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INEQUALITIES FOR THE GAMMA FUNCTION RELATING TO ASYMPTOTIC EXPANSIONS

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Abstract. Many inequalities for the gamma function can be deduced from monotonicity or convexity properties of $\log\Gamma(x)$ and related functions involving finite sums of the Stirling asymptotic series. Considering a particular case of a generalization of this classical expansion, we deduce further convexity results and inequalities which are similar to some other ones related to the usual form of the Stirling series. We give, among other things, inequalities which overvalue $\log\Gamma(x)$, whereas the corresponding finite sums of the classical expansion undervalue it or vice versa. Moreover we obtain bilateral inequalities also for the digamma and the polygamma functions. Finally, a few extensions of Gautschi-type inequalities are discussed.

1. Introduction and preliminary results

Several inequalities for the gamma function arise from monotonicity and convexity properties of $\log\Gamma(x)$, as well as of some functions connected to finite sums of the classical Stirling asymptotic series [13] for $x \to \infty$

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \tag{1.1}$$

where B_{2k}, ($k = 1, 2 \dots$), are the Bernoulli numbers. In this paper we present further results referring to a rather unusual form of the Stirling asymptotic series containing odd powers of $1/(x-\frac{1}{2})$, $x > 1/2$, for $x \to \infty$,

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right) - x + \frac{1}{2} + \frac{1}{2}\log(2\pi) - \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\left(1-\frac{1}{2^{2k-1}}\right)\frac{1}{(x-\frac{1}{2})^{2k-1}}. \tag{1.2}$$

Series (1.2), as well as (1.1), is a special case of a general asymptotic expansion of $\log\Gamma(z+h)$ for $0 \leqslant h \leqslant 1$ and $|\arg z| < \pi$, in terms of powers of $1/z$ [13, p. 295]. Replacing $z+h$ by x, and then getting $h = 1/2$ we deduce (1.2), taking into account that

$$B_k(1/2) = -\left(1-\frac{1}{2^{k-1}}\right)B_k, \qquad k = 0, 1, \dots. \tag{1.3}$$

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The starting point of our paper was a private communication of D. Kershaw, who called our attention to the following Theorem 1.1 and pointed out the intriguing connection with the asymptotic series (1.1) and (1.2).

THEOREM 1.1. *For* $x > \frac{1}{2}$

$$\left(x-\frac{1}{2}\right)\log x - x < \log\Gamma(x) - \frac{1}{2}\log(2\pi) < \left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right) - \left(x-\frac{1}{2}\right) \tag{1.4}$$

Proof. From the inequalities for the digamma function (see (14) in [9] and (4.5) below)

$$\log\left(x-\frac{1}{2}\right) < \psi(x) < \log x - \frac{1}{2x}, \qquad x > \frac{1}{2}$$

it follows for real $M > x$

$$\int_M^x \left(\log t - \frac{1}{2t}\right) dt < \int_M^x \psi(t)dt < \int_M^x \log\left(t-\frac{1}{2}\right) dt, \qquad x > \frac{1}{2}.$$

After some rearrangement, we get

$$\log\Gamma(M) - \left(M-\frac{1}{2}\right)\log M + M < \log\Gamma(x) - \left(x-\frac{1}{2}\right)\log x + x, \tag{1.5}$$

and

$$\log\Gamma(x) - \left(x-\frac{1}{2}\right)\left[\log\left(x-\frac{1}{2}\right)-1\right] < \log\Gamma(M) - \left(M-\frac{1}{2}\right)\left[\log\left(M-\frac{1}{2}\right)-1\right]. \tag{1.6}$$

Letting $M \to \infty$ in (1.5) and (1.6) and using (1.1) and (1.2) respectively, we obtain the result.

We point out that in (1.4) the left-hand inequality involves the first terms of the asymptotic expansion (1.1) whereas the right-hand one regards the corresponding terms of (1.2). The left-hand inequality in (1.4), valid for $x > 0$, is well-known and is a consequence of the complete monotonicity on $(0,\infty)$ of the function

$$A_0(x) = \log\Gamma(x) - \left(x-\frac{1}{2}\right)\log x + x - \frac{1}{2}\log(2\pi)$$

stated by Muldoon [12]. We recall that a function $f(x)$ is said to be completely monotonic on an interval I if $f(x)$ has derivatives of all order on I and $(-1)^n f^{(n)}(x) \geqslant 0$, $n = 0, 1, \ldots,$ for $x \in I$. If the inequality is strict for all $x \in I$ and for all $n \geqslant 0$, then $f(x)$ is said strictly completely monotonic.

