Mathematical
Inequalities
& Applications
Volume 5, Number 3 (2002), 573–582

GOLDEN–THOMPSON TYPE INEQUALITIES RELATED TO A GEOMETRIC MEAN VIA SPECHT'S RATIO

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Abstract. We prove a Golden-Thompson type inequality via Specht's ratio: Let H and K be selfadjoint operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI$ for some scalar $M > m$. Then

$$M_h(1)\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{\frac{1}{t}} \geqslant e^{(1-\lambda)H+\lambda K} \geqslant M_h(1)^{-1}M_h(t)^{-\frac{1}{t}}\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{\frac{1}{t}}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$, where $h = e^{M-m}$ and (generalized) Specht's ratio $M_h(t)$ is defined for $h > 0$ as

$$M_h(t) = \frac{(h^t-1)h^{\frac{t}{h^t-1}}}{e \log h^t} \quad (h \neq 1) \quad \text{and} \quad M_1(1) = 1.$$

1. Introduction

In the commutative case, if H and K are selfadjoint operators on a Hilbert space H, then $e^{H+K} = e^H e^K$. However, in the noncommutative case, it is entirely no relation between e^{H+K} and e^H, e^K under the usual order. The celebrated Golden-Thompson trace inequality, independently proved by Golden [6], Symanzik [11] and Thompson [12], says that $\mathrm{Tr}\, e^{H+K} \leqslant \mathrm{Tr}\, e^H e^K$ holds for Hermitian matrices H and K. Afterward, the Golden-Thompson trace inequality was complemented by Hiai and Petz [7]: Let H and K be Hermitian matrices and $0 \leqslant \lambda \leqslant 1$. Then the inequality

$$\mathrm{Tr}\left(e^{tH} \sharp_\lambda e^{tK}\right)^{1/t} \leqslant \mathrm{Tr}\, e^{(1-\lambda)H+\lambda K} \tag{1.1}$$

holds for all $t > 0$ and the left-hand side of (1.1) converges to the right-hand side as $t \downarrow 0$. Here $X \sharp_\lambda Y$ denotes the λ-geometric mean of nonnegative matrices X and Y (in particular, $X \sharp_{1/2} Y = X \sharp Y$ is the geometric mean), i.e.,

$$X \sharp_\lambda Y = X^{1/2}(X^{-1/2}YX^{-1/2})^\lambda X^{1/2} \qquad \text{for } 0 \leqslant \lambda \leqslant 1.$$

Moreover, Ando and Hiai [1] completed the complementary counterpart of the Golden-Thompson trace inequality by virtue of the log majorization.

The purpose of this paper is to investigate some relations between e^{H+K} and e^H, e^K under the usual order in terms of Specht's ratio. Let us recall Specht's ratio: Specht

Mathematics subject classification (2000): 47A30, 47A63.

Key words and phrases: operator inequality, Specht's ratio, Kantorovich type inequality, positive operator, Golden-Thompson inequality.

[10] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $M \geqslant m > 0$,

$$M_h(1)\sqrt[n]{x_1 \cdots x_n} \geqslant \frac{x_1 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 \cdots x_n},$$

where $h = \frac{M}{m}(\geqslant 1)$ is a generalized condition number in the sense of Turing [15] and (generalized) Specht's ratio $M_h(t)$ is defined for $h > 0$ as

$$M_h(t) = \frac{(h^t - 1)h^{\frac{t}{h^t-1}}}{e \log h^t} \quad (h \neq 1) \quad \text{and} \quad M_1(1) = 1 \tag{1.2}$$

for each $t > 0$ (cf. [16, 2, 13, 14]). We prove that if H and K are selfadjoint operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI$ for some scalar $M > m$, then

$$M_h(1)\left((1-\lambda)e^{tH} + \lambda e^{tK}\right)^{1/t} \geqslant e^{(1-\lambda)H+\lambda K} \geqslant M_h(1)^{-1}M_h(t)^{-1/t}\left((1-\lambda)e^{tH} + \lambda e^{tK}\right)^{1/t}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$, where $h = e^{M-m}$.

2. Preliminaries

We denote by $A \geqslant 0$ if A is a positive operator on a Hilbert space H. In particular, $A > 0$ means that A is positive and invertible. First of all, we consider the operator function derived from the family of power means. Let $B,\ C > 0$ and $\mu \in [0, 1]$ be given. Then it is defined by

$$F(s) = F_{B,C}(s) = ((1-\mu)B^s + \mu C^s)^{\frac{1}{s}} \quad (s \in \mathbb{R}).$$

It is known that $F(s)$ is monotone increasing on $[1, \infty)$, i.e., $F(s) \leqslant F(t)$ if $1 \leqslant s \leqslant t$, and $F(s)$ is not monotone increasing on $(0, 1]$ in general, see [3]. So we discuss the monotonicity of $F(s)$ under the chaotic order $A \gg B$, i.e., $\log A \geqslant \log B$ for $A,\ B > 0$. The following fact is basic in this paper:

