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A COMPUTATIONAL ROLE OF PARTIALLY RELAXED MONOTONE MAPPINGS IN APPROXIMATION SOLVABILITY OF NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. A computational role of a class of partially relaxed monotone mappings in the approximation-solvability of a class of nonlinear variational inequalities based on a variational inequality type algorithm is presented. We consider a class of nonlinear variational inequality (abbreviated as NVI) problems: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0 \text{ for all } x \in K,$$

where $T : K \to H$ is a γ-r-partially relaxed monotone mapping and K a closed convex subset of a real Hilbert space H.

1. Introduction

In recent years, there has been an enormous growth in applications of variational inequalities to problems arising from mathematical programming, optimization and control theory, mathematical finance, engineering sciences, and others. Of special interest is the explosion of new algorithms – a key ingredient to the approximation solvability of variational inequalities and computational mathematics in general. In the case of algorithms expressed as variational inequalities, Marcotte and Wu [9] applied such an algorithm to the approximation solvability of a class of variational inequalities involving cocoercive mappings [4,9] in $\mathbf{R}^n$. Inspired by the recent work [9], Verma [16] extended a class of the variational inequality type algorithms and applied them to the approximation solvability of a class of variational inequalities in a Hilbert space setting, which of course, have applications to the $\mathbf{R}^n$ space setting. Our plan in this paper is to present the approximation solvability of a class of variational inequalities involving the class of the partially relaxed monotone mappings along with our approach to some numerical applications in $\mathbf{R}^n$. To learn more details on the approximation solvability of variational inequalities and related recent algorithms, we recommend [1–19].

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $T : K \to H$ be any γ-r-partially relaxed monotone mapping and K a closed convex subset of H. We consider a class of nonlinear variational inequality (abbreviated as NVI) problems: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0 \text{ for all } x \in K, \tag{1.1}$$

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which is equivalent to a projection formula

$$x^* = P_K[x^* - \rho T(x^*)],$$

where P_K is the projection of H onto K, and $\rho > 0$ is a constant.

Now we need to recall the following auxiliary results, which are crucial to the development of the work on hand.

LEMMA 1.1. *An element $u \in K$ is a solution of the NVI problem (1.1) if and only if*

$$u = P_K[u - \rho T(u)] \text{ for } \rho > 0,$$

where $T : K \to H$ is a mapping on K.

LEMMA 1.2. *An element $u \in K$ is a solution of the NVI problem (1.1) if*

$$\langle T(u), x - u \rangle \geqslant 0 \text{ for all } x \in K.$$

A mapping $T : H \to H$ is said to be α-cocoercive [14] if for all x, $y \in H$, we have

$$||x - y||^2 \geqslant \alpha^2 ||T(x) - T(y)||^2 + ||\alpha(T(x) - T(y)) - (x - y)||^2,$$

where $\alpha > 0$ is a constant.

Alternatively, a mapping $T : H \to H$ is called α-cocoercive [4,9] if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geqslant \alpha ||T(x) - T(y)||^2 \text{ for all } x, y \in H.$$

T is called r-strongly monotone if for each x, $y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geqslant r||x - y||^2 \text{ for a constant } r > 0.$$

This implies that

$$||T(x) - T(y)|| \geqslant r||x - y||,$$

that is, T is r-expanding, and when $r = 1$, it is expanding. The mapping T is called β-Lipschitz continuous (or β-Lipschitzian) if there exists a constant $\beta \geqslant 0$ such that

$$||T(x) - T(y)|| \leqslant \beta ||x - y|| \text{ for all } x, y \in H.$$

We note that if T is α-cocoercive and expanding, then T is α-strongly monotone. On the other hand, if T is α-strongly monotone and β-Lipschitz continuous, then T is (α/β^2)-cocoercive for $\beta > 0$. Clearly every α-cocoercive mapping T is $(1/\alpha)$-Lipschitz continuous.

LEMMA 1.3. [9]. *For any two elements u, $v \in H$, we have*

$$||u||^2 + \langle u, v \rangle \geqslant -\frac{1}{4}||v||^2.$$

A mapping $T : H \to H$ is said to be γ-r-partially relaxed monotone if there exist constants γ, $r > 0$ such that

$$\langle T(x) - T(y), z - y \rangle \geqslant -\gamma ||z - x||^2 + r||x - y||^2 \text{ for all } x, y, z \in H.$$

A mapping $T : H \to H$ is said to be γ-partially relaxed monotone [14] *if there exists a constant $\gamma > 0$ such that*

$$\langle T(x) - T(y), z - y \rangle \geqslant -\gamma \|z - x\|^2 \text{ for all } x, y, z \in H.$$