Muldoon's result was recently extended by Alzer [2], who showed that for integer $N \geqslant 0$ the functions

$$A_N(x) = \log\Gamma(x) - \left(x-\frac{1}{2}\right)\log x + x - \frac{1}{2}\log(2\pi) - \sum_{k=1}^{2N} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}$$

and

$$B_N(x) = -\log\Gamma(x) + \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{2N+1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}$$

are strictly completely monotonic for $x > 0$.

Previously, Merkle [10] proved that the function

$$x \to \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x + x - \frac{1}{2}\log(2\pi) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \quad n = 0, 1, 2 \dots$$

is convex for even n and concave for odd n on $(0, \infty)$ and proposed as an application sharp bounds for the ratio $\Gamma(x+\beta)/\Gamma(x)$.

Then the authors [14] pointed out the n-convexity properties related to the functions $A_N(x)$ and $B_N(x)$ and deduced further inequalities for the ratio of gamma functions $\Gamma(x+1)/\Gamma(x+s)$. Moreover some authors improved inequalities for $\ln\Gamma(x)$ involving finite sums of the Stirling asymptotic series (1.1) to the purpose to derive more stringent bounds for large x. See for instance [4].

In Section 2 we prove inequalities for n-convex functions by means the second Euler-Maclaurin formula and in Section 3, in a similar way as in [14], we derive monotonicity and convexity results for some functions related to finite sum of (1.2). As a consequence we deduce bilateral inequalities for $\log\Gamma(x)$, digamma function $\psi(x)$ and polygamma function by using either finite sums of (1.2) or finite sums of both (1.1) and (1.2). Moreover we find that on the interval $(1/2, \infty)$ whereas finite sums of the Stirling formula (1.1) undervalue $\log\Gamma(x)$, the corresponding sums of (1.2) overvalue it or vice versa.

Finally, in Section 4 we consider Gautschi-type inequalities from the viewpoint of the convexity properties, giving some comments and extensions. We recall that the well-known Gautschi inequality ([6]; see also [7] for references)

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp[(1-s)\psi(x+1)] \tag{1.7}$$

for $x > 0$ and $0 \leqslant s \leqslant 1$ is a straightforward consequence of the convexity of $\log\Gamma(x)$ [14] and this remark allows us to deduce the more symmetric inequality

$$(y-x)\psi(x) < \log\frac{\Gamma(y)}{\Gamma(x)} < (y-x)\psi(y) \quad 0 < x < y. \tag{1.8}$$

In Section 4 we give some comments and estensions with regard to the inequalities for the function $\log[\Gamma(y)/\Gamma(x)]$ involving digamma and polygamma functions.

2. The second Euler-Maclaurin formula and n-convexity

We recall that a function $f(x)$ which has derivatives up the order n on an interval I is said to be n-convex for $n \geqslant 2$ on I if $f^{(n)}(x) \geqslant 0$ for $x \in I$. If the inequality is strict for all $x \in I$, then $f(x)$ is said to be strictly n-convex .

THEOREM 2.1. *Let $f(x) \in C^{2r}[0,\infty)$ be a $(2r)$-convex function, with primitive function $F(x)$ such that*

$$\lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to\infty} f^{(2k-1)}(x) = 0 \quad (k = 1,\dots,r-1), \quad \lim_{x\to\infty} F(x) = 0.$$

If r is odd, we have for $x \geqslant \frac{1}{2}$

$$\sum_{k=0}^{\infty} f(x+k) \leqslant -F\left(x-\frac{1}{2}\right) - \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}\left(x-\frac{1}{2}\right). \tag{2.1}$$

If f is strictly $(2r)$-convex, then the inequality is strict. For even r, we have reverse inequalities.

