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DOUBLE INTEGRAL INEQUALITIES BASED ON MULTI–BRANCH PEANO KERNELS

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Abstract. The Ostrowski inequality in one dimension has been known for about seventy years. In the last two or three years an Ostrowski type inequality in two dimensions has been developed. In this paper we extend these ideas in two dimensions to obtain double integral inequalities that are based on multi-branch Peano kernels.

1. Introduction

Ostrowski [1], in his now classical result, obtained an integral inequality in one dimension that indicates a two-sided bound between a function evaluated at an interior point x and the average of that function over a finite interval. As with a lot of mathematical ideas, that are extended in various directions, so too has the Ostrowski result been generalised and extended in a multitude of ways:- one such way has been to impose more stringent demands on the mapping function f.

Dragomir [2], [3] and [4] obtained the following result.

THEOREM 1. *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *be a division of the interval* $[a,b]$, α_i $(i = 0,\dots,k+1)$ *be* $k+2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,\dots,k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbf{R}$ *is absolutely continuous on* $[a,b]$, *then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{1.1}$$

$$\leqslant \begin{cases} \left[\dfrac{1}{4} \displaystyle\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \dfrac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty, \\ \dfrac{\|f'\|_p}{(q+1)^{\frac{1}{q}}} \left[\displaystyle\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right] \right]^{\frac{1}{q}} \\ \left[\dfrac{1}{2} \nu(h) + \max \left\{ \left| \alpha_{i+1} - \dfrac{x_i + x_{i+1}}{2} \right|, \; i = 0,\dots,k-1 \right\} \right] \|f'\|_1, \end{cases}$$

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$$\leqslant \begin{cases} \frac{1}{2}\|f'\|_\infty \sum_{i=0}^{k-1} h_i^2 \leqslant \frac{1}{2}(b-a)\,\nu(h)\,\|f'\|_\infty\,, \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{k-1} h_i^{q+1}\right]^{\frac{1}{q}} \leqslant \nu(h)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_p\,, \\ \qquad\qquad\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \nu(h)\,\|f'\|_1\,, \end{cases}$$

where $h_i := x_{i+1} - x_i \ \ (i = 0,\ldots,k-1),\ \nu(h) := \max\{h_i | i = 0,\ldots,n\}$ *and*

$$\|f'\|_\infty \ : = ess \sup_{t\in[a,b]} |f'(t)|\,,$$

$$\|f'\|_p \ : = \left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}},$$

$$\|f'\|_1 \ : = \int_a^b |f'(t)|\,dt$$

are the usual $L_\infty[a,b]$, $L_p[a,b]$ *and* $L_1[a,b]$ *norms.*

Dragomir [2] further extended Theorem 1. to include functions of bounded variation.

For a smooth mapping function f, Sofo [6] generalised Theorem 1 to the following:

THEOREM 2. *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *be a division of the interval* $[a,b]$, *and* $\alpha_i \ (i = 0,\ldots,k+1)$ *be* $k+2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i] \ (i = 1,\ldots,k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbf{R}$ *is a mapping such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$, *then for all* $x_i \in [a,b]$ *we have the inequality:*

$$\left|\int_a^b f(x)\,dx - \sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right| \tag{1.2}$$
$$\leqslant \frac{\|f^{(n)}\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1}-x_i)^{n+1} - (x_{i+1}-\alpha_{i+1})^{n+1}\right\}$$
$$\leqslant \frac{\|f^{(n)}\|_\infty}{(n+1)!}\sum_{i=0}^{k-1} h_i^{n+1} \leqslant \frac{\|f^{(n)}\|_\infty}{(n+1)!}(b-a)\,\nu^n(h)\,.$$

In the paper [7], Barnett and Dragomir developed a two dimensional version of the Ostrowski inequality, which Dragomir, Barnett and Cerone [8] further developed for the $L_p[a,b]$ norm.

