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EXTENSIONS OF FATOU'S INEQUALITY

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Abstract. Using some compacity techniques in the space of integrable functions we obtain an expression of the gap in the Fatou's inequality. Also, we derived as corollaries some results of H.-A. Klei.

1. Introduction

For a sequence of integrable functions there are two impediments which, when its occur, lead to a bad comportation of the sequence with respect to the strong convergence:

- the concentration of mass disturb the weak convergence,
- the asymptotic oscillatory behaviour troubles the convergence in measure.

The following example (inspired by [8]) seems to be typical:

Let $u_n : [-2\pi, 2\pi] \to \mathbf{R}$, $u_n(x) = n \cdot \chi_{[-\frac{1}{n}, 0]}(x) + \sin(nx) \cdot \chi_{(0, 2\pi]}(x)$, $\forall x \in [-2\pi, 2\pi], \forall n \in \mathbf{N}$.

The sequence concentrates the mass in $\{0\}$ and has an asymptotic oscillatory behaviour on $(0, 2\pi]$.

There are two main instruments for control of these two deviations: the modulus of uniform integrability gives a measure of the concentration of mass meanwhile the asymptotic oscillatory behaviour is controlled by a Young measure.

M. Saadoune et M. Valadier ([8]) use two compacity results ("biting lemma" and Prohorov's compacity theorem for Young measures) for obtain a structural result which is the most complet result about the comportation of a bounded sequence of integrable functions.

We present the result of Saadoune and Valadier and we obtain some importants consequences. Finally, we obtain as corollaries some convergence results of H.-A. Klei.

2. Biting lemma

Let $(\Omega, \mathscr{A}, \mu)$ be a space with a positive bounded measure μ on the σ–algebra $\mathscr{A}$ and let $L^1(A)$ be the space of all real–valued integrable functions on the set $A \in \mathscr{A}$.

For every sequence $(u_n)_{n \in \mathbf{N}} \subseteq L^1(\Omega)$,

$$\eta((u_n)) = \lim_{\varepsilon \to 0} \sup_{\mu(E) < \varepsilon} \sup_{n \in \mathbf{N}} \int_E |u_n| d\mu$$

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is the *modulus of uniform integrability* of (u_n) (H. P. Rosenthal, [7]).

Obviously, (u_n) is uniform integrable if and only if $\eta((u_n)) = 0$.

DEFINITION 2.1. ([1]). A sequence $(u_n) \subseteq L^1(\Omega)$ is *w^2–convergent to $u \in L^1(\Omega)$* if there exists a sequence of "bits" $(B_p) \subseteq \mathscr{A}$, $B_p \supseteq B_{p+1}, \mu(B_p) \downarrow 0$ such that, $\forall p \in \mathbf{N}$, $\left(u_n|_{\Omega\setminus B_p}\right)_{n\in\mathbf{N}}$ is weakly convergent to $u|_{\Omega\setminus B_p}$ in $L^1(\Omega \setminus B_p)$.

We denote in this case $u_n \xrightarrow{w^2} u$ and we can proof that

$$\eta((u_n)) = \lim_p \overline{\lim_n} \int_{B_p} |u_n| d\mu$$

([2, Proposition 5] and [3, Corollary of Proposition 4]).

So, if $u_n \xrightarrow{w^2} u$, (u_n) concentrates the mass on the sets $B_p, \forall p \in \mathbf{N}$.

A very useful result concerning w^2–convergence is the Brooks–Chacon's lemma or biting lemma ([1]).

We give an improvement of this lemma as it appears in [3, Theorem 6].

THEOREM 2.2. *For every bounded sequence (u_n) in $L^1(\Omega)$ there exists a subsequence (u_n^1) w^2–convergent such that $\eta((u_n)) = \eta((u_n^2))$, for every subsequence (u_n^2) of (u_n^1).*

Particularly this means that, for (u_n^1), the concentration of mass is maximale among all subsequences of (u_n).

3. Young measures

DEFINITION 3.1. ([9]). Let $\mathscr{B}$ be the σ–algebra of all Borel sets of $\mathbf{R}$; a *Young measure* is a positive measure $\tau : \mathscr{A} \otimes \mathscr{B} \to \mathbf{R}_+$ such that $\tau(A \times \mathbf{R}) = \mu(A), \forall A \in \mathscr{A}$.

