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QUASILINEARITY & HADAMARD'S INEQUALITY

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Abstract. We establish a number of functional results arising out of Hadamard's inequalities. Striking nonnegativity, superadditivity and supermultiplicativity properties are derived.

1. Introduction

Let I be a finite interval of real numbers consisting of more than one point and $f : I \to \mathbb{R}$ a convex function. If $a, b \in I$ with $a < b$, then we have the well-known Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities are reversed if f is concave.

A large number of generalizations and improvements of this classical result have been made in the last decade. Thus, for example, it has been extended to more general classes of function [3], [7]-[9], [12], [13], [15]. The generalization in [9] can be interpreted as a relation between different means, which leads to further extensions [11]. A different interpretation of (1.1) as a relation between means has suggested a Hadamard's inequality for convex functions on the three-sphere [4]. In another direction, Hadamard's inequality has been interpolated by suitably defined maps [2], [14]. Some results for associated Lipschitzian maps are given in [5], [10].

In this article we consider a different aspect and examine the differences between the two sides in each of the inequalities in (1.1). This approach has already proved fruitful in connection with the Cauchy-Schwarz inequality [1]. We uncover a number of quasilinearity and monotonicity properties underlying Hadamard's inequality. Two different motifs are explored.

In Section 2 we introduce a functional

$$H(f;a,b) := \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)$$

and a similar functional relating to the right-hand inequality in (1.1). Their basic properties are given in Theorem 1. Here the motif is superadditive. When f is logarithmically convex, supermultiplicative versions of these results may be derived for suitably defined maps. These are presented in Corollary 1.

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An analogous development is made in Section 3 starting with

$$V(f;a,b) := \left[\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right]^{b-a}$$

in place of $H(f;a,b)$. Here the motif is supermultiplicative, though superadditive results are also deduced by taking logarithms.

We end in Section 4 with some concluding remarks.

Throughout the paper $I \subset \mathbb{R}$ will be taken to represent a proper interval and $a,b \in I$ with $a < b$.

2. Functions with an additive motif

Suppose $f : I \to \mathbb{R}$ and for $f \in L_1[a,b]$ define

$$H(f;a,b) := \int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)$$

and

$$L(f;a,b) := \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx.$$

We have the following theorem.

THEOREM 1. *Let $f : I \to \mathbb{R}$ be convex. Then*
(i) for all $a,b,c \in I$ with $a \leqslant c \leqslant b$, we have

$$0 \leqslant H(f;a,c) + H(f;c,b) \leqslant H(f;a,b) \tag{2.1}$$

and

$$0 \leqslant L(f;a,c) + L(f;c,b) \leqslant L(f;a,b); \tag{2.2}$$

(ii) for all $[c,d] \subseteq [a,b] \subseteq I$, we have

$$0 \leqslant H(f;c,d) \leqslant H(f;a,b) \tag{2.3}$$

and

$$0 \leqslant L(f;c,d) \leqslant L(f;a,b). \tag{2.4}$$

Proof. (i) The first inequality in each of (2.1), (2.2) follows from (1.1).

Let $c \in [a,b]$ and put $\alpha := (c-a)/(b-a)$, $\beta = (b-c)/(b-a)$. Then $\alpha + \beta = 1$ with $\alpha, \beta \geqslant 0$ and by the convexity of f we have with $x = (a+c)/2$, $y = (b+c)/2 \in I$ that

$$\begin{aligned}
\frac{c-a}{b-a} f\left(\frac{a+c}{2}\right) + \frac{b-c}{b-a} f\left(\frac{b+c}{2}\right) &= \alpha f(x) + \beta f(y) \\
&\geqslant f(\alpha x + \beta y) \\
&= f\left(\frac{c-a}{b-a}\cdot\frac{c+a}{2} + \frac{b-c}{b-a}\cdot\frac{b+c}{2}\right) \\
&= f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Hence

$$\begin{aligned} &H(f;a,b)-H(f;a,c)-H(f;c,b)\\ &\quad=(c-a)f\left(\frac{a+c}{2}\right)+(b-c)f\left(\frac{b+c}{2}\right)-(b-a)f\left(\frac{a+b}{2}\right)\geqslant 0 \end{aligned}$$

and the second part of (2.1) is proved.

