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INEQUALITIES INVOLVING THE NORM OF SOME LINEAR OPERATORS WITH APPLICATION TO APPROXIMATION PROCEDURES

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Abstract. Inequalities regarding the norm of some generalized projection operators are established, in order to describe the topological structure of the set of unbounded divergence for corresponding approximation procedures.

1. Introduction

Many problems regarding the convergence or the divergence of the approximation procedures require inequalities which refer to the norm of the corresponding operators in order to make use of some principles of functional analysis.

The aim of the paper is to prove inequalities of this type for generalized projection operators and to describe the topological structure of the set of unbounded divergence of the corresponding approximation procedures.

2. Inequalities involving the norm of a class of generalized projection operators

Let $\mathcal{E}_n$ be the space of all trigonometric polynomials having the degree at most n and $C_{2\pi}$ the Banach space of all periodic continuous functions $f:\mathbb{R}\to\mathbb{C}$, with the period 2π, endowed with the uniform norm.

Denote by $U_n : C_{2\pi}\to\mathcal{E}_n$, $n\geqslant 1$, linear and continuous operators given by

$$\begin{cases} (U_nf)(x) = \dfrac{1}{2}\varphi_{n0}(f) + \displaystyle\sum_{k=1}^{n}[\varphi_{nk}(f)\cos kx + \psi_{nk}(f)\sin kx] \\ f\in C_{2\pi},\ x\in\mathbb{R}, \end{cases}$$

where φ_{nk} and ψ_{nk}, $0\leqslant k\leqslant n$, with $\psi_{n0}=0$, are linear and continuous functionals.

Suppose that, for each $n\geqslant 1$, there exists the natural number

$$m_n = \max\{k:\ 0\leqslant k\leqslant n \text{ with } U_nf=f,\ \forall f\in\mathcal{E}_k\}.$$

Putting $\alpha_{nk}=\varphi_{nk}(\cos kx)$ and $\beta_{nk}=\psi_{nk}(\sin kx)$, it is clear that $\|\varphi_{nk}\|\geqslant|\alpha_{nk}|$, $\|\psi_{nk}\|\geqslant|\beta_{nk}|$, for $0\leqslant k\leqslant n$, and $\alpha_{n0}=\alpha_{nk}=\beta_{nk}=1$ if $1\leqslant k\leqslant m_n$, $n\geqslant 1$.

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Let us introduce the positive numbers τ_{nk}, $n \geqslant 1$, $0 \leqslant k \leqslant n$ so that

$$\begin{cases} 0 < \tau_{nk} \leqslant \|\varphi_{nk}\| + \|\psi_{nk}\|, \text{ if } n \geqslant 1;\ 0 \leqslant k \leqslant n \\ \tau_{nk} = 2, \text{ if } 1 \leqslant k \leqslant m_n;\ n \geqslant 1 \\ \tau_{n0} = 1,\ n \geqslant 1 \end{cases}$$

For each $n \geqslant 1$, let

$$\delta_n = \begin{cases} \min\{\tau_{nk} : m_{n+1} \leqslant k \leqslant n\}, & \text{if } m_n < n \\ 2, & \text{if } m_n = n \end{cases} ; \quad n \geqslant 1$$

$$r_n = n - m_n; \quad n \geqslant 1$$

and denote by γ_n the arithmetic mean of the numbers τ_{nk}, $m_n + 1 \leqslant k \leqslant n$, $n \geqslant 1$, $m_n < n$.

In what follows, M_s, $s \geqslant 1$, will be some positive constants which do not depend on n.

Now, we are in a position to prove the following statements.

THEOREM 2.1. *The norm of the operator* U_n, $n \geqslant 1$, *satisfies the inequality:*

$$\|U_n\| \geqslant M_1 \min(2, \delta_n) \ln n.$$

Proof. It follows from [5] the inequality

$$\|U_n\| \geqslant M_2 \sum_{k=0}^{n} \frac{\|\varphi_{nk}\| + \|\psi_{nk}\|}{n-k+1}$$

so that

$$\|U_n\| \geqslant M_2 \sum_{k=0}^{n} \frac{\tau_{nk}}{n-k+1}.$$

If $m_n < n$, we get:

$$\|U_n\| \geqslant M_2 \left(\sum_{k=0}^{m_n} \frac{\tau_{nk}}{n-k+1} + \sum_{k=m_n+1}^{n} \frac{\tau_{nk}}{n-k+1} \right). \tag{2.1}$$

Taking into account the definitions of τ_{nk} and δ_n, we obtain:

$$\|U_n\| \geqslant M_2 \left(2 \sum_{k=1}^{m_n} \frac{1}{n-k+1} + \delta_n \sum_{k=m_n+1}^{n} \frac{1}{n-k+1} + \frac{1}{n+1} \right)$$

$$\geqslant M_2 \min(2, \delta_n) \sum_{k=1}^{n} \frac{1}{k}.$$

If $m_n = n$, we get $\|U_n\| \geqslant 2M_2 \sum_{k=1}^{n} \frac{1}{k}$.