Let $\mathscr{Y}$ be the space of all Young measures on $\Omega \times \mathbf{R}$.

For every measure $\tau \in \mathscr{Y}$ there exists a family $(\tau_x)_{x\in\Omega}$ of probabilities on $\mathbf{R}$ such that, $\forall \Psi \in L^1(\Omega \times \mathbf{R}, \mathscr{A} \otimes \mathscr{B}, \tau)$,

$$\int_{\Omega\times\mathbf{R}} \Psi(x,y) d\tau(x,y) = \int_{\Omega} \left[\int_{\mathbf{R}} \Psi(x,y) d\tau_x(y) \right] d\mu(x).$$

$(\tau_x)_{x\in\Omega}$ is *the disintegration* of τ ([9]).

In an equivalent form, a Young measure is a $(\mathscr{A} - \mathscr{C})$–measurable mapping $\tau : \Omega \to \mathscr{P}, \tau(x) = \tau_x, \forall x \in \Omega$, where $\mathscr{P}$ is the family of all probabilities on $\mathbf{R}$ and $\mathscr{C}$ is the family of all Borel sets in the narrow topology on $\mathscr{P}$ ([4, Théorème 2.2]).

For each $u \in L^1(\Omega)$, the Young measure associated to u is $\tau^u : \Omega \to \mathscr{P}$, where $\tau^u(x) = \tau^u_x = \delta_{u(x)}$ (the Dirac mass concentrated in $u(x)$). So, the mapping $u \mapsto \tau^u$ is an embeding of $L^1(\Omega)$ in $\mathscr{Y}$ ($L^1(\Omega) \hookrightarrow \mathscr{Y}$).

The narrow topology on $\mathscr{Y}$, $\mathscr{T}$, is the weakest topology on $\mathscr{Y}$ making continuous the mappings

$$\tau \mapsto \int_{\Omega\times\mathbf{R}} \chi_A(x)\cdot f(y)d\tau(x,y),$$

$\forall A \in \mathscr{A}$ and $f \in C_0(\mathbf{R})$ (the space of all real continuous mappings f with $\lim_{|x|\to\infty} f(x) = 0$) (see [9, Theorem 3]).

If $(u_n) \subseteq L^1(\Omega) \hookrightarrow \mathscr{Y}$ and $\tau = (\tau_x)_{x\in\Omega} \in \mathscr{Y}$ then $u_n \xrightarrow{\mathscr{T}} \tau$ iff, $\forall f \in C_0(\mathbf{R}), \forall A \in \mathscr{A}$,

$$\int_{\Omega\times\mathbf{R}} \chi_A(x) f(y)d\tau^{u_n}(x,y) = \int_A f(u_n(x))d\mu(x) \longrightarrow \int_A \left(\int_{\mathbf{R}} f(y)d\tau_x(y)\right) d\mu(x)$$

If we denote $u_f(x) = \int_{\mathbf{R}} f(y)d\tau_x(y)$ then $u_n \xrightarrow{\mathscr{T}} \tau$ iff $(f(u_n))_{n\in\mathbf{N}}$ is weakly convergent to u_f in $L^1(\Omega), \forall f \in C_0(\mathbf{R})$ (see [4, Théorème 5.4]).

PROPOSITION 3.2. *If $u_n \xrightarrow{\mathscr{T}} \tau$ then*

$$\tau(G'_u) = \sup_{a>0} \overline{\lim_n}\, \mu(|u_n - u| \geqslant a),$$

$\forall u : \Omega \to \mathbf{R}$ measurable, where $G_u = \{(x, u(x)) : x \in \Omega\}$ is the graph of u and $G'_u = (\Omega \times \mathbf{R}) \setminus G_u$.

Proof. For every measurable mapping $u : \Omega \to \mathbf{R}$ and for every $a > 0, \Psi_i : \Omega \times \mathbf{R} \to \mathbf{R}_+, \Psi_1 = \chi_{(|y - u(x)| < a)}, \Psi_2 = \chi_{(|y - u(x)| > a)}$ are measurables in (x, y) and l.s.c. in y. Then

$$\int_{\Omega\times\mathbf{R}} \Psi_i d\tau \leqslant \underline{\lim_n} \int_\Omega \Psi_i(x, u_n(x))d\mu(x)$$

(see [9, Lemma 5]).