Further, since f is convex on $[a,b]$, we have for all $c\in[a,b]$ that

$$\begin{vmatrix} 1 & 1 & 1\\ a & c & b\\ f(a) & f(c) & f(b) \end{vmatrix}\geqslant 0,$$

that is,

$$f(a)(b-c)-f(c)(b-a)+f(b)(c-a)\geqslant 0. \tag{2.5}$$

Because

$$\begin{aligned} L(f;a,b) &- L(f;a,c)-L(f;c,b)\\ &= \frac{f(a)+f(b)}{2}(b-a)-\frac{f(a)+f(c)}{2}(c-a)-\frac{f(c)+f(b)}{2}(b-c)\\ &= \frac{1}{2}\left[f(a)(b-c)-f(c)(b-a)+f(b)(c-a)\right], \end{aligned}$$

we have therefore that the second part of (2.2) holds also.

(ii) Using the first part of the theorem we have successively, for $[c,d]\subseteq[a,b]$, that

$$\begin{aligned} H(f;a,b) &\geqslant H(f;a,c)+H(f;c,b)\\ &\geqslant H(f;a,c)+H(f;c,d)+H(f;d,b) \end{aligned}$$

which gives us

$$H(f;a,b)-H(f;c,d)\geqslant H(f;a,c)+H(f;d,b)\geqslant 0$$

and (2.3) is proved. The argument of (2.4) goes likewise and we omit the details. □

REMARK 1. Inequalities (2.1)-(2.4) are reversed if f is concave. Similar comments apply to our subsequent results.

Now suppose that $f:I\to(0,\infty)$ is logarithmically convex. We can define the two mappings

$$\phi(f;a,b):=\exp\left[\int_a^b \ln\left[\frac{f(x)}{f\left(\frac{a+b}{2}\right)}\right]dx\right]$$

and

$$\psi(f;a,b):=\exp\left[\int_a^b \ln\left[\frac{\sqrt{f(a)f(b)}}{f(x)}\right]dx\right]$$

to derive the following corollary.

COROLLARY 1. *Let $f : I \to (0,\infty)$ be logarithmically convex. Then*
(i) for all $a,b,c \in I$ with $a \leqslant c \leqslant b$, we have

$$\phi(f;a,b) \geqslant \phi(f;a,c)\cdot\phi(f;c,b) \geqslant 1$$

and

$$\psi(f;a,b) \geqslant \psi(f;a,c)\cdot\psi(f;c,b) \geqslant 1;$$

(ii) for all $a,b,c,d \in I$ with $[c,d] \subseteq [a,b]$, we have

$$\phi(f;a,b) \geqslant \phi(f;c,d) \geqslant 1 \quad \text{and} \quad \psi(f;a,b) \geqslant \psi(f;c,d) \geqslant 1.$$

Proof. The results follow from the above theorem, since

$$\begin{aligned} H(\ln f;a,b) &:= \int_a^b \ln f(x)dx - (b-a)\ln f\left(\frac{a+b}{2}\right) \\ &= \int_a^b \ln\left[\frac{f(x)}{f\left(\frac{a+b}{2}\right)}\right]dx, \end{aligned}$$

that is,

$$\phi(f;a,b) = \exp[H(\ln f;a,b)]$$

and similarly

$$\psi(f;a,b) = \exp[L(\ln f;a,b)]. \qquad \square$$

For an arbitary function $f : I \to \mathbb{R}$ we introduce the mapping

$$S(f;a,b) := (b-a)\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right].$$

The following theorem holds.

THEOREM 2. *Suppose $f : I \to \mathbb{R}$ is convex. Then*
(i) for all $a \leqslant c \leqslant b$ ($a,b,c \in I$), we have

$$0 \leqslant S(f;a,c) + S(f;c,b) \leqslant S(f;a,b);$$

(ii) for all $[c,d] \subseteq [a,b]$ ($a,b,c,d \in I$), we have

$$0 \leqslant S(f;c,d) \leqslant S(f;a,b).$$

Proof. This is immediate from Theorem 1, since

$$S(f;a,b) = H(f;a,b) + L(f;a,b). \qquad \square$$

For positive mappings $f : I \to (0,\infty)$ we may introduce the functional

$$P(f;a,b) := \left[\frac{\sqrt{f(a)\cdot f(b)}}{f\left(\frac{a+b}{2}\right)}\right]^{b-a}.$$

Since

$$P(f;a,b) = \phi(f;a,b)\cdot\psi(f;a,b),$$

we have also the following corollary.

