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GENERALISED TRAPEZOIDAL RULES WITH ERROR INVOLVING BOUNDS OF THE nTH DERIVATIVE

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Abstract. Inequalities obtained for the generalised trapezoidal rules in terms of the upper and lower bounds of the nth derivative of the integrand. The Hayashi inequality, or more appropriately an inequality due to Steffensen, is utilised to procure the results which contain earlier expressions as particular cases.

1. Introduction

Using Hayashi's inequality (see [6, pp. 311-312]) Cerone and Dragomir [2] obtained the following result for the trapezoidal rule where the bound is in terms of the upper and lower bound of the first derivative.

THEOREM 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and $[a,b] \subset \mathring{I}$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ and $M > m$. If f' is integrable on $[a,b]$, then the following inequalities hold:*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)]\right| \leqslant \frac{(b-a)^2}{2(M-m)}(S-m)(M-S) \tag{1.1}$$

$$\leqslant \frac{M-m}{2}\left(\frac{b-a}{2}\right)^2, \tag{1.2}$$

where $S = \frac{f(b)-f(a)}{b-a}$.

The result (1.1) was also obtained previously in a similar fashion by Agarwal and Dragomir [2] however, their formulation did not reveal (1.2). A prior result obtained by Iyengar [4] (see also [7, p. 471]) is recovered if we take in (1.1): $m = -M$.

Cerone and Dragomir [2] also obtained non-symmetric bounds for a generalised trapezoidal rule.

THEOREM 2. *Let f satisfy the conditions of Theorem 1, then the following result holds*

$$\beta_L \leqslant \int_a^b f(x)\,dx - (b-a)\left[\left(\frac{\theta-a}{b-a}\right)f(a) + \left(\frac{b-\theta}{b-a}\right)f(b)\right] \leqslant \beta_U, \tag{1.3}$$

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where

$$\beta_U = \frac{(b-a)^2}{2(M-m)}\left[S(2\gamma_U - S) - mM\right],$$

$$\beta_L = \frac{(b-a)^2}{2(M-m)}\left[S(S - 2\gamma_L) + mM\right],$$

$$\gamma_U = \left(\frac{\theta - a}{b-a}\right)M + \left(\frac{b-\theta}{b-a}\right)m,\ \gamma_L = M + m - \gamma_U,$$

where $S = \frac{f(b)-f(a)}{b-a}$.

Milovanović and Pečarić [5] proved the following specialization of a more general result in which $f^{(n-1)}$ satisfy the Lipschitz condition.

THEOREM 3. *Let the function* $f : [a,b] \to \mathbf{R}$ *have a continuous derivative of order* $n-1$ *and* $\left|f^{(n)}(x)\right| \leqslant M$ *for* $x \in (a,b)$.
If $f^{(k)}(a) = f^{(k)}(b) = 0 \ (k = 1, 2, \ldots, n-1)$, *then the inequality*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}(f(a) + f(b))\right| \leqslant \frac{M(b-a)^{n+1}}{(n+1)!}\left[\zeta^{n+1} - \frac{q}{2}\left(1 + \frac{n}{2\zeta} - 1\right)\right]$$

holds, where

$$\zeta \ \textit{satisfies}\ \zeta^n - (1-\zeta)^n = q := \frac{n!}{M(b-a)^n}(f(b) - f(a)).$$

In this paper generalised trapezoidal type rules involving a parameter θ are obtained in assuming that the nth derivative is bounded both above and below. Further, the restrictive assumption of vanishing lower order derivatives at the end points is not made in the current work. Some of the results are compared with those obtained in Qi [8] where a Taylor approach is utilised. An expression involving function and derivative evaluation at three points is also given.

2. Integral inequalities

The following theorem due to Hayashi [6, pp. 311-312] will be required and thus it is stated for convenience.