It follows that, $\forall a > 0$

$$\tau(|y - u(x)| < a) \leqslant \underline{\lim_n}\, \mu(|u_n - u| < a), \text{ and} \tag{1}$$

$$\tau(|y - u(x)| > a) \leqslant \underline{\lim_n}\, \mu(|u_n - u| > a) \leqslant \overline{\lim_n}\, \mu(|u_n - u| > a). \tag{2}$$

It follows that

$$\tau(G_u) = \inf_{a>0} \tau(|y - u(x)| < a) \leqslant \inf_{a>0} \underline{\lim_n}\, \mu(|u_n - u| < a) \tag{3}$$

and

$$\begin{aligned}\tau(G'_u) = \sup_{a>0} \tau(|y - u(x)| > a) &\leqslant \sup_{a>0} \overline{\lim_n}\, \mu(|u_n - u| > a) \\ &= \sup_{a>0} \overline{\lim_n}\, \mu(|u_n - u| \geqslant a).\end{aligned} \tag{4}$$

From (3) we obtain

$$\tau(G'_u) \geqslant \sup_{a>0} \overline{\lim_n}\, \mu(|u_n - u| \geqslant a) \tag{5}$$

and from (4) and (5) we have $\tau(G'_u) = \sup_{a>0} \overline{\lim}_n \mu(|u_n - u| \geqslant a)$. $\square$

REMARK 3.3. $u_n \xrightarrow{\mu} u$ iff $\tau(G'_u) = 0$ iff $\int_{\Omega\times\mathbf{R}} |y - u(x)| d\tau(x,y) = 0 \Leftrightarrow \int_\Omega \left(\int_{\mathbf{R}} |y - u(x)| d\tau_x(y)\right) d\mu(x) = 0 \Leftrightarrow \int_{\mathbf{R}} |y - u(x)| d\tau_x(y) = 0$ μ–a.e. $\Leftrightarrow \tau_x(y \neq u(x)) = 0 \Leftrightarrow \tau_x = \delta_{u(x)}$, μ–a.e.

Particularly, if $\tau \in L^1(\Omega)(\exists u \in L^1(\Omega)$ such that $\tau = \tau^u)$ then $u_n \xrightarrow{\mathscr{T}} \tau^u$ iff $u_n \xrightarrow{\mu} u$ ((u_n) is convergent in measure to u).

So $\mathscr{T}|_{L^1(\Omega)}$ is the topology of convergence in measure on $L^1(\Omega)$.

PROPOSITION 3.4. *Let $(u_n) \subseteq L^1(\Omega)$ be a bounded sequence such that $u_n \xrightarrow{\mathscr{T}} \tau \in \mathscr{Y}$. Then τ has a barycenter $u \in L^1(\Omega)$.*

Proof. The mapping $\Psi : \Omega \times \mathbf{R} \to \mathbf{R}_+, \Psi(x,y) = |y|, \forall y \in \mathbf{R}$ is $(\mathscr{A} \otimes \mathscr{B})$–measurable and lower semi–continuous in y. Then

$$\int_{\Omega\times\mathbf{R}} \Psi(x,y) d\tau(x,y) \leqslant \underline{\lim}_n \int_{\Omega\times\mathbf{R}} \Psi(x,y) d\tau^{u_n}(x,y) \tag{$*$}$$

(see [9, Lemma 5]).

But $\int_{\Omega\times\mathbf{R}} |y| d\tau^{u_n}(x,y) = \int_\Omega \left(\int_{\mathbf{R}} |y| d\delta_{u_n(x)}\right) d\mu(x) = \int_\Omega |u_n(x)| d\mu(x)$
$\leqslant \sup_n \|u_n\|_1 < +\infty$.

Therefore, from $(*)$,

$$\int_\Omega \left(\int_{\mathbf{R}} |y| d\tau_x(y)\right) d\mu(x) < +\infty. \tag{$**$}$$

so that τ has a barycenter $u : \Omega \to \mathbf{R}, u(x) = \int_{\mathbf{R}} y d\tau_x(y)$ and, from $(**)$,

$$\int_\Omega |u| d\mu \leqslant \int_\Omega \left(\int_{\mathbf{R}} |y| d\tau_x(y)\right) d\mu(x) < +\infty$$

and therefore $u \in L^1(\Omega)$. $\square$

If $u_n \xrightarrow{\mathscr{T}} \tau \in \mathscr{Y}$ then τ contains some informations about the asymptotic oscillatory behaviour of (u_n).