Proof. The second Euler-Maclaurin summation formula [15, p.135] gives

$$\sum_{k=0}^{n-1} f\left(x+k+\frac{1}{2}\right) = \int_x^{x+n} f(t)\,dt + \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!}[f^{(2k-1)}(x+n) - f^{(2k-1)}(x)] + \frac{nB_{2r}(1/2)}{(2r)!} f^{(2r)}(\xi)$$

where $\xi \in (x, x+n)$. Since $(-1)^{r+1}B_{2r} > 0$, from (1.3) we have that $(-1)^r B_{2r}(1/2) > 0$. Hence, for $(2r)$-convex functions on $[1/2,\infty)$ we deduce for odd r

$$\sum_{k=0}^{n-1} f\left(x+k+\frac{1}{2}\right) \leqslant F(x+n) - F(x) + \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!}[f^{(2k-1)}(x+n) - f^{(2k-1)}(x)].$$

From this, as $n \to \infty$, we get

$$\sum_{k=0}^{\infty} f\left(x+k+\frac{1}{2}\right) \leqslant -F(x) - \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}(x)$$

and replacing x by $x - \frac{1}{2}$, we obtain (2.1) for $x \geqslant 1/2$. The proof of the fact that the inequality is strict for strictly $2r$-convex functions is similar to that given in [14]. If r is even, we have reverse inequalities.

REMARK 2.1. From the previous Theorem 2.1 and the corresponding Theorem 2.2 in [14] we derive, for odd r

$$\begin{aligned}\frac{1}{2}f(x) - F(x) - \sum_{k=1}^{r-1} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x) &\leqslant \sum_{k=0}^{\infty} f(x+k) \\ &\leqslant -F\left(x-\frac{1}{2}\right) + \sum_{k=1}^{r-1} \frac{B_{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right) f^{(2k-1)}\left(x-\frac{1}{2}\right)\end{aligned}$$

and vice versa for even r.

THEOREM 2.2. *Let* $f(x) \in C^{2r+2}[0,\infty)$ *and let* $F(x)$ *be a primitive function of* $f(x)$. *Further let* $f(x)$ *be* $(2r)$*-convex and* $(2r+2)$*-convex, and let*

$$\lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to\infty} f^{(2k-1)}(x) = 0 \quad (k = 1,\dots,r), \quad \lim_{x\to\infty} F(x) = 0.$$

If r is odd, we have for $x \geqslant \frac{1}{2}$

$$-F\left(x-\frac{1}{2}\right) - \sum_{k=1}^{r} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}\left(x-\frac{1}{2}\right) \leqslant \sum_{k=0}^{\infty} f(x+k) \leqslant -F\left(x-\frac{1}{2}\right) - \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}\left(x-\frac{1}{2}\right). \tag{2.2}$$

If f is strictly $(2r)$*-convex and strictly* $(2r+2)$*-convex, then the inequalities are strict. If r is even, we have reversed inequalities.*

Proof. This is an immediate consequence of Theorem 2.1.

REMARK 2.2. We note that (2.2) can be rewritten for odd r as

$$\frac{B_{2r}}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right) f^{(2r-1)}\left(x-\frac{1}{2}\right) \leqslant \sum_{k=0}^{\infty} f(x+k) + F\left(x-\frac{1}{2}\right) + \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}\left(x-\frac{1}{2}\right) \leqslant 0$$

while for even r we have reverse inequalities. So we can write for each r

$$\left|\sum_{k=0}^{\infty} f(x+k) + F\left(x-\frac{1}{2}\right) + \sum_{k=1}^{r-1} \frac{B_{2k}(1/2)}{(2k)!} f^{(2k-1)}\left(x-\frac{1}{2}\right)\right| \leqslant (-1)^r \frac{B_{2r}}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right) f^{(2r-1)}\left(x-\frac{1}{2}\right).$$

3. Inequalities for the gamma and polygamma functions

The results in Section 2 allow us to obtain inequalities for the gamma, digamma and polygamma functions.