Hanna, Dragomir and Cerone [9] obtained the following result of Ostrowski type in two dimensions

THEOREM 3. *Let $f : [a,b] \times [c,d] \to \mathbf{R}$ be a continuous mapping such that the partial derivatives $\frac{\partial^{l+k} f(\cdot,\cdot)}{\partial x^k \partial y^l}$, $k = 0,1,\ldots,n-1$, $l = 0,1,\ldots,m-1$ exist and are continuous on $[a,b] \times [c,d]$, then the following inequality holds*

$$\begin{aligned} &\left| \int_a^b \int_c^d f(t,s)\,dsdt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k(x) \cdot Y_l(y) \frac{\partial^{l+k} f(x,y)}{\partial x^k \partial y^l} \right. \\ &- (-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S_m(y,s) \frac{\partial^{k+m} f(x,s)}{\partial x^k \partial y^m} ds \\ &\left. - (-1)^n \sum_{l=0}^{m-1} Y_l(y) \int_a^b K_n(x,t) \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l} dt \right| \\ \leqslant\ & \frac{1}{(n+1)!\,(m+1)!} \left((x-a)^{n+1} + (b-x)^{n+1} \right) \\ &\times \left((y-c)^{m+1} + (d-y)^{m+1} \right) \left\| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right\|_\infty \end{aligned} \tag{1.3}$$

if $\frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b] \times [c,d])$ *for all* $(x,y) \in [a,b] \times [c,d]$, *where*

$$\left\| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right\|_\infty := \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| < \infty,$$

$$\begin{cases} X_k(x) := \dfrac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \\ \\ Y_l(y) := \dfrac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}. \end{cases} \quad \text{and} \tag{1.4}$$

Furthermore, for $K_n : [a,b]^2 \to \mathbf{R}$ and $S_m : [c,d]^2 \to \mathbf{R}$ we define the two branch kernels as

$$\begin{cases} K_n(x,t) := \begin{cases} \dfrac{(t-a)^n}{n!}, & t \in [a,x] \\ \dfrac{(t-b)^n}{n!}, & t \in (x,b] \end{cases} \\ S_m(y,s) := \begin{cases} \dfrac{(s-c)^m}{m!}, & s \in [c,y] \\ \dfrac{(s-d)^m}{m!}, & s \in (y,d] \end{cases} \end{cases} . \tag{1.5}$$

In this paper we use the results of Theorem 2, with a multi-branch version of the Peano kernels (1.5), to further generalise the double integral inequality of Theorem 3.

2. A double integral identity

The following theorem holds.

THEOREM 4. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0,\ldots,k+1)$ be $k+2$ points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ and $\alpha_{k+1} = b$. Let $J_l : c = y_0 < y_1 < \cdots < y_{l-1} < y_l = d$ be a division of the interval $[c,d]$ and β_ρ $(\rho = 0,\ldots,l+1)$ be $l+2$ points such that $\beta_0 = c$, $\beta_\rho \in \left[y_{\rho-1}, y_\rho\right]$ and $\beta_{\rho+1} = d$. Further, let $f : [a,b]\times[c,d] \to \mathbf{R}$ be a continuous mapping such that all the partial derivatives $\frac{\partial^{r+j} f\left(x_i, y_\rho\right)}{\partial x_i^j \partial y_\rho^r}$, $r = 0,\ldots,m$, $j = 0,\ldots,n$; $m,n \in \mathbf{N}$ exist and are continuous on $[a,b]\times[c,d]$ and let $K_{n,k} : [a,b]^2 \to \mathbf{R}$, $S_{m,l} : [c,d]^2 \to \mathbf{R}$ be defined as*

$$\begin{cases} K_{n,k}(x,t) := \begin{cases} \dfrac{(t-\alpha_1)^n}{n!}, & t \in [a, x_1] \\ \vdots & \\ \dfrac{(t-\alpha_k)^n}{n!}, & t \in [x_{k-1}, b] \end{cases} \\ S_{m,l}(y,s) := \begin{cases} \dfrac{(s-\beta_1)^m}{m!}, & s \in [c, y_1] \\ \vdots & \\ \dfrac{(s-\beta_l)^m}{m!}, & s \in [y_{l-1}, d] \end{cases} \end{cases} \tag{2.1}$$

then for all $(x_i, y_i) \in [a,b]\times[c,d]$ we have the identity

$$\begin{aligned} &\int_a^b\int_c^d f(t,s)\,ds\,dt - \sum_{j=1}^{n}\sum_{i=0}^{k} X^j(x_i) \sum_{r=1}^{m}\sum_{\rho=0}^{l} Y^r\left(y_\rho\right) \frac{\partial^{r+j-2} f\left(x_i, y_\rho\right)}{\partial x_i^{j-1} \partial y_\rho^{r-1}} \\ &\quad + (-1)^m \sum_{j=1}^{n}\sum_{i=0}^{k} X^j(x_i) \int_c^d S_{m,l}(y,s) \frac{\partial^{j+m-1} f(x_i, s)}{\partial x_i^{j-1} \partial s^m} ds \\ &\quad + (-1)^n \sum_{r=1}^{m}\sum_{\rho=0}^{l} Y^r\left(y_\rho\right) \int_a^b K_{n,k}(x,t) \frac{\partial^{n+r-1} f\left(t, y_\rho\right)}{\partial t^n \partial y_\rho^{r-1}} dt \\ &= (-1)^{n+m} \int_a^b\int_c^d K_{n,k}(x,t)\, S_{m,l}(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds\,dt, \end{aligned} \tag{2.2}$$