The following compacity result is a variant with parameter of Prohorov's theorem.

THEOREM 3.5. ([9, Theorem 7]). *For every norm–bounded sequence $(u_n) \subseteq L^1(\Omega) \hookrightarrow \mathscr{Y}$ there exists a subsequence (u_n^1) $\mathscr{T}$–convergent to a Young measure $\tau \in \mathscr{Y}$.*

4. Convergence results

Combining the biting lemma and Prohorov's theorem, M. Saadoune and M. Valadier proved the following general result:

THEOREM 4.1. ([8, Theorem 4.5]). *Let (u_n) be a bounded sequence in $L^1(\Omega)$. There exist a subsequence (u_n^1) and a Young measure $\tau \in \mathscr{Y}$ such that:*

(1) $u_n^1 \xrightarrow{\mathscr{T}} \tau$ *and* $\tau_x\left(\mathrm{Ls}(u_n(x))\right) = 1$ *a.e.*
$\left(\mathrm{Ls}(u_n(x)) = \bigcap_{n\in\mathbf{N}} \overline{\{u_m(x) : m \geqslant n\}}\right)$.

(2) *The mapping* $u : \Omega \to \mathbf{R}, u(x) = bar(\tau_x) = \int_{\mathbf{R}} y d\tau_x(y), \forall x \in \Omega$, *is integrable on* Ω.

(3) $u_n^1 \xrightarrow{w^2} u$ *and* $\eta((u_n^2)) = \eta((u_n)), \forall (u_n^2)$ *a subsequence of* (u_n^1).

(4) (*a*) $u_n^1|_M \xrightarrow{\mu} u|_M$ *where* $M = \{x \in \Omega : \int_{\mathbf{R}} |y - u(x)| d\tau_x(y) = 0\}$;

(*b*) $u_n^1|_A \overset{\mu}{\not\to} u|_A, \forall A \in \mathscr{A}$, *with* $A \subseteq \Omega \setminus M$ *and* $\mu(A) > 0$.

(5) $\lim_n \|u_n^1 - v\|_1 = \eta((u_n)) + \int_{\Omega\times\mathbf{R}} |y - v(x)| d\tau(x,y), \forall v \in L^1(\Omega)$.

REMARK 4.2. Even if the original sequence (u_n) is $\mathscr{T}$–convergent the result does not hold without extraction. Indeed, if $\Omega = [0,1]$, μ is the Lebesgue's mesure on Ω and $u_n : \Omega \to \mathbf{R}$ is defined by $u_{2n} = 2n\chi_{[0,\frac{1}{2n}]}$ and $u_{2n+1} = 0, \forall n \in \mathbf{N}$, then $u_n \xrightarrow{\mu} 0 = u$ so that $u_n \xrightarrow{\mathscr{T}} \mu \otimes \delta_0 \in \mathscr{Y}$.

$M = \Omega$, $u_n \xrightarrow{w^2} 0$ but $\eta((u_n)) = 1 \neq 0 = \eta((u_{2n+1}))$ so that (3) is false for (u_n).

(5) is also false because the sequence $(\|u_n - u\|_1)_{n\in\mathbf{N}} = (\|u_n\|_1)_{n\in\mathbf{N}}$ has not a limit.

PROPOSITION 4.3. *Let* $(u_n) \subseteq L^1(\Omega)$ *be a bounded sequence such that* $u_n \xrightarrow{\mathscr{T}} \tau \in \mathscr{Y}$ *and* $\eta((u_n)) = \eta((u_n^1))$, *for each subsequence* (u_n^1) *of* (u_n).

Then (u_n) *satsfies the condition* (1), (2), (4) *and* (5) *of the theorem* 4.1 without extraction a subsequence.

Proof. If $u_n \xrightarrow{\mathscr{T}} \tau$ then τ_x is carried by $\mathrm{Ls}(u_n(x))$, μ–almost everywhere (see the proof of [9, Theorem 10]). From the proposition 3.4, τ has a barycenter $u \in L^1(\Omega)$.

Obviously, the condition (4) is fulfiled.