COROLLARY 3.1. *For* $x > \frac{1}{2}$, *real* $p > 1$ *and integer* $N \geqslant 0$,

$$\frac{1}{(p-1)(x-\frac{1}{2})^{p-1}} - \sum_{k=1}^{2N+1} \frac{B_{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\frac{p(p+1)\cdots(p+2k-2)}{(x-\frac{1}{2})^{p+2k-1}} < \sum_{k=0}^{\infty} \frac{1}{(x+k)^p} < \frac{1}{(p-1)(x-\frac{1}{2})^{p-1}} - \sum_{k=1}^{2N} \frac{B_{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\frac{p(p+1)\cdots(p+2k-2)}{(x-\frac{1}{2})^{p+2k-1}} \tag{3.1}$$

where the sum is zero for $N=0$*. We have reverse inequalities replacing* $2N$ *by* $2N-1$.

Proof. The inequalities (3.1) are an immediate consequence of Theorem 2.2 by setting $f(x)=x^{-p}$ and $r=2N+1$. In the case $r=2N$ we have reverse inequalities.

Similarly from Corollary 3.1 and the corresponding Corollary 2.2 in [14] we deduce

REMARK 3.1. For $x>\frac{1}{2}$, real $p>1$ and integer $N\geqslant 0$,

$$\frac{1}{(p-1)(x-\frac{1}{2})^{p-1}}-\sum_{k=1}^{2N+1}\frac{B_{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\frac{p(p+1)\cdots(p+2k-2)}{(x-\frac{1}{2})^{p+2k-1}}$$
$$<\sum_{k=0}^{\infty}\frac{1}{(x+k)^p}<\frac{1}{2x^p}+\frac{1}{(p-1)x^{p-1}}+\sum_{k=1}^{2N+1}\frac{B_{2k}}{(2k)!}\frac{p(p+1)\cdots(p+2k-2)}{x^{p+2k-1}}, \tag{3.2}$$

and

$$\frac{1}{2x^p}+\frac{1}{(p-1)x^{p-1}}+\sum_{k=1}^{2N}\frac{B_{2k}}{(2k)!}\frac{p(p+1)\cdots(p+2k-2)}{x^{p+2k-1}}<\sum_{k=0}^{\infty}\frac{1}{(x+k)^p}$$
$$<\frac{1}{(p-1)(x-\frac{1}{2})^{p-1}}-\sum_{k=1}^{2N}\frac{B_{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\frac{p(p+1)\cdots(p+2k-2)}{(x-\frac{1}{2})^{p+2k-1}}, \tag{3.3}$$

where the sums are zero for $N=0$.

REMARK 3.2. Let us consider the polygamma function

$$\psi^{(n)}(x)=\frac{d^{n+1}}{dx^{n+1}}\log\Gamma(x)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}} \qquad n=1,2,\ldots.$$

As a consequence of Corollary 3.1 setting $p=n+1$, we have for $x>1/2$

$$\frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}}<(-1)^{n+1}\psi^{(n)}(x)$$
$$<\frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}} \tag{3.4}$$

and reverse inequalities replacing $2N$ by $2N-1$.

From (3.4) we obtain, in particular, for $n=1$ and $N=0$,

$$\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^2}<\psi'(x)<\frac{1}{x-\frac{1}{2}}, \quad x>\frac{1}{2}.$$

Other inequalities for the polygamma function can be similarly deduced from (3.2) and (3.3).

Now we investigate the n-convexity properties of the functions

$$F_1(x)=\log\Gamma(x)-\left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}}$$

and

$$F_2(x) = \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log \left(x - \frac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}},$$

for $N \geqslant 0$ and $x > 1/2$ ... .

THEOREM 3.1. *If n is even, then $F_1(x)$ is strictly n-convex and $F_2(x)$ strictly n-concave, whereas if n is odd, then $F_1(x)$ is strictly n-concave and $F_2(x)$ is strictly n-convex.*

Proof. By differentiation for $n \geqslant 1$,

$$\begin{aligned} F_1^{(n)}(x) =& (-1)^n (n-1)! \left[\sum_{k=0}^{\infty} \frac{1}{(x+k)^n} - \frac{1}{(n-1)(x-\frac{1}{2})^{n-1}} \right. \\ & \left. - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{(2k)!} \frac{n(n+1)\cdots(n+2k-2)}{(x-\frac{1}{2})^{n+2k-1}} \right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} F_2^{(n)}(x) =& (-1)^{n+1} (n-1)! \left[\sum_{k=0}^{\infty} \frac{1}{(x+k)^n} - \frac{1}{(n-1)(x-\frac{1}{2})^{n-1}} \right. \\ & \left. - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{(2k)!} \frac{n(n+1)\cdots(n+2k-2)}{(x-\frac{1}{2})^{n+2k-1}} \right]. \end{aligned} \tag{3.6}$$