where

$$\begin{cases} X^j(x_i) := \dfrac{(-1)^j}{j!}\left((x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right), \\ Y^r\left(y_\rho\right) := \dfrac{(-1)^r}{r!}\left(\left(y_\rho - \beta_\rho\right)^r - \left(y_\rho - \beta_{\rho+1}\right)^r\right). \end{cases} \tag{2.3}$$

Proof. The following identity is valid (see [6])

$$\begin{aligned} &\int_a^b g(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k} \left((x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right) g^{(j-1)}(x_i) \\ &\quad = (-1)^n \int_a^b K_{n,k}(x,t)\, g^{(n)}(t)\,dt, \end{aligned} \tag{2.4}$$

where $K_{n,k}(x,t)$ is given by the first part of (2.1).

For the partial mapping $f(\cdot,s)$, $s\in[c,d]$ we have from (2.4)

$$\int_a^b f(t,s)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{k}\left((x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right)\frac{\partial^{j-1}f(x_i,s)}{\partial x_i^{j-1}} = (-1)^n\int_a^b K_{n,k}(x,t)\,\frac{\partial^n f(t,s)}{\partial t^n}dt \tag{2.5}$$

for every $x_i\in[a,b]$ and $s\in[c,d]$.

Now, integrating over s on $[c,d]$ we have

$$\int_a^b\int_c^d f(t,s)\,dsdt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{k}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\} \times\int_c^d\frac{\partial^{j-1}f(x_i,s)}{\partial x_i^{j-1}}ds=(-1)^n\int_a^b K_{n,k}(x,t)\left(\int_c^d\frac{\partial^n f(t,s)}{\partial t^n}ds\right)dt \tag{2.6}$$

for all $x_i\in[a,b]$.

Utilising (2.4) again for the partial mapping $\frac{\partial^{j-1}f(x_i,\cdot)}{\partial x_i^{j-1}}$ on $[c,d]$ we obtain

$$\int_c^d\frac{\partial^{j-1}f(x_i,s)}{\partial x_i^{j-1}}ds+\sum_{r=1}^{m}\frac{(-1)^r}{r!}\sum_{\rho=0}^{l}\left\{(y_\rho-\beta_\rho)^r-(y_\rho-\beta_{\rho+1})^r\right\} \times\frac{\partial^{r-1}}{\partial y_\rho^{r-1}}\left(\frac{\partial^{j-1}f(x_i,y_\rho)}{\partial x_i^{j-1}}\right)=(-1)^m\int_c^d S_{m,l}(y,s)\,\frac{\partial^m}{\partial s^m}\left(\frac{\partial^{j-1}f(x_i,s)}{\partial x_i^{j-1}}\right)ds, \tag{2.7}$$

hence

$$\int_c^d\frac{\partial^{j-1}f(x_i,s)}{\partial x_i^{j-1}}ds+\sum_{r=1}^{m}\frac{(-1)^r}{r!}\sum_{\rho=0}^{l}\left\{(y_\rho-\beta_\rho)^r-(y_\rho-\beta_{\rho+1})^r\right\} \times\frac{\partial^{r+j-2}f(x_i,y_\rho)}{\partial x_i^{j-1}\partial y_\rho^{r-1}}=(-1)^m\int_c^d S_{m,l}(y,s)\,\frac{\partial^{j-1+m}f(x_i,s)}{\partial x_i^{j-1}\partial s^m}ds. \tag{2.8}$$

Again, utilising (2.4) for the partial mapping $\frac{\partial^n f(t,\cdot)}{\partial t^n}$ on $[c,d]$, we obtain

$$\int_c^d\frac{\partial^n f(t,s)}{\partial t^n}ds+\sum_{r=1}^{m}\frac{(-1)^r}{r!}\sum_{\rho=0}^{l}\left\{(y_\rho-\beta_\rho)^r-(y_\rho-\beta_{\rho+1})^r\right\} \times\frac{\partial^{n+r-1}f(t,y_\rho)}{\partial t^n\partial y_\rho^{r-1}}=(-1)^m\int_c^d S_{m,l}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}ds. \tag{2.9}$$