Let $v \in L^1(\Omega)$; for every subsequence $(\|u_n^1 - v\|_1)$ of $(\|u_n - v\|_1)$, from the theorem 4.1, there exists a subsequence (u_n^2) of (u_n^1) such that $u_n^2 \xrightarrow{\mathscr{T}} \tau$ and $\lim_n \|u_n^2 - v\|_1 = \eta((u_n^1)) + \int_{\Omega\times\mathbf{R}} |y - v(x)| d\tau(x,y) = \eta((u_n)) + \int_{\Omega\times\mathbf{R}} |y - v(x)| d\tau(x,y) = L$.

Then, every subsequence $(\|u_n^1 - v\|_1)$ has a subsequence $(\|u_n^2 - v\|_1)$ convergent to L so that $\|u_n - v\|_1 \to L$. □

REMARK 4.4. Generally, the condition (3) is not fulfiled in the conditions of previous proposition.

Indeed, $\forall n \in \mathbf{N}^*, k = 0, \ldots, n-1$ let $f_n^k = n \cdot \chi_{[\frac{k}{n}, \frac{k+1}{n})} \in L^1([0,1])$ and let $(\varphi_m)_{m\in\mathbf{N}}$ be the sequence: $f_1^0, f_2^0, f_2^1, \ldots, f_n^0, f_n^1, \ldots, f_n^{n-1}, \ldots$ Then $\varphi \xrightarrow{\mu} \underline{0}$ hence

$\varphi_m \xrightarrow{\mathscr{T}} \mu \otimes \delta_0 \in \mathscr{Y}$ (μ is the Lebesgue's measure on $[0,1]$) and $\eta((\varphi_m)) = \eta((\varphi_m^1)) = 1$ for each subsequence (φ_m^1) of (φ_m), but (φ_m) is not w^2–convergent in $L^1([0,1])$ (if I suppose that $\varphi_m \xrightarrow{w^2} f$ then, from [3, Proposition 3], $f = 0$, a.e. so that there exists (B_p) with $\mu(B_p) \downarrow 0$ such that $\int_{\Omega\setminus B_p} \varphi_m d\mu \to 0$; but, if $\mu(B_{p_0}) < \frac{1}{2}$, then, $\forall n \in \mathbf{N}^*, \exists k_n \in \{0,\ldots,n-1\}$ such that $\mu\left([\frac{k_n}{n}, \frac{k_n+1}{n}) \setminus B_{p_0}\right) > \frac{1}{2n}$; so $\int_{\Omega\setminus B_{p_0}} f_n^{k_n} d\mu > \frac{1}{2}$).

REMARK 4.5. From the condition (5) of the theorem 4.1, if $v = u$ then we obtain:

$$\lim_n \|u_n^1 - u\|_1 = \eta((u_n)) + \int_{\Omega\times\mathbf{R}} |y - u(x)| d\tau. \qquad (*)$$

We remark that $\int_{\Omega\times\mathbf{R}} |y-u(x)|d\tau = \int_{G'_u} |y-u(x)|d\tau = 0$ iff $\tau(G'_u) = 0$ and, from 3.3, iff $u_n \xrightarrow{\mu} u$.

Therefore the relation $(*)$ gives a decomposition of the deficiency of strong convergence in a "weak part" and a "measure part".

Of course, $(*)$ is a generalization of Lebesgue–Vitali theorem because $\|u_n - u\|_1 \to 0$ iff $\eta((u_n)) = 0 = \int_{\Omega\times\mathbf{R}} |y-u(x)|d\tau = 0$ hence iff (u_n) is uniform integrable and convergent in measure to u.

EXEMPLE 4.6. We take again the sequence $u_n : [-2\pi, 2\pi] \to \mathbf{R}$

$$u_n(x) = n\chi_{[-\frac{1}{n},0]}(x) + \sin nx \chi_{(0,2\pi]}(x).$$

Then $u_n \xrightarrow{\mathscr{T}} \tau$ where the disintegration of τ, $(\tau_x)_{x\in[-2\pi,2\pi]}$ is given by

$$\tau_x(B) = \begin{cases} \delta_0(B), & x \in [-2\pi, 0], \\ \frac{1}{\pi}\int_{B\cap[-1,1]} \frac{1}{\sqrt{1-t^2}} d\mu(t), & x \in (0, 2\pi]. \end{cases}$$

$u(x) = \mathrm{bar}(\tau_x) = 0, u_n \xrightarrow{w^2} \underline{0}$ $(B_p = [-\frac{1}{p}, 0], \forall p \in \mathbf{N})$.

$$\mathrm{Ls}(u_n(x)) = \begin{cases} \{0\}, & x \in [-2\pi, 0), \\ \emptyset, & x = 0, \\ [-1,1], & x \in (0, 2\pi]. \end{cases}$$

$M = [-2\pi, 0]$ and $\int_A |\sin(nx)|d\mu(x) = \frac{2}{\pi}\mu(A) \not\to 0, \forall A \in \mathscr{A}, A \subseteq (0, 2\pi]$.