THEOREM 4. *Let* $h : [a,b] \to \mathbf{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $g : [a,b] \to \mathbf{R}$ *an integrable mapping on* $[a,b]$ *with*

$$0 \leqslant g(x) \leqslant A,\ \textit{for all}\ x \in [a,b],$$

then

$$A\int_{b-\lambda}^b h(x)\,dx \leqslant \int_a^b h(x)g(x)\,dx \leqslant A\int_a^{a+\lambda} h(x)\,dx, \tag{2.1}$$

where

$$\lambda = \frac{1}{A}\int_a^b g(x)\,dx.$$

Theorem 4 is attributed to be a generalisation of Steffensen's inequality [6, p. 311-312] obtained by taking $A = 1$ in the above theorem. Inspection of the original paper of Steffensen [9] reveals that the more general situation depicted by the following theorem was also treated.

THEOREM 5. *Let* $h : [a,b] \to \mathbf{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $g : [a,b] \to \mathbf{R}$ *be an integrable mapping on* $[a,b]$ *with*

$$\phi \leqslant g(x) \leqslant \Phi, \text{ for all } x \in [a,b],$$

then

$$\phi \cdot \int_a^{b-\lambda} h(x)\,dx + \Phi \int_{b-\lambda}^b h(x)\,dx \tag{2.2}$$
$$\leqslant \int_a^b h(x)\,g(x)\,dx \leqslant \Phi \cdot \int_a^{a+\lambda} h(x)\,dx + \phi \int_{a+\lambda}^b h(x)\,dx,$$

where

$$\lambda = \int_a^b G(x)\,dx, \quad G(x) = \frac{g(x)-\phi}{\Phi-\phi}, \quad \Phi \neq \phi. \tag{2.3}$$

REMARK 1. We note that result (2.2) may be obtained upon simplification and using Steffensen's more well known result that

$$\int_{b-\lambda}^b h(x)\,dx \leqslant \int_a^b h(x)\,G(x)\,dx \leqslant \int_a^{a+\lambda} h(x)\,dx, \tag{2.4}$$

where λ is as given by (2.3) and $0 \leqslant G(x) \leqslant 1$. Contrarily, if we take $\phi = 0$ and $\Phi = 1$ we obtain (2.4) from (2.2). Also, if we take $\phi = 0$ in (2.2) then the Hayashi result (2.1) is seen to be included.

Equation (2.4) has the pleasant interpretation, as noted by Steffensen, that if we divide by λ then

$$\frac{1}{\lambda}\int_{b-\lambda}^b h(x)\,dx \leqslant \frac{\int_a^b G(x)\,h(x)\,dx}{\int_a^b G(x)\,dx} \leqslant \frac{1}{\lambda}\int_a^{a+\lambda} h(x)\,dx.$$

Thus, the weighted integral mean of $h(x)$ is bounded by the integral means over the end intervals of length λ, the total weight.

Further, it should be stated here that discrete versions of (2.2) and (2.4) were also treated in Steffensen [9].

The following theorem gives trapezoid type rules using the above results.

THEOREM 6. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *and* $f^{(n-1)}$ *be absolutely continuous on* $\mathring{I}$ *(*$\mathring{I}$ *is the interior of* I *) and* $[a,b] \subset \mathring{I}$ *with* $m = \inf_{x\in[a,b]} f^{(n)}(x) > -\infty$, $M = \sup_{x\in[a,b]} f^{(n)}(x) < \infty$ *and* $M > m$. *If* $f^{(n)}$ *is integrable on* $[a,b]$, *then the following inequalities hold*

$$\left| \int_a^b f(x)\,dx - T_n(\theta;a,b) - P_n(\theta;a,b) \right| \leqslant Q_n^-(\theta;a,b), \tag{2.5}$$

where

$$T_n(\theta;a,b)=\sum_{k=1}^{n}\frac{(\theta-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-\theta)^k f^{(k-1)}(b)}{k!}, \qquad (2.6)$$

$$P_n(\theta;a,b)=-\frac{m}{(n+1)!}\left[(\theta-a)^{n+1}+(-1)^n(b-\theta)^{n+1}\right]+Q_n^{+}(\theta;a,b), \qquad (2.7)$$