COROLLARY 2. *If* $f : I \to (0,\infty)$ *is logarithmically convex, then*

(i) for all $a,b,c \in I$ *with* $a \leqslant c \leqslant b$*, we have*

$$P(f;a,b) \geqslant P(f;a,c)\cdot P(f;c,b) \geqslant 1;$$

(ii) for all $[c,d] \subseteq [a,b] \subseteq I$*, we have*

$$P(f;c,d) \leqslant P(f;a,b).$$

3. Functionals based on a multiplicative motif

For $f : I \to \mathbb{R}$ convex we can also define the functionals

$$V(f;a,b) := \left[\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right]^{b-a},$$

$$W(f;a,b) := \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]^{b-a}.$$

For these functionals we have the following theorem.

THEOREM 3. *Let* $f : I \to \mathbb{R}$ *be convex. Then for all* $a,b,c \in I$ *with* $a < c < b$*, we have*

$$V(f;a,c)\cdot V(f;c,b) \leqslant V(f;a,b), \tag{3.1}$$

$$W(f;a,c)\cdot W(f;c,b) \leqslant W(f;a,b). \tag{3.2}$$

Proof. We have

$$V(f;a,b) = \left[\frac{H(f;a,b)}{b-a}\right]^{b-a}$$

and

$$W(f;a,b) = \left[\frac{L(f;a,b)}{b-a}\right]^{b-a}$$

and by (2.1)

$$V(f;a,b) = \left[\frac{H(f;a,b)}{b-a}\right]^{b-a} \geqslant \left[\frac{H(f;a,c)+H(f;c,b)}{b-a}\right]^{b-a}.$$

Since

$$H(f;a,b) = (b-a)[V(f;a,b)]^{1/(b-a)},$$

we get

$$[V(f;a,b)]^{1/(b-a)} \geqslant \frac{(c-a)[V(f;a,c)]^{1/(c-a)} + (b-c)[V(f;c,b)]^{1/(b-c)}}{(c-a)+(b-c)}. \tag{3.3}$$

By application of the well-known arithmetico-geometric inequality

$$\frac{px+qy}{p+q} \geqslant x^{p/(p+q)} \cdot y^{q/(p+q)}$$

with $p = c-a > 0$, $q = b-c > 0$ and $x = [V(f\,;a,c)]^{1/(c-a)}$, $y = [V(f\,;c,b)]^{1/(b-c)}$, we deduce that the right-hand side of (3.3) is greater than or equal to

$$[V(f\,;a,c)V(f\,;c,b)]^{1/(b-a)}.$$

Inequality (3.1) follows at once. Inequality (3.2) follows likewise *via* (2.2). □

In what follows, suppose that $f : I \to (0,\infty)$ is logarithmically convex. We define the mappings

$$\overline{\sigma}(f\,;a,b) := (b-a)\ln\left[\frac{1}{b-a}\int_a^b \ln\left[\frac{f(x)}{f((a+b)/2)}\right]dx\right]$$

and

$$\sigma(f\,;a,b) := (b-a)\ln\left[\frac{1}{b-a}\int_a^b \ln\left[\frac{\sqrt{f(a)f(b)}}{f(x)}\right]dx\right].$$

COROLLARY 3. *Let f be as above. Then for all $a<c<b$, we have*

$$\overline{\sigma}(f\,;a,b) \geqslant \overline{\sigma}(f\,;a,c) + \overline{\sigma}(f\,;c,b)$$

and

$$\sigma(f\,;a,b) \geqslant \sigma(f\,;a,c) + \sigma(f\,;c,b).$$

Proof. Since f is logarithmically convex, we have that $\ln f$ is convex,

$$\begin{aligned} V(\ln f\,;a,b) &= \left[\frac{1}{b-a}\int_a^b \ln f(x)dx - \ln f\left(\frac{a+b}{2}\right)\right]^{b-a} \\ &= \left[\frac{1}{b-a}\int_a^b \ln\left[\frac{f(x)}{f((a+b)/2)}\right]dx\right]^{b-a} \\ \text{and} \quad W(\ln f\,;a,b) &= \left[\frac{1}{b-a}\int_a^b \ln\left[\frac{\sqrt{f(a)f(b)}}{f(x)}\right]dx\right]^{b-a}. \end{aligned}$$