Setting $p = n$ in (3.1) we see that the expression bracketed in (3.5) is positive, whereas the expression bracketed in (3.6) is negative. Therefore

$$(-1)^n F_1^{(n)}(x) > 0 \quad \text{and} \quad (-1)^{n+1} F_2^{(n)}(x) > 0.$$

Furthermore let us consider the functions

$$\begin{aligned} F_3(x) = & \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log \left(x - \frac{1}{2}\right) + x - \frac{1}{2} - \frac{1}{2} \log(2\pi) \\ & - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}} \end{aligned}$$

and

$$\begin{aligned} F_4(x) = & -\log \Gamma(x) + \left(x - \frac{1}{2}\right) \log \left(x - \frac{1}{2}\right) - x + \frac{1}{2} + \frac{1}{2} \log(2\pi) \\ & + \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}}, \end{aligned}$$

which are clearly related to the asymptotic series (1.2). We shall investigate the complete monotonicity of these functions, taking into account that

$$F_3(x) = F_1(x) + x - \frac{1}{2} - \frac{1}{2}\log(2\pi)$$

and

$$F_4(x) = -F_2(x) - x + \frac{1}{2} + \frac{1}{2}\log(2\pi).$$

THEOREM 3.2. *For integer $N \geqslant 0$ the functions $F_3(x)$ and $F_4(x)$ are strictly completely monotonic on* $(1/2, \infty)$.

Proof. From Theorem 3.2 we have $F_3''(x) = F_1''(x) > 0$ and $F_4''(x) = [-F_2(x)]'' > 0$. By integration on $[x, M]$, real $M > x > 1/2$, we get

$$F_3'(M) - F_3'(x) > 0, \quad \text{and} \quad F_4'(M) - F_4'(x) > 0. \tag{3.7}$$

By means of the Stirling asymptotic expansion for $\psi(x)$ as $x \to \infty$, we have for $M \to \infty$ in (3.7), $F_3'(x) < 0$ and $F_4'(x) < 0$. Further integration on $[x, M]$ gives the inequalities

$$F_3(M) - F_3(x) < 0, \quad \text{and} \quad F_4(M) - F_4(x) < 0.$$

Again for $M \to \infty$ we get also $F_3(x) > 0$ and $F_4(x) > 0$. So we conclude, according to the results in Theorem 3.2, that the functions $F_3(x)$ and $F_4(x)$ are strictly completely monotonic on $(\frac{1}{2}, \infty)$.

Monotonicity properties of F_3 and F_4 enable us to deduce the following inequalities for $x > \frac{1}{2}$, $N = 0, 1, 2 \ldots$

$$\begin{aligned}
&\left(x - \frac{1}{2}\right)\log\left(x - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}} \\
&< \log\Gamma(x) < \left(x - \frac{1}{2}\right)\log\left(x - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) + \frac{1}{2}\log(2\pi) \\
&+ \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}}
\end{aligned} \tag{3.8}$$

and

$$\log\left(x - \frac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} < \psi(x) < \log\left(x - \frac{1}{2}\right) - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}}$$

Furthermore, taking also into account the complete monotonicity of $A_N(x)$ and $B_N(x)$ defined in Section 1, we obtain inequalities for $\log\Gamma(x)$ and $\psi(x)$ involving finite sums of both aymptotic expansions (1.1) and (1.2) on $(\frac{1}{2},\infty)$,

$$\left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{2N} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} < \log\Gamma(x)$$
$$< \left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right) - \left(x-\frac{1}{2}\right) + \frac{1}{2}\log(2\pi)$$
$$+ \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}} \tag{3.9}$$

and

$$\log\left(x-\frac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} < \psi(x) < \log\left(x-\frac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2kx^{2k}}$$

and the reverse inequalities replacing $2N$ by $2N+1$. Bilateral inequality (3.9) coincides with (1.4) in the case $N=0$.