with

$$\frac{2(n+1)!}{M-m}Q_n^{\pm}(\theta;a,b) \qquad (2.8)$$
$$=\begin{cases}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j}\left(\lambda_n^0\right)^j\left[(\theta-a)^{n+1-j}\pm(-1)^n(b-\theta)^{n+1-j}\right], & n \text{ odd},\\ \sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j}\left[(\lambda_n^a)^j(\theta-a)^{n+1-j}+(-1)^n\left(\lambda_n^b\right)^j(b-\theta)^{n+1-j}\right] & \\ \qquad\pm\left[(\lambda_n^a)^{n+1}+(-1)^n\left(\lambda_n^b\right)^{n+1}\right], & n \text{ even}\end{cases}$$

and

$$\lambda_n^0=\lambda_n(a,b),\ \lambda_n^a=\lambda_n(a,\theta),\ \lambda_n^b=\lambda_n(\theta,b), \qquad (2.9)$$

where

$$\lambda_n(a,b)=\frac{b-a}{M-m}\left(S_{n-1}(a,b)-m\right) \qquad (2.10)$$

and

$$S_{n-1}(a,b)=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}. \qquad (2.11)$$

Proof. Let $h(x)=\frac{(\theta-x)^n}{n!}$, $\theta\in[a,b]$ and $g(x)=f^{(n)}(x)-m$. Assume for the time being that n is odd, then $h(x)$ is a nonincreasing function and so from Hayashi's inequality

$$L_o\leqslant I_n\leqslant U_o, \qquad (2.12)$$

where

$$I_n = I_n(\theta;a,b)=\int_a^b\frac{(\theta-x)^n}{n!}\left(f^{(n)}(x)-m\right)dx,$$
$$\lambda_n(a,b) = \frac{1}{M-m}\int_a^b\left(f^{(n)}(x)-m\right)dx=\lambda_n^0 \quad (\text{for } n \text{ odd})$$

and

$$L_o=W\left(b-\lambda_n^0,b\right),\quad U_o=W\left(a,a+\lambda_n^0\right)$$

with

$$W(l,u)=(M-m)\int_l^u h(x)\,dx=(M-m)\int_l^u\frac{(\theta-x)^n}{n!}dx.$$

In a straightforward fashion, the above expressions may be simplified to give

$$\begin{aligned} I_n(\theta;a,b) &= \int_a^b f(x)\,dx - T_n(\theta;a,b) \\ &\quad + \frac{m}{(n+1)!}\left[(\theta-a)^{n+1} + (-1)^n (b-\theta)^{n+1}\right], \end{aligned} \tag{2.13}$$

where λ_n^0 is as given by (2.11) and (2.10), and

$$W(l,u) = \frac{M-m}{(n+1)!}\left[(\theta-l)^{n+1} - (\theta-u)^{n+1}\right] \tag{2.14}$$

with

$$L_o = \frac{M-m}{(n+1)!}\left[\left(\theta-\left(b-\lambda_n^0\right)\right)^{n+1} - (\theta-b)^{n+1}\right] \tag{2.15}$$

and

$$U_o = \frac{M-m}{(n+1)!}\left[(\theta-a)^{n+1} - \left(\theta-\left(a+\lambda_n^0\right)\right)^{n+1}\right]. \tag{2.16}$$

Further, it may be noticed from (2.12) that

$$\left|I_n - \frac{U_o+L_o}{2}\right| \leqslant \frac{U_o-L_o}{2}, \tag{2.17}$$

where

$$\begin{aligned} \frac{U_o \pm L_o}{2} &= \frac{M-m}{2(n+1)!}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j} \\ &\quad \times \left(\lambda_n^0\right)^j\left[(\theta-a)^{n+1-j} \pm (-1)^n (b-\theta)^{n+1-j}\right] \end{aligned} \tag{2.18}$$

Combining (2.13), (2.17) and (2.18) produces the results (2.5) – (2.11) for n, odd.