As $\overline{\sigma}(f\,;a,b) = \ln[V(\ln f\,;a,b)]$ and $\sigma(f\,;a,b) = \ln[W(\ln f\,;a,b)]$, the desired result follows from (3.1) and (3.2). □

Also, for a convex map $f : I \to \mathbb{R}$ we can consider the functional

$$Z(f\,;a,b) := \left\{\frac{1}{b-a}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]\right\}^{b-a}.$$

By an argument similar to that of Theorem 3, we have the following theorem.

THEOREM 4. *Let $f : I \to \mathbb{R}$ be convex. Then for all $a,b,c \in I$ with $a<c<b$, we have*

$$Z(f\,;a,c)\cdot Z(f\,;c,b) \leqslant Z(f\,;a,b).$$

A similar property holds for the mapping

$$\theta(f;a,b) := (b-a)\ln\left[\frac{\sqrt{f(a)f(b)}}{f((a+b)/2)}\right]^{b-a}.$$

COROLLARY 4. *Under the above assumptions, we have for all* $a<c<b$ *that*

$$\theta(f;a,b) \geqslant \theta(f;a,c)+\theta(f;c,b).$$

4. Concluding remarks

It is natural to seek extensions of the foregoing results to generalisations of convex functions. However we were unable to prove some of the most direct and simple extensions.

Thus suppose $f:I\to\mathbb{R}$ is a P-function, that is, the convexity definition

$$f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y) \tag{4.1}$$

for all $x,y\in I$ and $\lambda\in[0,1]$ is relaxed to

$$f(\lambda x+(1-\lambda)y) \leqslant f(x)+f(y). \tag{4.2}$$

Setting $x=y$ shows that f is then necessarily nonnegative. It is shown in [8] that if f is integrable, then

$$\frac{1}{2}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant f(a)+f(b)$$

applies. This suggests we might seek to establish a version of Theorem 1 with the functionals

$$H(f;a,b) := \int_a^b f(x)dx - \frac{b-a}{2}f\left(\frac{a+b}{2}\right)$$

and

$$L(f;a,b) := [f(a)+f(b)]\,(b-a) - \int_a^b f(x)dx.$$

The first inequalities in (2.1)-(2.4) all hold trivially as before. The problem lies with the second inequalities. For the second inequality in (2.2), for example, we need as before to establish (2.5). However the use of (4.2) in place of (4.1) would appear too weak to give (2.5) in general, and we can prove only a weakened version of the second inequalities in (2.1)-(2.4).

The results of Sections 2 and 3 may be used for particular choices of convex function to provide transparent derivations of some quite complicated inequalities that would be tedious to deconvolve or synthesise from first principles. Some examples are given below, all based on the simple choice $f(x)=1/x$.

Assume that $0<a<b$ and consider the mapping $f:(0,\infty)\to(0,\infty)$ given by $f(x)=1/x$. A simple calculation shows us that

$$H(f\;;\;a,b) = \ln\left[\frac{b}{a}\exp\left(\frac{2(a-b)}{a+b}\right)\right] =: A(a,b)$$

and

$$L(f\ ;\ a,b) = \ln\left[\frac{a}{b}\exp\left(\frac{b^2-a^2}{2ab}\right)\right] =: B(a,b).$$

By Theorem 1 we have that

$$A(a,b) \geqslant A(a,c) + A(c,b) \geqslant 0 \quad \text{for all} \quad c \in [a,b] \tag{4.3}$$

and

$$B(a,b) \geqslant B(a,c) + B(c,b) \geqslant 0 \quad \text{for all} \quad c \in [a,b]. \tag{4.4}$$