Now, substituting (2.8) and (2.9) into (2.6) we have

$$\int_a^b \int_c^d f\,(t,s)\,dsdt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} \tag{2.10}$$
$$\times \left[(-1)^m \int_c^d S_{m,l}\,(y,s)\, \frac{\partial^{j-1+m} f\,(x_i,s)}{\partial x_i^{j-1} \partial s^m} ds \right.$$
$$\left. - \sum_{r=1}^{m} \frac{(-1)^r}{r!} \sum_{\rho=0}^{l} \left\{ \left(y_\rho - \beta_\rho\right)^r - \left(y_\rho - \beta_{\rho+1}\right)^r \right\} \frac{\partial^{r+j-2} f\,\left(x_i, y_\rho\right)}{\partial x_i^{j-1} \partial y_\rho^{r-1}} \right]$$
$$= (-1)^n \int_a^b K_{n,k}\,(x,t) \left[(-1)^m \int_c^d S_{m,l}\,(y,s)\, \frac{\partial^{n+m} f\,(t,s)}{\partial t^n \partial s^m} ds \right.$$
$$\left. - \sum_{r=1}^{m} \frac{(-1)^r}{r!} \sum_{\rho=0}^{l} \left\{ \left(y_\rho - \beta_\rho\right)^r - \left(y_\rho - \beta_{\rho+1}\right)^r \right\} \frac{\partial^{n+r-1} f\,\left(t, y_\rho\right)}{\partial t^n \partial y_\rho^{r-1}} \right] dt.$$

Rewriting (2.10) we have

$$\int_a^b \int_c^d f\,(t,s)\,dsdt = \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\} \tag{2.11}$$
$$\times \sum_{r=1}^{m} \frac{(-1)^r}{r!} \sum_{\rho=0}^{l} \left\{ \left(y_\rho - \beta_\rho\right)^r - \left(y_\rho - \beta_{\rho+1}\right)^r \right\} \frac{\partial^{r+j-2} f\,\left(x_i, y_\rho\right)}{\partial x_i^{j-1} \partial y_\rho^{r-1}}$$
$$- (-1)^m \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k} \left\{ (x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j \right\}$$
$$\times \int_c^d S_{m,l}\,(y,s)\, \frac{\partial^{j-1+m} f\,(x_i,s)}{\partial x_i^{j-1} \partial s^m} ds$$
$$- (-1)^n \sum_{r=1}^{m} \frac{(-1)^r}{r!} \sum_{\rho=0}^{l} \left\{ \left(y_\rho - \beta_\rho\right)^r - \left(y_\rho - \beta_{\rho+1}\right)^r \right\}^n$$
$$\times \int_a^b K_{n,k}\,(x,t)\, \frac{\partial^{n+r-1} f\,\left(t, y_\rho\right)}{\partial t^n \partial y_\rho^{r-1}} dt$$
$$+ (-1)^{n+m} \int_a^b \int_c^d K_{n,k}\,(x,t)\, S_{m,l}\,(y,s)\, \frac{\partial^{n+m} f\,(t,s)}{\partial t^n \partial s^m} dsdt$$

and using (2.3) we arrive at the required identity (2.2). □

REMARK 1. For $k = 2$ and $l = 2$ the Peano kernels of (2.1) reduce to those of (1.5) in which case the identity (2.2) reduces to an identity obtained in [9].

If we now assume that the points of the rectangular division I_k and J_l are fixed, we obtain the following corollary.

COROLLARY 1. *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *and* $J_l : c = y_0 < y_1 < \cdots < y_{l-1} < y_l = d$ *be a division of the rectangle* $[a,b] \times [c,d]$. *If* $f : [a,b] \times [c,d] \to \mathbf{R}$ *is as defined in Theorem 4, then we have the equality*