Now, for the situation in which n is even. It should be noted that the inequality (2.1) is reversed for $h(x)$ nondecreasing and so for n even $\frac{(\theta-x)^n}{n!}$ is nonincreasing for $x \in [a,\theta]$ and nondecreasing for $x \in (\theta,b]$. Let a superscript of a or b represent these intervals.

Then on the interval $[a,\theta]$ we have

$$L^a \leqslant I_n^a \leqslant U^a, \tag{2.19}$$

where

$$\begin{aligned} I_n^a &= I_n(\theta;a,\theta), \\ L^a &= W(\theta-\lambda_n^a,\theta), \quad U^a = W(a,a+\lambda_n^a) \end{aligned}$$

with

$$\lambda_n^a = \lambda_n(a,\theta) = \frac{\theta-a}{M-m}\left(S_{n-1}(a,\theta)-m\right).$$

Similarly, on $(\theta,b]$ we have

$$L^b \leqslant I_n^b \leqslant U^b, \tag{2.20}$$

where

$$\begin{aligned} I_n^b &= I_n(\theta;\theta,b), \\ L^b &= W\left(\theta,\theta+\lambda_n^b\right), \quad U^b = W\left(b-\lambda_n^b, b\right) \end{aligned}$$

with

$$\lambda_n^b = \lambda_n(\theta,b) = \frac{b-\theta}{M-m}\left(S_{n-1}(\theta,b)-m\right).$$

Thus, combining (2.19) and (2.20) gives

$$L_e \leqslant I_n \leqslant U_e, \tag{2.21}$$

where

$$\begin{aligned} I_n &= I_n^a + I_n^b \\ L_e &= L^a + L^b, \\ U_e &= U^a + U^b. \end{aligned} \tag{2.22}$$

That is, I_n is as given by (2.10) and, on using (2.14),

$$\begin{aligned} L_e &= W\left(\theta-\lambda_n^a,\theta\right) + W\left(\theta,\theta+\lambda_n^b\right) \\ &= \frac{M-m}{(n+1)!}\left[\left(\lambda_n^a\right)^{n+1} + (-1)^n\left(\lambda_n^b\right)^{n+1}\right] \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} U_e &= W\left(a,a+\lambda_n^a\right) + W\left(b-\lambda_n^b,b\right) \\ &= \frac{M-m}{(n+1)!}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j} \\ &\quad \times\left[\left(\lambda_n^a\right)^j(\theta-a)^{n+1-j} + (-1)^n\left(\lambda_n^b\right)^j(b-\theta)^{n+1-j}\right]. \end{aligned} \tag{2.24}$$

Further, from (2.17), we have

$$\left|I_n - \frac{U_e+L_e}{2}\right| \leqslant \frac{U_e-L_e}{2}, \tag{2.25}$$

where

$$\begin{aligned} &\frac{U_e \pm L_e}{2} \\ &= \frac{M-m}{2(n+1)!}\left\{\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n+1}{j}\left[\left(\lambda_n^a\right)^j(\theta-a)^{n+1-j}\right.\right. \\ &\quad \left.\left. + (-1)^n\left(\lambda_n^b\right)^j(b-\theta)^{n+1-j}\right] \pm \left[\left(\lambda_n^a\right)^{n+1} + (-1)^n\left(\lambda_n^b\right)^{n+1}\right]\right\}. \end{aligned} \tag{2.26}$$

Combining (2.13), (2.25) and (2.26) produces the results (2.6) – (2.11) for n even and thus the theorem is now completely proved. □

We may use Steffensen's inequality (2.2) with $g(x) = f^{(n)}(x)$ and $h(x) = \frac{(\theta-x)^n}{n1}$ to prove the above theorem. This will not be pursued here though.