LEMMA 2.1. [3] *The operator function $F(s)$ is monotone increasing under the chaotic order, i.e., $F(s) \ll F(t)$ if $s < t$. In particular,*

$$\text{s-}\lim_{h\to 0} F(h) = e^{(1-\mu)\log B + \mu \log C}.$$

Proof. For readers' convenience, we cite a proof. It suffices to show that

$$\frac{1}{s}\log((1-\mu)B^s + \mu C^s) \leqslant \frac{1}{t}\log((1-\mu)B^t + \mu C^t)$$

for $s < t$ with $s, t \neq 0$. To prove this, the operator concavity of x^r for $r \in [0, 1]$ is available. We first assume $0 < s < t$. Then

$$\log((1-\mu)B^t + \mu C^t)^{\frac{s}{t}} \geqslant \log((1-\mu)B^s + \mu C^s),$$

and so $\log F(t) \geqslant \log F(s)$. Next, if $s < t < 0$, then $\frac{t}{s} \in (0, 1)$ and hence

$$\log((1-\mu)B^s + \mu C^s)^{\frac{t}{s}} \geqslant \log((1-\mu)B^t + \mu C^t).$$

Noting $t < 0$, we have $\log F(s) \leqslant \log F(t)$.

Now we prove the second assertion. By the operator concavity of $\log x$ and $x - 1 \geqslant \log x$ for $x > 0$, it implies that for any $t > 0$

$$\begin{aligned}(1-\mu)\log B + \mu \log C &= \frac{1}{t}((1-\mu)\log B^t + \mu \log C^t) \\ &\leqslant \frac{1}{t}\log((1-\mu)B^t + \mu C^t) \\ &\leqslant \frac{1}{t}((1-\mu)B^t + \mu C^t - 1) \\ &= (1-\mu)\frac{B^t - 1}{t} + \mu \frac{C^t - 1}{t} \\ &\to (1-\mu)\log B + \mu \log C \quad (t \to +0).\end{aligned}$$

Therefore it follows that

$$\text{s-}\lim_{t\to+0} \log((1-\mu)B^t + \mu C^t)^{\frac{1}{t}} = (1-\mu)\log B + \mu \log C,$$

so that

$$\text{s-}\lim_{t\to+0} ((1-\mu)B^t + \mu C^t)^{\frac{1}{t}} = e^{(1-\mu)\log B + \mu \log C}.$$

On the other hand, it follows from the identity obtained above that for $s > 0$

$$\begin{aligned}F_{B,C}(-s) &= F_{B^{-1},C^{-1}}(s)^{-1} \\ &\to [e^{(1-\mu)\log B^{-1} + \mu \log C^{-1}}]^{-1} \\ &= e^{(1-\mu)\log B + \mu \log C}.\end{aligned}$$

Hence we have the second assertion, which says that $\text{s-}\lim_{h\to 0} F(h)$ can be regarded as $F(0)$. Therefore, if $s < 0 < t$, then

$$F(s) \ll F(0) \ll F(t).$$

Consequently we have the monotonicity of $F(s)$. □

For the sake of convenience, Nakamoto and one of the authors [3] defined a geometric mean different from the μ-geometric mean in the sense of Kubo-Ando: For $B, C > 0$ and $\mu \in [0, 1]$,

$$B \diamondsuit_\mu C = e^{(1-\mu)\log B + \mu \log C}$$

is said to be the chaotically μ-geometric mean of B and C.

3. Lemmas

Jensen's inequality says that if $f(t)$ is a real valued continuous convex (resp. concave) function and A is a selfadjoint operator on a Hilbert space H, then

$$(f(A)x, x) \geqslant f((Ax, x)) \quad (\text{resp. } f((Ax, x)) \geqslant (f(A)x, x))$$

holds for every unit vector $x \in H$. Mond and Pečarić [9] pointed out that the problem of determining the upper estimates of the difference and the ratio in Jensen's inequality is reduced to solving a single variable maximization (resp. minimization) problem by using the convexity (resp. concavity) of $f(t)$, cf. [8]. We cite the following complementary inequality to Jensen's inequality for the exponential function [4] ([8, Corollary 11], [2]), based on the Mond-Pečarić method.

LEMMA 3.1. (Furuta). *Let A be a selfadjoint operator on a Hilbert space H satisfying $MI \geqslant A \geqslant mI$ for some scalar $M > m$. Then*

$$M_h(t)e^{(tAx,x)} \geqslant (e^{tA}x,x)$$

holds for every unit vector $x \in H$ and for all $t > 0$, where $h = e^{M-m}$ and $M_h(t)$ is defined as (1.2).

Since the exponential function is not operator monotone, the assumption $A \geqslant B$ does not always assure $e^A \geqslant e^B$. However, Lemma 3.1 shows that e^t is order preserving in the following sense via Specht's ratio.