$\eta((u_n)) = 1 = \eta((u_n^1))$, for every subsequence (u_n^1) of (u_n).

From the proposition 4.3, we can write (5) without extraction a subsequence:

$$\lim_n \|u_n - v\|_1 = 1 + \int_{\Omega\times\mathbf{R}} |y - u(x)| d\tau(x,y).$$

If $v = u = 0$ we obtain $\lim_n \|u_n\|_1 = 5$.

We remark again the concentration of mass on the sets $[-\frac{1}{p}, 0]$ and an asymptotic oscilatory behaviour on $(0, 2\pi]$ (where τ_x is not a Dirac measure).

5. Improvements of Fatou's lemma

We try to obtain some localization of w^2–limit u in the theorem 4.1.

THEOREM 5.1. *Let* (u_n) *be a bounded sequence in* $L^1(\Omega)$*; then there exist a subsequence* (u_n^1) *and a Young measure* $\tau = (\tau_x)_{x\in\Omega} \in \mathscr{Y}$ *such that* $u_n^1 \xrightarrow{\mathscr{T}} \tau$ *and*

$$\underline{\lim}\, u_n \leqslant u \leqslant \overline{\lim}\, u_n \text{ where } u(x) = \mathrm{bar}(\tau_x), \forall x \in \Omega.$$

If $u_n \geqslant 0, \forall n \in \mathbf{N}$ *then*

$$\lim \int_\Omega u_n^1 d\mu = \eta((u_n)) + \int_\Omega u d\mu, \quad \textit{hence}$$

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u_n)) \leqslant \lim \int_\Omega u_n^1 d\mu \leqslant \int_\Omega \overline{\lim}\, u_n d\mu + \eta((u_n)).$$

In addition, if we suppose that there exists $\lim_n \|u_n\|_1 \in \mathbf{R}_+$ *then*

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u_n)) \leqslant \lim \int_\Omega u_n d\mu \leqslant \int_\Omega \overline{\lim}\, u_n d\mu + \eta((u_n)).$$

Proof. From the theorem 4.1 there exist a subsequence (u_n^1) and a Young measure τ which accomplish the conditions (1)–(5).

Because $\mathrm{Ls}(u_n^1(x)) \subseteq \mathrm{Ls}(u_n(x))$, from (1) we obtain $\tau_x(\mathrm{Ls}(u_n(x)) = 1$ a.e.

$\forall x \in \Omega, \mathrm{Ls}(u_n(x)) \subseteq \left[\underline{\lim}\, u_n(x), \overline{\lim}\, u_n(x)\right]$ hence

$$u(x) = \int_{\mathbf{R}} y d\tau_x(y) = \int_{\mathrm{Ls}(u_n(x))} y d\tau_x(y) = \int_{[\underline{\lim}\, u_n(x), \overline{\lim}\, u_n(x)]} y d\tau_x(y).$$

Therefore

$$\underline{\lim}\, u_n(x) \leqslant u(x) \leqslant \overline{\lim}\, u_n(x) \text{ a.e.} \tag{$*$}$$

If $u_n \geqslant 0, \forall n \in \mathbf{N}$ then $\mathrm{Ls}(u_n^1(x)) \subseteq [0, +\infty)$, hence $\tau_x(-\infty, 0] = 0$, a.e. Therefore, from (5) of 4.1,

$$\begin{aligned}
\lim \int_\Omega u_n^1 d\mu &= \lim \|u_n^1\|_1 = \eta((u_n)) + \int_{\Omega\times\mathbf{R}} |y| d\tau(x,y) \\
&= \eta((u_n)) + \int_\Omega \left(\int_{\mathbf{R}} |y| d\tau_x(y)\right) d\mu(x) \\
&= \eta((u_n)) + \int_\Omega \left(\int_{[0,+\infty)} y d\tau_x(y)\right) d\mu(x) \\
&= \eta((u_n)) + \int_\Omega \left(\int_{\mathbf{R}} y d\tau_x(y)\right) d\mu(x) = \eta((u_n)) + \int_\Omega u d\mu
\end{aligned}$$

and from $(*)$

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u_n)) \leqslant \lim \int_\Omega u_n^1 d\mu \leqslant \int_\Omega \overline{\lim}\, u_n d\mu + \eta((u_n)).$$