COROLLARY 1. *Let the conditions of Theorem 6 be valid. Then*

$$L-R\leqslant\int_a^b f(x)\,dx-T_n(\theta;a,b)\leqslant U-R \tag{2.27}$$

holds, where $T_n(\theta;a,b)$ *is as given by (2.6),*

$$\frac{(n+1)!}{m}R=(\theta-a)^{n+1}+(-1)^n(b-\theta)^{n+1}, \tag{2.28}$$

$$\frac{(n+1)!}{M-m}L=\begin{cases}(-1)^{n+1}\left[\left(b-\theta-\lambda_n^0\right)^{n+1}-(b-\theta)^{n+1}\right], & n \text{ odd,}\\ (\lambda_n^a)^{n+1}+(-1)^n\left(\lambda_n^b\right)^{n+1}, & n \text{ even}\end{cases} \tag{2.29}$$

and

$$\frac{(n+1)!}{M-m}U=\begin{cases}(\theta-a)^{n+1}-\left(\theta-a-\lambda_n^0\right)^{n+1}, & n \text{ odd,}\\ (\theta-a)^{n+1}-(\theta-a-\lambda_n^a)^{n+1} & \\ +(-1)^{n+1}\left[\left(b-\theta-\lambda_n^b\right)^{n+1}-(b-\theta)^{n+1}\right], & n \text{ even}\end{cases} \tag{2.30}$$

with

$$\begin{aligned}\lambda_n^0&=\frac{b-a}{M-m}\left(S_{n-1}(a,b)-m\right),\\ \lambda_n^a&=\frac{\theta-a}{M-m}\left(S_{n-1}(a,\theta)-m\right),\\ \lambda_n^b&=\frac{b-\theta}{M-m}\left(S_{n-1}(\theta,b)-m\right),\end{aligned}$$

and

$$S_{n-1}(a,b)=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}.$$

Proof. The corollary follows readily from the results of Theorem 6. From (2.12), (2.13) and (2.21) we obtain the lower bound as stated on using (2.15) and (2.23) and the stated upper bound results from (2.16) and (2.24) on utilising (2.14). □

REMARK 2. It should be noticed that $U>0$ and $L>0$ since $0<\lambda_n^0<b-a$, $0<\lambda_n^a<\theta-a$ and $0<\lambda_n^b<b-\theta$ as $0<\frac{S_{n-1}(a,b)-m}{M-m}<1$. Further, $R>0$ for n even or for $\theta>\frac{a+b}{2}$ and n odd. Now, $R<0$ for $\theta<\frac{a+b}{2}$ and n odd.

REMARK 3. Corollary 1 gives non symmetric bounds for the generalised trapezoidal rule $T_n(\theta;a,b)$ as defined by (2.6) while Theorem 4 gives symmetric bounds for a perturbed trapezoidal rule. The bounds involve the upper and lower bounds M and m of $f^{(n)}(x)$, $x\in[a,b]$ and some arbitrary point $\theta\in[a,b]$. If θ is taken to be at the midpoint, that is, $\theta=\frac{a+b}{2}$, some simplification occurs. In particular, for n odd, $P_n\left(\frac{a+b}{2};a,b\right)=0$ and so there is no perturbation. For n odd and $\theta=\frac{a+b}{2}$ then $R=0$ in (2.28).

REMARK 4. If $n=1$ in Theorem 6, then we recapture Theorem 1 on taking $\theta=\frac{a+b}{2}$. Further, Theorem 2 is reproduced from Corollary 1 on taking $n=1$. Thus, the results of this section are an extension of the work of Cerone and Dragomir [2] to involve bounds for the generalised trapezoidal rule in terms of bounds on $f^{(n)}$.
If $n=1$ in (2.5) then

$$\begin{aligned}&\left|\int_a^b f(x)\,dx-[(\theta-a)f(a)+(b-\theta)f(b)]\right.\\&\left.-(b-a)\left[S_0(a,b)-m\right]\left(\theta-\frac{a+b}{2}\right)\right|\\ \leqslant\ &\frac{(b-a)^2}{2(M-m)}\left(S_0(a,b)-m\right)\left(M-S_0(a,b)\right).\end{aligned}$$

Now, using the definition of $S_0(a,b)=\frac{f(b)-f(a)}{b-a}$, then the above result may be simplified to

$$\begin{aligned}&\left|\int_a^b f(x)\,dx-\frac{b-a}{2}[f(a)+f(b)]+m(b-a)\left(\theta-\frac{a+b}{2}\right)\right|\\ \leqslant\ &\frac{(b-a)^2}{2(M-m)}\left(S_0(a,b)-m\right)\left(M-S_0(a,b)\right).\end{aligned}$$

It may be noticed that the above result is a perturbed formula which has the same bounds (1.1) independent of θ. Further, the perturbation vanishes if $\theta=\frac{a+b}{2}$.