Set

$$\alpha(a,b) := \frac{b}{a}\exp\left(\frac{2(a-b)}{a+b}\right), \quad 0 < a \leqslant b$$

and

$$\beta(a,b) := \frac{a}{b}\exp\left(\frac{b^2-a^2}{2ab}\right), \quad 0 < a \leqslant b.$$

From (4.3) and (4.4) we derive

$$\alpha(a,b) \geqslant \alpha(a,c)\cdot\alpha(c,b) \quad \text{and} \quad \beta(a,b) \geqslant \beta(a,c)\beta(c,b). \tag{4.3}$$

Also

$$V(f\ ;\ a,b) = \left(\ln\left[\left(\frac{b}{a}\right)^{b-a}\exp\left(-\frac{a+b}{2}\right)\right]\right)^{b-a} =: C(a,b)$$

and

$$W(f\ ;\ a,b) = \left(\ln\left[\left(\frac{a}{b}\right)^{b-a}\exp\left(\frac{a+b}{2ab}\right)\right]\right)^{b-a} =: D(a,b).$$

Now by Theorem 3

$$C(a,b) \geqslant C(a,c)\cdot C(c,b) \quad \text{for all} \quad 0 < a < c < b$$

and

$$D(a,b) \geqslant D(a,c)\cdot D(c,b) \quad \text{for all} \quad 0 < a < c < b.$$

Moreover, if we define

$$\gamma(a,b) := (b-a)\ln\left(\ln\left[\left(\frac{b}{a}\right)^{b-a}\exp\left(-\frac{a+b}{2}\right)\right]\right)$$

and

$$\delta(a,b) := (b-a)\ln\left(\ln\left[\left(\frac{a}{b}\right)^{b-a}\exp\left(\frac{a+b}{2ab}\right)\right]\right),$$

then for all $0 < a < c < b$ we have

$$\gamma(a,b) \geqslant \gamma(a,c) + \gamma(c,b) \quad \text{and} \quad \delta(a,b) \geqslant \delta(a,c) + \delta(c,b).$$

REFERENCES

[1] Y. J. CHO, S. S. DRAGOMIR, S. S. KIM AND C. E. M. PEARCE, Cauchy-Schwarz functionals, *Bull. Austral. Math. Soc.* **62** (2000), 479–491.

[2] S. S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. & Appl.* **167** (1992), 49–56.

[3] S. S. DRAGOMIR, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang J. Math.* **24** (1993), 101–106.

[4] S. S. DRAGOMIR, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, *Math. Ineq. & Applic.* **3** (2000), 177–187.

[5] S. S. DRAGOMIR, Y. J. CHO AND S. S. KIM, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. & Applic.* **245** (2000), 489–501.

[6] S. S. DRAGOMIR AND N. M. IONESCU, Some converse of Jensen's inequality and applications, *Anal. Numér. Théor Approx.* **23** (1994), 71–78.

[7] S. S. DRAGOMIR AND C. E. M. PEARCE, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377–385.

[8] S. S. DRAGOMIR, J. E. PEČARIĆ AND L. E. PERSSON, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995), 335–341.

[9] P. GILL, C. E. M. PEARCE AND J. PEČARIĆ, Hadamard's inequality for r-convex functions, *J. Math. Anal. Applic.* **215** (1997), 461–470.

[10] M. MATIĆ AND J. PEČARIĆ, Notes on inequalities of Hadamard's type for Lipschitzian mappings, *Tamkang J. Math.* **32** (2001), 127–130.

[11] C. E. M. PEARCE, J. PEČARIĆ AND V. SIMIČ, Stolarsky means and Hadamard's inequality, *J. Math. Anal. and Applic.* **220** (1998), 99–109.

[12] C. E. M. PEARCE AND A. M. RUBINOV, P-functions, quasi-convex functions and Hadamard-type inequalities, *J. Math. Anal. & Applic.* **240** (1999), 92–104.

[13] J. E. PEČARIĆ AND S. S. DRAGOMIR, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Math.* **7** (1991), 103–107.

[14] G.-S. YANG AND M.-C. HONG, A note on Hadamard's inequality, *Tamkang J. Math.* **28** (1997), 33–37.

[15] G.-S. YANG AND H.-L. WU, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang J. Math.* **27** (1996), 327–336.

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