$$\begin{aligned}\int_a^b \int_c^d f(t,s)\,dsdt &= \sum_{j=1}^{n} \frac{1}{2^j j!} \sum_{i=0}^{k} \left\{-h_i^j + (-1)^j h_{i-1}^j\right\} \\ &\times \sum_{r=1}^{m} \frac{(-1)^r}{2^r r!} \sum_{\rho=0}^{l} \left\{-\lambda_\rho^r + (-1)^r \lambda_{\rho-1}^r\right\} \frac{\partial^{r+j-2} f(x_i, y_\rho)}{\partial x_i^{j-1} \partial y_\rho^{r-1}} \\ &- (-1)^m \sum_{j=1}^{n} \frac{1}{2^j j!} \sum_{i=0}^{k} \left\{-h_i^j + (-1)^j h_{i-1}^j\right\} \int_c^d S_{m,l}(y,s) \frac{\partial^{j+m-1} f(x_i,s)}{\partial x_i^{j-1} \partial s^m} ds \\ &- (-1)^n \sum_{r=1}^{m} \frac{(-1)^r}{2^r r!} \sum_{\rho=0}^{l} \left\{-\lambda_\rho^r + (-1)^r \lambda_{\rho-1}^r\right\} \int_a^b K_{n,k}(x,t) \frac{\partial^{n+r-1} f(t, y_\rho)}{\partial t^n \partial y_\rho^{r-1}} dt \\ &+ (-1)^{n+m} \int_a^b \int_c^d K_{n,k}(x,t) S_{m,l}(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt,\end{aligned} \tag{2.12}$$

where $h_i := x_{i+1} - x_i$, $h_{-1} := 0$, $h_k := 0$ *and* $\lambda_\rho := y_{\rho+1} - y_\rho$, $\lambda_{-1} = 0$ *and* $\lambda_\rho := 0$.

Proof. Choose

$$\alpha_0 = a, \quad \alpha_1 = \frac{a + x_1}{2}, \quad \alpha_1 = \frac{x_1 + x_2}{2}, \ldots$$

$$\alpha_{k-1} = \frac{x_{k-2} + x_{k-1}}{2}, \quad \alpha_k = \frac{x_{k-1} + x_k}{2}, \quad \alpha_{k+1} = b,$$

$$\beta_0 = c, \quad \beta_1 = \frac{c + y_1}{2}, \quad \beta_1 = \frac{y_1 + y_2}{2}, \ldots$$

$$\beta_{l-1} = \frac{y_{l-2} + y_{l-1}}{2}, \quad \beta_l = \frac{y_{l-1} + y_l}{2}, \quad \beta_{l+1} = d.$$

Substituting these values into (2.11), we obtain the result (2.12). □

The case of equidistant partitioning is important in practice, and with this in mind we obtain the following corollary.

COROLLARY. *Let*

$$\begin{cases} I_k : x_i = a + i\left(\dfrac{b-a}{k}\right), & i = 0, \ldots, k \quad \text{and} \\ J_l : y_\rho = c + \rho\left(\dfrac{d-c}{l}\right), & \rho = 0, \ldots, l \end{cases} \tag{2.13}$$

be an equidistant partitioning of the rectangle $[a,b]\times[c,d]$. *Then we have the equality*

$$\begin{aligned}
&\int_a^b\int_c^d f\,(t,s)\,dsdt \qquad (2.14)\\
&=\sum_{j=1}^{n}\left(\frac{b-a}{2k}\right)^j\frac{1}{j!}\left(1-(-1)^j\right)\sum_{r=1}^{m}\left(\frac{d-c}{2l}\right)^r\frac{1}{r!}\left(1-(-1)^r\right)\\
&\quad\times\sum_{i=0}^{k}\sum_{\rho=0}^{l}\frac{\partial^{r+j-2}f\,\left(x_i,y_\rho\right)}{\partial x_i^{j-1}\partial y_\rho^{r-1}}\\
&\quad-(-1)^m\sum_{j=1}^{n}\left(\frac{b-a}{2k}\right)^j\frac{\left(1-(-1)^j\right)}{j!}\sum_{i=0}^{k}\int_c^d S_{m,l}\,(y,s)\,\frac{\partial^{j+m-1}f\,(x_i,s)}{\partial x_i^{j-1}\partial s^m}ds\\
&\quad-(-1)^n\sum_{r=1}^{m}\left(\frac{d-c}{2l}\right)^r\frac{\left(1-(-1)^r\right)}{r!}\sum_{\rho=0}^{l}\int_a^b K_{n,k}\,(x,t)\,\frac{\partial^{n+r-1}f\,\left(t,y_\rho\right)}{\partial t^n\partial y_\rho^{r-1}}dt\\
&\quad+(-1)^{n+m}\int_a^b\int_c^d K_{n,k}\,(x,t)\,S_{m,l}\,(y,s)\,\frac{\partial^{n+m}f\,(t,s)}{\partial t^n\partial s^m}dsdt.
\end{aligned}$$

Proof. From (2.13) we note that

$$h_i := \frac{b-a}{k} \qquad i=0,\ldots,k$$

and

$$\lambda_\rho := \frac{d-c}{l} \qquad \rho=0,\ldots,l$$

and substituting into (2.12) we arrive at the equality (2.14). □

3. The inequalities

The following theorem will now be proved.