By using the n-convexity (concavity) properties of the functions $F_1(x)$ and $F_2(x)$, and the complete monotonicity of $F_3(x)$ and $F_4(x)$ we can obtain inequalities of the same type as in [14].

Recalling that a function $\exp(-h(x))$ is completely monotonic on an interval I if $h'(x)$ is completely monotonic on I, we deduce that the functions

$$F_5(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}\left(\frac{e}{x-\frac{1}{2}}\right)^{x-\frac{1}{2}} \exp\{-\sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}}\}$$

and

$$F_6(x) = \frac{\sqrt{2\pi}}{\Gamma(x)}\left(\frac{x-\frac{1}{2}}{e}\right)^{x-\frac{1}{2}} \exp\{\sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(2k-1)(x-\frac{1}{2})^{2k-1}}\}$$

are completely monotonic for $x>\frac{1}{2}$ as an immediate consequence of the complete monotonicity of $-F_3'(x)$ and $-F_4'(x)$.

4. Inequalities for the gamma function involving digamma and polygamma functions

In [14] we show that Gautschi's inequality (1.7) is a straighforward consequence of the convexity of $\log\Gamma(x)$ and this remark allows us to deduce, in particular, the more symmetric inequality (1.8). We also consider the following improvement of Gautschi's inequality, proposed but seemingly not published by Kershaw,

$$(a-b)\,\psi\left(x+\sqrt{ab}\right) < \log\frac{\Gamma(x+a)}{\Gamma(x+b)} < (a-b)\,\psi\left(x+\frac{a+b}{2}\right) \tag{4.1}$$

with $x > 0$ and $0 \leqslant b \leqslant a$. The proof of (4.1) may be obtained by means the same reasoning in [8] to prove

$$(1-s)\,\psi(x+\sqrt{s}) < \log\frac{\Gamma(x+1)}{\Gamma(x+s)} < (1-s)\,\psi\left(x+\frac{s+1}{2}\right), \tag{4.2}$$

where $x > 0$ and $0 < s < 1$. In fact, (4.2) is a particular case of (4.1) by setting $b = s$ and $a = 1$. Recently, on investigating about the best bounds in Gautschi-type inequalities, Elezović, Giordano and Pečarić [5] have proved (4.1).

Inequalities (4.1) and (1.8) can be easily compared. In fact (1.8) can be rewritten in the form

$$(a-b)\,\psi(x+b) < \log\frac{\Gamma(x+a)}{\Gamma(x+b)} < (a-b)\,\psi(x+a)$$

with $z = t+a$ and $x = t+b$. Since $\psi(x)$ is an increasing function, it is evident that (4.1) is sharper than (1.8).

Many authors have been interestd in studying and extending Gautschi-type inequalities. For example, Bustoz and Ismail [3] proved that some inequalities for the gamma function in particular (4.2), follow from the complete monotonicity of certain functions involving the ratio $\Gamma(x+1)/\Gamma(x+s)$. In this Section we deduce inequalities for $\log[\Gamma(y)/\Gamma(x)]$ as consequences of $2r$-concavity properties of $\psi(x)$ and by applying Hadamard-type inequalities.

Merkle [10] obtains the inequalities

$$(y-x)\,\frac{\psi(x)+\psi(y)}{2} < \log\frac{\Gamma(y)}{\Gamma(x)} < (y-x)\psi\left(\frac{x+y}{2}\right) \tag{4.3}$$

espressed in terms of the digamma function, as an application of some conditions for the convexity of the derivative of a function f. We note (see also [5]) that (4.3) is an immediate consequence of the Hadamard inequalities for a concave function.