REMARK 5. In [8], Feng Qi obtains in our notation using a Taylor series approach

$$Q_L\leqslant\int_a^b f(x)\,dx-T_n(\theta;a,b)\leqslant Q_U, \tag{2.31}$$

where, if we define

$$Q(u,v):=u\frac{(\theta-a)^{n+1}}{(n+1)!}+(-1)^n v\frac{(b-\theta)^{n+1}}{(n+1)!}, \tag{2.32}$$

then

$$Q_L=\begin{cases}Q(m,m), & n\ \text{even}\\ Q(m,M), & n\ \text{odd}\end{cases} \tag{2.33}$$

and

$$Q_U=\begin{cases}Q(M,M), & n\ \text{even}\\ Q(M,m), & n\ \text{odd.}\end{cases} \tag{2.34}$$

Here, as in the rest of the paper, $m\leqslant f^{(n)}(x)\leqslant M$, $x\in[a,b]$.

It is interesting, although difficult, to compare (2.31) with (2.27). The difficulty arises from the fact that the λ_n's are not known explicitly, although we have their bounds as discussed in Remark 2. We note that R as defined in (2.28) is equivalent to $Q(m,m)$ which is the lower bound in (2.31) for n even.

LEMMA 1. *Let $C_U = U - R$ and $C_L = L - R$ where R, L and U are as defined in (2.28) – (2.30). Further, let Q_L and Q_U be as defined by (2.33) and (2.34). Consider $D_U := C_U - Q_U$ and $D_L := C_L - Q_L$, then we have*

$$D_U \leqslant 0 \quad \text{for} \begin{cases} n \text{ even and } M \geqslant (1+2\kappa)\, m \\ \qquad\text{and,} \\ n \text{ odd and } \theta \geqslant \frac{a+b}{2} \text{ or } M \geqslant [1+2\psi(\theta-a)]\, m \end{cases} \tag{2.35}$$

and

$$D_L \geqslant 0 \quad \text{for} \begin{cases} n \text{ even and } M \geqslant (1+2\chi)\, m \\ \qquad\text{and,} \\ n \text{ odd and } \theta \leqslant \frac{a+b}{2} \text{ or } M \geqslant [1+2\psi(b-\theta)]\, m, \end{cases} \tag{2.36}$$

where

$$\begin{aligned} \kappa &= \frac{(\theta-a)^{n+1}+(b-\theta)^{n+1}}{(\theta-a-\lambda_n^a)^{n+1}+(b-\theta-\lambda_n^b)^{n+1}}, \qquad (2.37)\\ \chi &= \frac{(\theta-a)^{n+1}+(b-\theta)^{n+1}}{(\lambda_n^a)^{n+1}+(\lambda_n^b)^{n+1}} \quad \text{and} \\ \psi(x) &= \frac{(b-\theta)^{n+1}-(a-\theta)^{n+1}}{(x-\lambda_n^0)^{n+1}}. \end{aligned}$$

Here λ_n's are as given by (2.9) and (2.10).
The inequalities are reversed if the conditions do not hold.