EXAMPLE 1.4. [1,2]. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$T(x) = cI(x) + v,$$

where $c > 0$ is a constant, x, $v \in \mathbf{R}^n$ with v fixed, and I is the $n \times n$ identity matrix. Then T is a γ-partially relaxed monotone mapping for $c = \gamma$. On the top of that, T is c-Lipschitz continuous. Since

$$\|y - z\|^2 + \|y - x\|^2 + \|x - z\|^2 \geqslant 0 \text{ for all } x, y, z \in \mathbf{R}^n,$$

we have

$$\langle y - z, y - z \rangle + \langle y - x, y - x \rangle + \langle x - z, x - z \rangle \geqslant 0$$

or

$$-\langle x, y \rangle - \langle y, z \rangle + \langle y, y \rangle + \langle z, z \rangle - \langle z, x \rangle + \langle x, x \rangle \geqslant 0$$

or

$$\gamma[\langle x - y, z - y \rangle + \langle z - x, z - x \rangle] \geqslant 0$$

or

$$\langle \gamma x - \gamma y, z - y \rangle + c\|z - x\|^2 \geqslant 0 \text{ (since } \gamma = c\text{)}$$

or

$$\langle T(x) - T(y), z - y \rangle \geqslant -\gamma \|z - x\|^2,$$

that is, T is γ-partially relaxed monotone.

The partially relaxed monotone mappings [16] are weaker than the cocoercive [9] and strongly monotone mappings and on the top of that, are more computation-oriented. This class of relaxed monotone mappings satisfy the following implications:

γ-r-partial relaxed monotonicity
↓
γ-partial relaxed monotonicity

2. Algorithms and the NVI problem (1.1)

This section deals with the approximation-solvability of the NVI problem (1.1) based on an iterative algorithm [9], which is represented by a class of variational inequalities, while it does preserve an equivalence to a class of projection formulas.

ALGORITHM 2.1. [9]. For an arbitrarily chosen initial point $x^0 \in K$, we consider an iterative algorithm generated as follows (for $k \geqslant 0$):

$$\langle \rho T(x^0) + x^1 - x^0, x - x^1 \rangle \geqslant 0$$

$$\vdots$$

$$\langle \rho T(x^k) + x^{k+1} - x^k, x - x^{k+1} \rangle \geqslant 0 \text{ for all } x \in K \text{ and for } \rho > 0.$$

Algorithm 2.1 is equivalent to the projection formula

$$x^{k+1} = P_K[x^k - \rho T(x^k)],$$

where P_K is the projection of H onto K.

Before we present our main result on the approximation-solvability of the NVI problem (1.1), we need to recall the following auxiliary result.

LEMMA 2.1. *For $v,\ w \in H$, we have*

$$\langle v, w\rangle = \frac{1}{2}[||v+w||^2 - ||v||^2 - ||w||^2].$$

Now, we present, based on Algorithm 2.1, the approximation-solvability of the NVI problem (1.1) involving the γ-r-partially relaxed monotone mappings in a Hilbert space setting.

THEOREM 2.1. *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $x^* \in K$ be a solution of the NVI problem (1.1) and the sequence $\{x^k\}$ be generated by Algorithm 2.1. Suppose that a mappings $T : K \to H$ satisfy the following assumptions:*
(i) T is γ-r-partially relaxed monotone.
Then we have:
(a) The estimates:
(ii) $||x^{k+1} - x^*||^2 \leqslant (1-2\rho r)||x^k - x^*||^2 - (1-2\rho\gamma)||x^k - x^{k+1}||^2.$
(iii) $||x^{k+1} - x^*||^2 \leqslant (1-2\rho r)||x^k - x^*||^2$ *for* $1-2\rho\gamma > 0$ *and* $1-2\rho r > 0$.
(b) The sequence $\{x^k\}$ converges to x^* for $0 < \rho < 1/2\gamma$ and $0 < \rho < 1/2r$.

Proof. First, we compute the estimate and then show the convergence of the sequence $\{x^k\}$ to x^*, a solution of the NVI problem (1.1). Since x^k satisfies Algorithm 2.1, we have

$$\langle \rho T(x^k) + x^{k+1} - x^k, x - x^{k+1}\rangle \geqslant 0 \text{ for all } x \in K. \tag{2.1}$$

On the top of that, x^* is a solution of the NVI problem (1.1), that is, we can have, for a constant $\rho > 0$ that

$$\langle \rho T(x^*), x - x^*\rangle \geqslant 0 \text{ for all } x \in K. \tag{2.2}$$

Replacing x by x^* in (2.1) and x by x^{k+1} in (2.2), and adding, we obtain

$$0 \leqslant -\rho\langle T(x^k) - T(x^*), x^{k+1} - x^*\rangle + \langle x^{k+1} - x^k, x^* - x^{k+1}\rangle.$$