Given a concave function $f : [x,y] \to R$, the well-known Hadamard's inequalities state

$$(y-x)\frac{f(x)+f(y)}{2} \leqslant \int_x^y f(t)dt \leqslant (y-x)f\left(\frac{x+y}{2}\right)$$

and (see, e.g., [1])

$$0 \leqslant (y-x)f\left(\frac{x+y}{2}\right) - \int_x^y f(t)dt \leqslant \int_x^y f(t)dt - (y-x)\frac{f(x)+f(y)}{2}. \tag{4.4}$$

Choosing $f(t) = \psi(t)$, which is strictly concave, we have (4.3) and from (4.4) the more informative inequality

$$0 < (y-x)\psi\left(\frac{x+y}{2}\right) - \log\frac{\Gamma(y)}{\Gamma(x)} < \log\frac{\Gamma(y)}{\Gamma(x)} - (y-x)\frac{\psi(x)+\psi(y)}{2}.$$

REMARK 4.1. Letting $y = x+1$ in (4.3), we get [see also 10]

$$\psi(x) + \frac{1}{2x} < \log x < \psi\left(x+\frac{1}{2}\right)$$

and, from this, there follows

$$\log\left(x-\frac{1}{2}\right) < \psi(x) < \log x - \frac{1}{2x}, \qquad x > \frac{1}{2} \tag{4.5}$$

which is used in Theorem 1.1, and further

$$0 < \log\left(x-\frac{1}{2}\right) - \psi(x) < \psi(x) - \log x + \frac{1}{2x} \qquad x > \frac{1}{2}.$$

THEOREM 4.1. *Let x and y be real numbers, $0 < x < y$, and $h = (y-x)/n$. Then*

$$\frac{h}{2}\left[\psi(x)+\psi(y)\right] + h\sum_{k=1}^{n-1}\psi(x+kh) < \log\frac{\Gamma(y)}{\Gamma(x)} < h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right]$$

and, more in general,

$$0 < h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right] - \log\frac{\Gamma(y)}{\Gamma(x)}$$
$$< \log\frac{\Gamma(y)}{\Gamma(x)} - \frac{h}{2}\left[\psi(x)+\psi(y)\right] - h\sum_{k=1}^{n-1}\psi(x+kh).$$

Proof. These inequalities are straightforward consequences of the extensions of Hadamard's inequalities proved in [1], Theorem 1.1.

THEOREM 4.2. *Let x and y be real numbers, $0 < x < y$, and $h = (y-x)/n$. Then for odd $r \geqslant 1$,*

$$\frac{h}{2}\left[\psi(x)+\psi(y)\right] + h\sum_{k=1}^{n-1}\psi(x+kh) - \sum_{\nu=1}^{r-1}\frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right]$$
$$< \log\frac{\Gamma(y)}{\Gamma(x)} < h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right] - \sum_{\nu=1}^{r-1}\frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right]$$

and more generally,

$$0 < h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right] - \sum_{\nu=1}^{r-1}\frac{B_{2\nu}(1/2)h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right]$$
$$-\log\frac{\Gamma(y)}{\Gamma(x)} < \log\frac{\Gamma(y)}{\Gamma(x)} - \frac{h}{2}[\psi(x)+\psi(y)] - h\sum_{j=1}^{n-1}\psi(x+jh)$$
$$+\sum_{\nu=1}^{r-1}\frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right].$$

If r is even, we have reverse inequalities.

Proof. Since $\psi(x)$ is $2r$-concave, these inequalities clearly follow from Theorem 2.1 in [1].

THEOREM 4.3. *Let x and y be real numbers, $0 < x < y$, and $h = (y-x)/n$. Then for odd $r \geqslant 1$,*

$$h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right]-\sum_{\nu=1}^{r-2}\frac{B_{2\nu}(1/2)h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right]<\log\frac{\Gamma(y)}{\Gamma(x)}$$
$$<h\sum_{k=0}^{n-1}\psi\left[x+\left(k+\frac{1}{2}\right)h\right]-\sum_{\nu=1}^{r-1}\frac{B_{2\nu}(1/2)h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right],$$

and

$$\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{j=1}^{n-1}\psi(x+jh)-\sum_{\nu=1}^{r-1}\frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right]<\log\frac{\Gamma(y)}{\Gamma(x)}$$
$$<\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{j=1}^{n-1}\psi(x+jh)-\sum_{\nu=1}^{r-2}\frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\psi^{(2\nu-1)}(y)-\psi^{(2\nu-1)}(x)\right].$$

If r is even, we have reverse inequalities.

Proof. These inequalities immediately follow from Hadamard-type inequalities for $(2r)$-concave and $(2r+2)$-concave functions, which are proved in [1], Theorem 2.2.

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