Proof. We shall consider the upper bound first and use a superscript of e and o to denote n to be either even or odd respectively. Then from (2.28), (2.30) and (2.34)

$$\begin{aligned} &(n+1)!D_U^e \\ = \ &(M-m)\left[(\theta-a)^{n+1}-(\theta-a-\lambda_n^a)^{n+1}+(b-\theta)^{n+1}-\left(b-\theta-\lambda_n^b\right)^{n+1}\right] \\ &-m\left[(\theta-a)^{n+1}+(b-\theta)^{n+1}\right]-M(\theta-a)^{n+1}-M(b-\theta)^{n+1} \\ = \ &-(M-m)\left[(\theta-a-\lambda_n^a)^{n+1}+\left(b-\theta-\lambda_n^b\right)^{n+1}\right] \\ &-2m\left[(\theta-a)^{n+1}+(b-\theta)^{n+1}\right] \\ \leqslant \ &0 \qquad \text{for } M \geqslant (1+2\kappa)\, m, \end{aligned}$$

where κ is as defined in (2.37).

Further,

$$\begin{aligned} (n+1)!D_U^o &= (M-m)\left[(\theta-a)^{n+1}-(\theta-a-\lambda_n^a)^{n+1}\right] \\ &\quad -m\left[(\theta-a)^{n+1}-(b-\theta)^{n+1}\right]-M(\theta-a)^{n+1}+m(b-\theta)^{n+1} \\ &= 2m\left[(b-\theta)^{n+1}-(\theta-a)^{n+1}\right]-(M-m)(\theta-a-\lambda_n^o)^{n+1} \\ &\leqslant 0 \ \text{ for } \theta \geqslant \frac{a+b}{2} \text{ or } M \geqslant [1+2\psi(\theta-a)]\, m, \end{aligned}$$

where $\psi(\cdot)$ is defined in (2.37). Thus, the result (2.35) is now completely demonstrated for both even and odd n.

Now for the lower bounds. From (2.28), (2.29) and (2.33)

$$\begin{aligned}(n+1)!D_L^e &= (M-m)\left[(\lambda_n^a)^{n+1}+\left(\lambda_n^b\right)^{n+1}\right]-2m\left[(\theta-a)^{n+1}+(b-\theta)^{n+1}\right]\\ &\geqslant 0 \quad \text{for } M\geqslant(1+2\chi)\,m,\end{aligned}$$

where χ is as defined in (2.37).

Further,

$$\begin{aligned}(n+1)!D_L^o &= (M-m)\left[\left(b-\theta-\lambda_n^0\right)^{n+1}-(b-\theta)^{n+1}\right]\\ &\quad -m\left[(\theta-a)^{n+1}-(b-\theta)^{n+1}\right]-m\,(\theta-a)^{n+1}+M\,(b-\theta)^{n+1}\\ &= (M-m)\,(b-\theta-\lambda_n^o)^{n+1}+2m\left[(b-\theta)^{n+1}-(a-\theta)^{n+1}\right]\\ &\geqslant 0 \quad \text{for } \theta\leqslant\frac{a+b}{2} \text{ or } M\geqslant[1+2\psi\,(b-a)]\,m,\end{aligned}$$

where $\psi(\cdot)$ is defined in (2.37). If the conditions hold then the inequalities (2.35) and (2.36) are valid and the bounds of Corollary 1 are tighter than those of (2.31). The opposite is true otherwise. □

REMARK 6. If n is odd and $\theta=\frac{a+b}{2}$, then (2.35) and (2.36) hold and the results of Corollary 1 provide tighter bounds than those from (2.31) – (2.34).

The following produces a three point rule where evaluation of the function and its derivatives occurs at a,ξ and b.

COROLLARY 2. *Let the conditions of Theorem 6 still be valid and let* $\alpha<\xi<\beta$ *with* $\alpha,\xi,\beta\in[a,b]$, *then*

$$\begin{aligned}&\left|\int_a^b f\,(x)\,dx-T_n\,(a,\alpha,\xi,\beta,b)-P_n\,(a,\alpha,\xi,\beta,b)\right| \\ &\leqslant Q_n^-\,(a,\alpha,\xi,\beta,b),\end{aligned}\tag{2.38}$$