THEOREM 5. *Let* $f:[a,b]\times[c,d]\to\mathbf{R}$ *be continuous on* $[a,b]\times[c,d]$ *and* *assume that* $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ *exists on* $(a,b)\times(c,d)$. *The following inequality is valid:*

$$
\begin{aligned}
&|V| \qquad (3.1)\\
&:= \left| \int_a^b \int_c^d f\,(t,s)\,dsdt - \sum_{j=1}^{n}\sum_{i=0}^{k} X^j(x_i) \sum_{r=1}^{m}\sum_{\rho=0}^{l} Y^r(y_\rho) \frac{\partial^{r+j-2} f\,(x_i,y_\rho)}{\partial x_i^{j-1}\partial y_\rho^{r-1}} \right.\\
&\quad + (-1)^m \sum_{j=1}^{n}\sum_{i=0}^{k} X^j(x_i) \int_c^d S_{m,l}(y,s) \frac{\partial^{j+m-1} f\,(x_i,s)}{\partial x_i^{j-1}\partial s^m} ds\\
&\quad \left. + (-1)^n \sum_{r=1}^{m}\sum_{\rho=0}^{l} Y^r(y_\rho) \int_a^b K_{n,k}(x,t) \frac{\partial^{n+r-1} f\,(t,y_\rho)}{\partial t^n \partial y_\rho^{r-1}} dt \right|\\
&\leqslant \begin{cases}
\frac{1}{(n+1)!\,(m+1)!} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1}-x_i)^{n+1} - (x_{i+1}-\alpha_{i+1})^{n+1} \right\} \\
\times \sum_{\rho=0}^{l-1} \left\{ (\beta_{\rho+1}-y_\rho)^{m+1} - (y_{\rho+1}-\beta_{\rho+1})^{m+1} \right\} \left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_\infty, \\
\text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \in L_\infty([a,b]\times[c,d]); \\
\frac{1}{n!m!} \left(\frac{\sum_{i=0}^{k-1}\left\{ (\alpha_{i+1}-x_i)^{nq+1} - (x_{i+1}-\alpha_{i+1})^{nq+1} \right\}}{nq+1} \right)^{\frac{1}{q}} \\
\times \left(\frac{\sum_{\rho=0}^{l-1}\left\{ (\beta_{\rho+1}-y_\rho)^{mq+1} - (y_{\rho+1}-\beta_{\rho+1})^{mq+1} \right\}}{mq+1} \right)^{\frac{1}{q}} \left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_p, \\
\text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ p^{-1}+q^{-1}=1; \\
\frac{1}{n!m!} \nu^n(h)\,\mu^m(\lambda) \cdot \left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_1, \quad \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \in L_1([a,b]\times[c,d])
\end{cases}
\end{aligned}
$$

for all $(x_i, y_i) \in [a,b]\times[c,d]$, *where*

$$\left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]} \left| \frac{\partial^{n+m}f\,(t,s)}{\partial t^n\partial s^m} \right| < \infty,$$

$$\left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^{n+m}f\,(t,s)}{\partial t^n\partial s^m} \right|^p dsdt \right)^{\frac{1}{p}} < \infty,$$

and

$$\left\| \frac{\partial^{n+m}f}{\partial t^n\partial s^m} \right\|_1 := \int_a^b \int_c^d \left| \frac{\partial^{n+m}f\,(t,s)}{\partial t^n\partial s^m} \right| dsdt < \infty,$$

$$
\begin{aligned}
\nu(h) &: = \max\{h_i | i=0,\ldots,k-1\},\\
\mu(\lambda) &: = \max\{\lambda_\rho | \rho=0,\ldots,l-1\}
\end{aligned}
$$

and

$$h_i := x_{i+1} - x_i, \quad \lambda_\rho := y_{\rho+1} - y_\rho.$$

Proof. From Theorem 4, we use (2.2) to obtain

$$\begin{aligned} |V| &= \left| \int_a^b \int_c^d K_{n,k}(x,t)\, S_{m,l}(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt \right| \\ &\leqslant \int_a^b \int_c^d |K_{n,k}(x,t)\, S_{m,l}(y,s)| \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| dsdt. \end{aligned} \tag{3.2}$$