In addition, if $\exists \lim \|u_n\|_1 \in \mathbf{R}$ then $\lim \int_\Omega u_n^1 d\mu = \lim \int_\Omega u_n d\mu$. $\square$

COROLLARY 5.2. *For every bounded sequence* $(u_n) \subseteq L^1_+(\Omega)$ *there exists a subsequence* (u^1_n) *such that*

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u^1_n)) \leqslant \underline{\lim} \int_\Omega u_n d\mu.$$

Proof. Let (u^1_n) be a subsequence of (u_n) such that

$$\underline{\lim} \int_\Omega u_n d\mu = \lim \int_\Omega u^1_n d\mu.$$

From the previous theorem, there exists a subsequence (u^2_n) of (u^1_n) such that

$$\lim \int_\Omega u^2_n d\mu = \eta((u^1_n)) + \int_\Omega u d\mu \text{ and } \underline{\lim}\, u^1_n \leqslant u \leqslant \overline{\lim}\, u^1_n.$$

Then

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u^1_n)) \leqslant \int_\Omega \underline{\lim}\, u^1_n d\mu + \eta((u^1_n)) \leqslant \int_\Omega u d\mu + \eta((u^1_n))$$

$$= \lim \int_\Omega u^2_n d\mu = \underline{\lim} \int_\Omega u_n d\mu. \quad \square$$

REMARKS 5.3. (i) Without extraction of a subsequence the result does not holds. Indeed, for the sequence in the remark 4.2,

$$\int_\Omega \underline{\lim}\, u_n d\mu + \eta((u_n)) = 1 > \underline{\lim} \int_\Omega u_n d\mu.$$

(ii) If $\theta((u_n)) = \inf\left\{\eta((u^1_n)) : (u^1_n) \text{ subsequence of } (u_n)\right\}$ then, for every bounded sequence $(u_n) \subseteq L^1_+(\Omega)$,

$$\int_\Omega \underline{\lim}\, u_n d\mu + \theta((u_n)) \leqslant \underline{\lim} \int_\Omega u_n d\mu.$$

Now we derive some convergence results of H.-A. Klei as corollaries.

COROLLARY 5.4. ([5, Theorem 3]. *Let* (u_n) *be a bounded sequence in* $L^1_+(\Omega)$ *such that* $\left(\int_\Omega u_n d\mu\right)_{n\in\mathbf{N}}$ *converges in* $\mathbf{R}_+$.

Then the following assertions are equivalents:

(i) $\lim \int_\Omega u_n d\mu = \eta((u_n)) + \int_\Omega \underline{\lim}\, u_n d\mu$ *and* $\eta((u_n)) = \eta((u^1_n))$ *for each subsequence* (u^1_n) *of* (u_n).

(ii) $u_n \xrightarrow{\mu} \underline{\lim}\, u_n$.

Proof. $(i) \Longrightarrow (ii)$. For each subsequence (u^1_n) of (u_n) there exist, from the theorem 5.1, a subsequence (u^2_n) and a Young measure $\tau = (\tau_x)_{x\in\Omega}$ such that $u^2_n \xrightarrow{\mathscr{T}} \tau$ and

$$\lim \int_\Omega u^2_n d\mu = \eta((u^1_n)) + \int_\Omega u d\mu \text{ and } \underline{\lim}\, u^1_n \leqslant u \leqslant \overline{\lim}\, u^1_n,$$

where $u(x) = \text{bar}(\tau_x)$.