Now, make use of the first inequality of

$$\ln(n+1) \leqslant \sum_{k=1}^{n} \frac{1}{k} \leqslant 1 + \ln n; \quad n \geqslant 1 \tag{2.2}$$

and complete the proof.

THEOREM 2.2. *If the inequalities*

$$m_n < n \text{ and } \tau_{n,m_n+1} \leqslant \tau_{n,m_n+2} \leqslant \cdots \leqslant \tau_{nn} \tag{2.3}$$

hold for $n \geqslant 1$, *then the norm of the operator* U_n *satisfies the following estimations:*

1° $\|U_n\| \geqslant M_2 \min(2, \gamma_n) \ln n$ *and*

2° $\|U_n\| \geqslant M_2 \ln\left(\frac{n}{er_n}(\sqrt{r_n})^{\gamma_n}\right)$.

Proof. 1° We begin with the relation (2.1), apply Chebyshev inequality to the second sum of this relation and obtain:

$$\|U_n\| \geqslant M_2 \left(\sum_{k=0}^{m_n} \frac{\tau_{nk}}{n-k+1} + \frac{1}{n-m_n} \sum_{k=m_n+1}^{n} \tau_{nk} \sum_{k=m_n+1}^{n} \frac{1}{n-k+1}\right)$$

so that:

$$\|U_n\| \geqslant M_2 \left(2\sum_{k=1}^{m_n} \frac{1}{n-k+1} + \gamma_n \sum_{k=m_n+1}^{n} \frac{1}{n-k+1} + \frac{1}{n+1}\right). \tag{2.4}$$

Now, similarly to Theorem 1.1 we get:

$$\|U_n\| \geqslant M_2 \min(\gamma_n, 2) \ln n.$$

2° According to (2.2) we obtain:

$$\begin{aligned}\sum_{k=1}^{m_n} \frac{1}{n-k+1} &= \sum_{k=1}^{n} \frac{1}{n-k+1} - \sum_{k=m_n+1}^{n} \frac{1}{n-k+1}\\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{r_n} \frac{1}{k} \geqslant \ln(n+1) - 1 - \ln r_n,\end{aligned}$$

so that:

$$\sum_{k=1}^{m_n} \frac{1}{n-k+1} \geqslant \ln \frac{n}{r_n} - 1 \tag{2.5}$$

and

$$\sum_{k=m_n+1}^{n} \frac{1}{n-k+1} = \sum_{k=1}^{r_n} \frac{1}{k} > \ln r_n. \tag{2.6}$$

Now, it follows from (2.4), (2.5) and (2.6):

$$\|U_n\| \geqslant M_2 \left(2 \ln \frac{n}{r_n} - 2 + \gamma_n \ln r_n\right) = M_3 \left(\ln \frac{n}{er_n}(\sqrt{r_n})^{\gamma_n}\right).$$

This completes the proof.

REMARK. A similar inequality to 2° of Theorem 2.2 can be deduced under the hypotheses of Theorem 2.1 with δ_n instead of γ_n.

Finally, making use of a principle of condensation of singularities, [1], see also [4], by standard arguments, we obtain the following statement of topological structure, via Theorems 2.1 and 2.2.

THEOREM 2.3. *In each of the following situations, the set of all functions* $f \in C_{2\pi}$ *for which the sequence of linear and continuous operators* $(U_n)_{n\geqslant 1}$ *unboundedly diverges (namely* $\limsup_{n\to\infty} \|U_n f\| = \infty$*) is superdense in* $C_{2\pi}$ *(i.e. it is residual, uncountable and dense in* $C_{2\pi}$*):*

(i) The sequence $(\min(2,\delta_n)\ln n)_{n\geqslant 1}$ *is unbounded;*

(ii) The sequence $(u_n)_{n\geqslant 1}$ *or the sequence* $(t_n)_{n\geqslant 1}$*, with* $u_n = \min(2,\gamma_n)\ln n$ *and* $t_n = \frac{n}{r_n}(\sqrt{r_n})^{\gamma_n}$*,* $n \geqslant 1$*, contains an unbounded subsequence* $(u_{n_k})_{k\geqslant 1}$ *respectively* $(t_{n_k})_{k\geqslant 1}$ *so that the relations (2.3) are valid for* $n = n_k$*,* $k \geqslant 1$*;*

(iii) There exists a strictly increasing sequence $(n_k)_{k\geqslant 1}$ *of natural numbers so that* $r_{n_k} \in \sigma(n_k)$*;*

(iv) The sequence $(r_n)_{n\geqslant 1}$ *is bounded.*

REMARK. The assertion of Theorem 2.3,(iv), refers to the so-called quasi-projections [4]; on the same topic see also [2]. If $m_n = n$, so $r_n = 0$, we obtain, essentially, the famous Theorem of Lozinski and Harsiladze, concerning the impossibility of the uniform convergence of trigonometric (or polynomial) projection, see [3].

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