Using the properties of the modulus and integral together with Hölder's inequality, we have

$$\begin{aligned} &\int_a^b \int_c^d |K_{n,k}(x,t)\, S_{m,l}(y,s)| \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| dsdt \\ &\leqslant \begin{cases} \displaystyle\int_a^b \int_c^d |K_{n,k}(x,t)\, S_{m,l}(y,s)|\, dsdt \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \\ \displaystyle\left(\int_a^b \int_c^d |K_{n,k}(x,t)\, S_{m,l}(y,s)|^q\, dsdt \right)^{\frac{1}{q}} \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \\ \displaystyle\sup_{(t,s)\in[a,b]\times[c,d]} |K_{n,k}(x,t)\, S_{m,l}(y,s)| \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1. \end{cases} \end{aligned} \tag{3.3}$$

Now

$$\int_a^b \int_c^d |K_{n,k}(x,t)\, S_{m,l}(y,s)|\, dsdt = \int_a^b |K_{n,k}(x,t)|\, dt \int_c^d |S_{m,l}(y,s)|\, ds, \tag{3.4}$$

$$\begin{aligned} \int_a^b |K_{n,k}(x,t)|\, dt &= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{|t-\alpha_{i+1}|^n}{n!} dt \\ &= \sum_{i=0}^{k-1} \left[\int_{x_i}^{\alpha_{i+1}} \frac{(\alpha_{i+1}-t)^n}{n!} dt + \int_{\alpha_{i+1}}^{x_{i+1}} \frac{(t-\alpha_{i+1})^n}{n!} dt \right] \\ &= \frac{1}{(n+1)!} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1}-x_i)^{n+1} - (x_{i+1}-\alpha_{i+1})^{n+1} \right\}, \end{aligned}$$

and similarly

$$\int_c^d |S_{m,l}(y,s)|\, ds = \frac{1}{(m+1)!} \sum_{\rho=0}^{l-1} \left\{ \left(\beta_{\rho+1}-y_\rho\right)^{m+1} - \left(y_{\rho+1}-\beta_{\rho+1}\right)^{m+1} \right\}.$$

From (3.4),

$$\begin{aligned}
&\int_a^b \int_c^d \left|K_{n,k}(x,t)\, S_{m,l}(y,s)\right| dsdt \\
= &\ \frac{1}{(n+1)!\,(m+1)!} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1}-x_i)^{n+1} - (x_{i+1}-\alpha_{i+1})^{n+1} \right\} \\
&\times \sum_{\rho=0}^{l-1} \left\{ \left(\beta_{\rho+1}-y_\rho\right)^{m+1} - \left(y_{\rho+1}-\beta_{\rho+1}\right)^{m+1} \right\}
\end{aligned}$$

and from (3.3) and (3.2) we obtain the first part of the inequality in (3.1).

From the second part of (3.3),

$$\begin{aligned}
&\left(\int_a^b \int_c^d \left|K_{n,k}(x,t)\, S_{m,l}(y,s)\right|^q dsdt \right)^{\frac{1}{q}} \\
= &\ \left(\int_a^b \left|K_{n,k}(x,t)\right|^q dt \right)^{\frac{1}{q}} \left(\int_c^d \left|S_{m,l}(y,s)\right|^q ds \right)^{\frac{1}{q}} \\
= &\ \frac{1}{n!m!} \left(\frac{\sum_{i=0}^{k-1} \left\{ (\alpha_{i+1}-x_i)^{nq+1} - (x_{i+1}-\alpha_{i+1})^{nq+1} \right\}}{nq+1} \right)^{\frac{1}{q}} \\
&\times \left(\frac{\sum_{\rho=0}^{l-1} \left\{ \left(\beta_{\rho+1}-y_\rho\right)^{mq+1} - \left(y_{\rho+1}-\beta_{\rho+1}\right)^{mq+1} \right\}}{mq+1} \right)^{\frac{1}{q}}
\end{aligned}$$

and using the second part of (3.3), we obtain the second part of the inequality in (3.1).