where

$$\begin{aligned}&T_n\,(a,\alpha,\xi,\beta,b)\\ &= T_n\,(\alpha;a,\xi)+T_n\,(\beta;\xi,b)\\ &= \sum_{k=1}^{n}\left[\frac{(\alpha-a)^k f^{(k-1)}(a)+\left[(\beta-\xi)^k+(-1)^{k-1}(\xi-\alpha)^k\right]f^{(k-1)}(\xi)}{k!}\right.\\ &\quad\left.+\frac{(-1)^k(\beta-\xi)^k f^{(k-1)}(b)}{k!}\right],\end{aligned}$$

$$P_n\,(a,\alpha,\xi,\beta,b)=P_n\,(\alpha;a,\xi)+P_n\,(\beta;\xi,b)$$

and

$$Q_n^-(a,\alpha,\xi,\beta,b) = Q_n^-(\alpha;a,\xi) + Q_n^-(\beta;\xi,b)$$

with $T_n(\theta;a,b)$, $P_n(\theta;a,b)$ *and* $Q_n(\theta;a,b)$ *being as defined in (2.6) – (2.8) respectively.*

Proof. Applying Theorem 6 on $[a,\xi]$ with $a \leqslant \alpha \leqslant \xi$ we obtain

$$\left| \int_a^{\xi} f(x)\,dx - T_n(\alpha;a,\xi) - P_n(\alpha;a,\xi) \right| \leqslant Q_n^-(\alpha;a,\xi). \tag{2.39}$$

Similarly, an application on $[\xi,b]$ with $\xi \leqslant \beta < b$ produces

$$\left| \int_{\xi}^{b} f(x)\,dx - T_n(\beta;\xi,b) - P_n(\beta;\xi,b) \right| \leqslant Q_n^-(\beta;\xi,b). \tag{2.40}$$

Adding (2.39) and (2.40) produces (2.38). □

REMARK 7. It should be emphasized that the upper and lower bounds may be expressed in terms of $W(l,u)$ as defined in (2.14). Thus, for example, the first representation in (2.23) and (2.24) would be favoured over the second representations which give the results in terms of $\theta - a$ and $b - \theta$.

Following this approach, the following corollary results.

COROLLARY 3. *Let the conditions of Theorem 6 hold, then*

$$\left| \int_a^b f(x)\,dx - T_n(\theta;a,b) - P_n(\theta;a,b) \right| \leqslant \frac{U-L}{2}, \tag{2.41}$$

where $T_n(\theta;a,b)$ *is as given by (2.6) and*

$$P_n(\theta;a,b) = -\frac{M-m}{m} W(a,b) + \frac{U+L}{2}$$

with, from (2.14)

$$(n+1)!\,W(l,u) = (M-m)\left[(\theta-l)^{n+1} + (-1)^n (u-\theta)^{n+1}\right]. \tag{2.42}$$

Here, $\frac{U \pm L}{2}$ *is given by*

$$\begin{aligned} &\frac{U_e \pm L_e}{2} \qquad\qquad (2.43)\\ &= \frac{1}{2}\left\{W(a,a+\lambda_n^a) + W\left(b-\lambda_n^b,b\right) \pm \left[W(\theta-\lambda_n^a,\theta) + W\left(\theta,\theta+\lambda_n^b\right)\right]\right\} \end{aligned}$$

or

$$\frac{U_o \pm L_o}{2} = \frac{W(a,a+\lambda_n^a) \pm W\left(b-\lambda_n^b,b\right)}{2}, \tag{2.44}$$

depending on whether n is even or odd respectively.

Proof. The proof follows directly from that of Theorem 6. For n odd, then from (2.13) and (2.17), L_0 and U_0 may be expressed in terms of $W(l,u)$ as given by (2.42) to produce (2.44) from (2.15) and (2.16). Similarly, for n even using the first representation in (2.23) and (2.24) in terms of $W(l,u)$ gives (2.43). The result follows on noting that

$$I_n - \frac{U+L}{2} = \int_a^b f(x)\,dx - T_n(\theta;a,b) - P_n(\theta;a,b). \quad \square$$

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