From (i), $\int_\Omega u d\mu = \int_\Omega \underline{\lim}\, u_n d\mu$ so that

$$u = \underline{\lim}\, u_n \ \text{ a.e.} \tag{a}$$

From the proposition 4.3, (u_n^2) satisfies the condition (5) of the theorem 4.1. Then

$$\lim \|u_n^2 - u\|_1 = \eta((u_n^2)) + \int_{\Omega\times\mathbf{R}} |y - u(x)| d\tau.$$

But, from (a)

$$\begin{aligned}\int_{\Omega\times\mathbf{R}} |y-u(x)|d\tau &= \int_\Omega \left(\int_{[\underline{\lim}\, u_n(x), \overline{\lim}\, u_n(x)]} |y - \underline{\lim}\, u_n(x)| d\tau_x(y) \right) d\mu(x) \\ &= \int_\Omega \left(\int_{[\underline{\lim}\, u_n(x), \overline{\lim}\, u_n(x)]} (y - \underline{\lim}\, u_n(x)) d\tau_x(y) \right) d\mu(x) \\ &= \int_\Omega \left(\int_{\mathbf{R}} (y - u(x)) d\tau_x(y) \right) d\mu(x) = 0 \quad \left(u(x) = \int_{\mathbf{R}} y d\tau_x(y) \right).\end{aligned}$$

Therefore, from 4.5, $u_n^2 \xrightarrow{\mu} u$. Hence each subsequence of (u_n) has a subsequence convergent in measure to u. So that $u_n \xrightarrow{\mu} u = \underline{\lim}\, u_n$.

$(ii) \Longrightarrow (i)$. Let (u_n) be convergent in measure to $u = \underline{\lim}\, u_n$. From the theorem 5.1 there exist a subsequence (u_n^1) and $\tau = (\delta_{u(x)})_{x\in\Omega} \in \mathscr{Y}$ such that

$$\lim \int_\Omega u_n d\mu = \lim \int_\Omega u_n^1 d\mu = \eta((u_n)) + \int_\Omega u d\mu \tag{b}$$

$$(\text{bar}(\tau_x) = \int_{\mathbf{R}} y d\tau_x(y) = \int_{\mathbf{R}} y d\delta_{u(x)}(y) = u(x)).$$

For every subsequence (u_n^2) of (u_n), using (b) we obtain

$$\lim \int_\Omega u_n^2 d\mu = \eta((u_n^2)) + \int_\Omega u d\mu.$$

Because $\lim \int_\Omega u_n d\mu = \lim \int_\Omega u_n^2 d\mu$, using (b) again,

$$\eta((u_n)) + \int_\Omega u d\mu = \eta((u_n^2)) + \int_\Omega u d\mu$$

so that $\eta((u_n)) = \eta((u_n^2))$. $\square$

COROLLARY 5.5. ([5, Theorem 5]). *Let (u_n) be a bounded sequence of $L_+^1(\mathbf{R})$. Then the following assertions are equivalents:*

(i) $\exists \lim_n \int_\Omega u_n d\mu = \int_\Omega \underline{\lim}_n\, u_n d\mu$.

(ii) $u_n \xrightarrow{\|\cdot\|_1} \underline{\lim}_n u_n$.

Proof. $(i) \Longrightarrow (ii)$. From the theorem 5.1 there exist a subsequence (u_n^1) of (u_n) and $\tau = (\tau_x)_{x\in\Omega} \in \mathscr{Y}$ such that

$$\lim \int_\Omega u_n d\mu = \lim \int_\Omega u_n^1 d\mu = \eta((u_n)) + \int_\Omega u d\mu,$$

where $u(x) = \mathrm{bar}(\tau_x) \geqslant \underline{\lim}\, u_n(x)$ a.e.

Hence, from (i),

$$\eta((u_n)) + \int_\Omega u d\mu = \int_\Omega \underline{\lim}\, u_n d\mu$$

so that $\eta((u_n)) = 0$ and $u = \underline{\lim}\, u_n$ a.e.

Now, from the corollary 5.4, $u_n \xrightarrow{\mu} \underline{\lim}\, u_n$ and, from Lebesgue–Vitali's theorem, $u_n \xrightarrow{\|\cdot\|_1} \underline{\lim}_n u_n$.

$(ii) \Longrightarrow (i)$ is obviously. □

COROLLARY 5.6. ([6, Proposition 3]). *Let (u_n) be a bounded sequence in $L^1_+(\Omega)$ converging in measure to u. Then*

$$\underline{\lim} \int_\Omega u_n d\mu = \theta((u_n)) + \int_\Omega u d\mu.$$

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