Finally, from (3.3),

$$\sup_{(t,s)\in[a,b]\times[c,d]} \left|K_{n,k}(x,t)\, S_{m,l}(y,s)\right| = \sup_{t\in[a,b]} \left|K_{n,k}(x,t)\right| \sup_{s\in[c,d]} \left|S_{m,l}(y,s)\right|, \tag{3.5}$$

$$\begin{aligned}
\sup_{t\in[a,b]} \left|K_{n,k}(x,t)\right| &\leqslant \sup_{t\in[x_i,x_{i+1}]} \left| \frac{(t-\alpha_{i+1})^n}{n!} \right| \\
&= \frac{1}{n!} \max_{i=0,\ldots,k-1} \left\{ \left|(\alpha_{i+1}-x_i)\right|^n, \left|(x_{i+1}-\alpha_{i+1})\right|^n \right\} \\
&\leqslant \frac{1}{n!} \left[\max_{i=0,\ldots,k-1} \left\{ (\alpha_{i+1}-x_i), (x_{i+1}-\alpha_{i+1}) \right\} \right]^n
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{x_{i+1}-x_i}{2}+\left|\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\right|\right\}\right]^n \\
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}+\max_{i=0,\dots,k-1}|\delta_i|\right]^n \\
&\leqslant \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}+\max_{i=0,\dots,k-1}\left\{\frac{h_i}{2}\right\}\right]^n \\
(\text{since } \delta_i &:= \alpha_{i+1}-\frac{x_{i+1}+x_i}{2} \text{ and therefore } |\delta_i|\leqslant\frac{h_i}{2}) \\
&= \frac{1}{n!}\left[\max_{i=0,\dots,k-1}\{h_i\}\right]^n=\frac{\nu^n(h)}{n!},
\end{aligned}
$$

and similarly,

$$
\sup_{s\in[c,d]}|S_{m,l}(y,s)|\leqslant\frac{\mu^m(\lambda)}{m!}.
$$

From (3.5)

$$
\sup_{(t,s)\in[a,b]\times[c,d]}|K_{n,k}(x,t)\,S_{m,l}(y,s)|\leqslant\frac{\nu^n(h)\,\mu^m(\lambda)}{n!m!}
$$

and from (3.3) we obtain the third line of the inequality (3.1), hence the proof is complete. □

When the points of the division I_k and J_l are fixed, we obtain the following inequality.

COROLLARY 3. *Let f, I_k and J_l be defined as in Corollary 1, then*

$$
\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(t,s)\,dsdt-\sum_{j=1}^n\frac{1}{2^j j!}\sum_{i=0}^k\left\{-h_i^j+(-1)^j h_{i-1}^j\right\} \qquad (3.6)\\
&\times\sum_{r=1}^m\frac{(-1)^r}{2^r r!}\sum_{\rho=0}^l\left\{-\lambda_\rho^r+(-1)^r\lambda_{\rho-1}^r\right\}\frac{\partial^{r+j-2}f(x_i,y_\rho)}{\partial x_i^{j-1}\partial y_\rho^{r-1}} \\
&+(-1)^m\sum_{j=1}^n\frac{1}{2^j j!}\sum_{i=0}^k\left\{-h_i^j+(-1)^j h_{i-1}^j\right\}\int_c^d S_{m,l}(y,s)\,\frac{\partial^{j+m-1}f(x_i,s)}{\partial x_i^{j-1}\partial s^m}ds \\
&+(-1)^n\sum_{r=1}^m\frac{(-1)^r}{2^r r!}\sum_{\rho=0}^l\left\{-\lambda_\rho^r+(-1)^r\lambda_{\rho-1}^r\right\}\int_a^b K_{n,k}(x,t)\,\frac{\partial^{n+r-1}f(t,y_\rho)}{\partial t^n\partial y_\rho^{r-1}}dt\Bigg|
\end{aligned}
$$

$$\leqslant \begin{cases} \frac{1}{(n+1)!\,(m+1)!2^n2^m} \sum_{i=0}^{k-1} h_i^{n+1} \sum_{\rho=0}^{l-1} \lambda_\rho^{m+1} \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \\ \qquad\qquad\qquad\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty\left([a,b]\times[c,d]\right); \\ \frac{1}{n!m!2^n2^m\,(nq+1)^{\frac{1}{q}}\,(mq+1)^{\frac{1}{q}}} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \cdot \left(\sum_{i=0}^{k-1} h_i^{nq+1} \right)^{\frac{1}{q}} \left(\sum_{\rho=0}^{l-1} \lambda_\rho^{mq+1} \right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p\left([a,b]\times[c,d]\right),\ p>1,\ p^{-1}+q^{-1}=1; \\ \frac{1}{n!m!} \nu^n(h)\,\mu^m(\lambda) \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1, \qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1\left([a,b]\times[c,d]\right). \end{cases}$$

Proof. The proof follows directly, using the same substitutions as those used in Corollary 1. □

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