, Austral. Math. Soc. Gazette, **23** (1996), No. 2, 67–70.

[16] B. MOND AND J. PEČARIĆ, *A Mixed Arithmetic-Mean-Harmonic-Mean Matrix Inequality*, Linear Algebra Appl. **237/238** (1996), 449–454.

[17] B. MOND AND J. PEČARIĆ, *Mixed means inequalities for positive linear operators*, Austral. Math. Soc. Gazette, **23** (1996), No. 5, 198–200.

[18] B. G. PACHPATTE, *On Hardy Type Integral Inequalities*, Tamkang Journal of Math. **18**, No. 2 (1987), 27–41.

[19] B. G. PACHPATTE, *On Multivariate Hardy Type Inequalities*, Analele Stiintifice ale Univ. "AL. I. CUZA" Iasi, Tom. XXXVIII, s.I.a, Matem., 1992, f. 3., 355–361.

[20] C. D. TARNAVAS AND D. D. TARNAVAS, *An inequality for mixed power means*, Math. Inequal. Appl. **2**, No. 2 (1999), 175–181.

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