Since T is γ-r-partially relaxed monotone, it implies that

$$0 \leqslant \langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \rho\gamma||x^{k+1} - x^k||^2 - \rho r||x^k - x^*||^2. \tag{2.3}$$

Taking $v = x^{k+1} - x^k$ and $w = x^* - x^{k+1}$ in Lemma 2.1, and applying to (2.3), we have

$$0 \leqslant \frac{1}{2}[||x^* - x^k||^2 - ||x^{k+1} - x^k||^2 - ||x^* - x^{k+1}||^2] + \rho\gamma||x^{k+1} - x^k||^2 - \rho r||x^k - x^*||^2.$$

It follows that

$$||x^{k+1}-x^*||^2 \leqslant ||x^k-x^*||^2 - ||x^{k+1}-x^k||^2 + 2\rho\gamma||x^{k+1}-x^k||^2 - 2\rho r||x^k-x^*||^2.$$

That means, we have

$$||x^{k+1}-x^*||^2 \leqslant (1-2\rho r)||x^k-x^*||^2 - (1-2\rho\gamma)||x^{k+1}-x^k||^2. \tag{2.4}$$

It follows from (2.4) for $1-2\rho\gamma > 0$ and $1-2\rho r > 0$ that

$$||x^{k+1}-x^*||^2 \leqslant (1-2\rho r)||x^k-x^*||^2. \tag{2.5}$$

It follows from (2.5) that

$$\lim_{k\to\infty} ||x^{k+1}-x^*|| = 0.$$

Thus, $x^k \to x^*$ strongly as $k \to \infty$, and this concludes the proof.

3. Applications

In this section we consider some applications of the result of Section 2 to the $\mathbf{R}^n$. Let $F : X \to \mathbf{R}^n$ be a mapping from a closed convex subset X of $\mathbf{R}^n$ into $\mathbf{R}^n$. We consider a variational inequality problem: find an element $u \in X$ such that

$$[F(u)]^t(x-u) \geqslant 0 \text{ for all } x \in X, \tag{3.1}$$

where $[F(u)]^t$ denotes the transpose of the vector $F(u)$. Based on Algorithm 2.1, we have:

ALGORITHM 3.1. For an arbitrarily chosen initial point $x^0 \in X$, a sequence $\{x^k\}$ is generated by an iterative scheme:

$$[\rho F(x^k) + D_\rho(x^{k+1}-x^k)]^t(x-x^{k+1}) \geqslant 0 \text{ for all } x \in X, \tag{3.2}$$

where D_ρ is a fixed positive-definite matrix.

In what follows, D_ρ shall denote a symmetric matrix in (3.2) for the convergence of the projection method. The symbol $\lambda_{\min}(S)$ shall denote the minimum eigenvalue of a symmetric matrix S.

Since D_ρ is symmetric, it implies that (3.2) is equivalent to

$$x^{k+1} = P_{D_\rho}[x^k - D_\rho^{-1}(\rho F(x^k))], \tag{3.3}$$

where P_{D_ρ} is the projection on the set X with respect to the norm $||\cdot||_{D_\rho}$ induced by the positive-definite symmetric matrix D_ρ.

THEOREM 3.1. *Let F be a γ-r-partially relaxed monotone mapping and $D_\rho = D$, where D is a symmetric positive-definite matrix. Suppose that the sequence $\{x^k\}$ is generated by Algorithm 3.1 for a constant $\rho > 0$, and x^* is a solution of variational inequality (3.1). Then we have the following conclusions:*

(i) $||x^{k+1}-x^*||_D^2 \leqslant [1-2\rho r/\lambda_{\min}(D)]||x^k-x^*||_D^2 - [1-2\rho\lambda/\lambda_{\min}(D)]||x^{k+1}-x^k||_D^2$;

(ii) the sequence $\{x^k\}$ *generated by Algorithm 3.1 converges to* x^**, a solution of the variational inequality (3.1) for* $0 < \rho < \lambda_{\min}(D)/2r$ *and* $0 < \rho < \lambda_{\min}(D)/2\gamma$.

Proof. The proof is similar to that of Theorem 2.1.

REMARK 3.1. An estimate similar to that of Theorem 2.1 can be achieved by applying the auxiliary problem principle of Cohen [3], but the convergence analysis will differ.

4. Numerical computation/experiment

It sounds interesting if we can come up with some adaptive linesearch rule which would work under the partial relaxed monotonicity condition. The difficulty we are faced with is the way this condition comes up in the analysis – it always involves an (unknown) solution point.This differs from how (for example) strong monotonicity and Lipschitz continuity conditions are usually applied to other projection methods. As a result the problem – how to develop a linesearch rule under the framework of the partial relaxed monotonicity condition – is still open.

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