LEMMA 3.2. *Let A and B be selfadjoint operators on a Hilbert space H satisfying either $MI \geqslant A \geqslant mI$ or $MI \geqslant B \geqslant mI$ for some scalar $M > m$. Then*

$$A \geqslant B \qquad \textit{implies} \quad M_h(t)e^{tA} \geqslant e^{tB} \qquad \textit{for all } t > 0,$$

where $h = e^{M-m}$ and $M_h(t)$ is defined as (1.2). In particular,

$$A \geqslant B \qquad \textit{implies} \quad M_h(1)e^{A} \geqslant e^{B}.$$

Proof. Suppose that $MI \geqslant B \geqslant mI$. Then it follows that for all $t > 0$

$$\begin{aligned} M_h(t)(e^{tA}x,x) &\geqslant M_h(t)e^{(tAx,x)} && \text{by the convexity of } e^t \\ &\geqslant M_h(t)e^{(tBx,x)} && \text{by } A \geqslant B \text{ and } t > 0 \\ &\geqslant (e^{tB}x,x) && \text{by Lemma 3.1 and } MI \geqslant B \geqslant mI \end{aligned}$$

holds for every unit vector $x \in H$.

Next, suppose that $MI \geqslant A \geqslant mI$. Then we have $-B \geqslant -A$ and $-mI \geqslant -A \geqslant -MI$. Hence it follows that $e^{-m-(-M)} = e^{M-m} = h$ and $M_h(t)e^{-tA} \geqslant e^{-tB}$ as stated above. By taking the inverse of both sides, we have $M_h(t)e^{tA} \geqslant e^{tB}$. □

The chaotic order $A \gg B$ for $A,\ B > 0$ is introduced by the operator monotonicity of the logarithmic function, i.e., $A \gg B$ if $\log A \geqslant \log B$. The following statement is equivalent to Lemma 3.2, it makes clear the difference between the usual order and the chaotic order:

If $A \gg B$ for $A,\ B > 0$, then $M_h(t)A^t \geqslant B^t$ for all $t > 0$.

The following lemma estimates the upper bound of the difference in Jensen's inequality [8, Corollary 12]:

LEMMA 3.3. *Let A_j be positive operators on a Hilbert space H satisfying $MI \geqslant A_j \geqslant mI > 0$ $(j = 1, 2, \cdots, k)$ for some scalar $M > m > 0$. Let $f(t)$ be a real valued continuous concave function on $[m, M]$ and also let $x_1, x_2, \cdots, x_k$ be any finite number of vectors such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then the following inequality holds;*

$$-\beta(m, M, f) \geqslant f\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right) - \sum_{j=1}^{k}(f(A_j)x_j, x_j) \ (\geqslant 0),$$

where

$$\beta(m, M, f) = \min_{m \leqslant t \leqslant M}\left\{\frac{f(M) - f(m)}{M - m}(t - m) + f(m) - f(t)\right\}.$$

Proof. For the sake of convenience, we cite a proof. Put $\bar{t} = \sum_{j=1}^{k}(A_j x_j, x_j)$ and $\mu = \frac{f(M)-f(m)}{M-m}$. Then we have $m \leqslant \bar{t} \leqslant M$. By the concavity of $f(t)$, we have

$$\begin{aligned}
&\sum_{j=1}^{k}(f(A_j)x_j, x_j) - f\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right) \\
&\geqslant \sum_{j=1}^{k}((\mu(A_j - m) + f(m))x_j, x_j) - f\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right) \\
&= \mu(\bar{t} - m) + f(m) - f(\bar{t}) \\
&\geqslant \beta(m, M, f). \quad \square
\end{aligned}$$

If we put $f(t) = \log t$ in Lemma 3.3, then we have Specht's ratio as the upper bound, (cf. [14]):

LEMMA 3.4. *Let A_j be positive operators on a Hilbert space H satisfying $MI \geqslant A_j \geqslant mI > 0$ $(j = 1, 2, \cdots, k)$ for some scalar $M > m > 0$. Let $x_1, x_2, \cdots, x_k$ be any finite number of vectors such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then*

$$\log M_h(1) \geqslant \log\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right) - \sum_{j=1}^{k}(\log A_j x_j, x_j) \ (\geqslant 0),$$

where $h = \frac{M}{m}$ and $M_h(1)$ is defined as (1.2).

Proof. If we put $f(t) = \log t$ in Lemma 3.3, then we have

$$\beta(m, M, f) = \frac{f(M) - f(m)}{M - m}(t_0 - m) + f(m) - f(t_0),$$

where $t_0 = \frac{M-m}{\log M - \log m}$. Therefore it follows that

$$\begin{aligned}
\frac{f(M)-f(m)}{M-m}&(t_0-m)+f(m)-f(t_0)\\
&= 1+\frac{M\log m - m\log M}{M-m} - \log\left(\frac{M-m}{\log M-\log m}\right)\\
&= 1-\frac{\log h}{h-1} - \log(h-1)+\log(\log h)\\
&= -\log\left(\frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}\right)\\
&= -\log M_h(1). \quad \square
\end{aligned}$$

Since $\log t$ is operator concave, we have

$$\log((1-\lambda)A+\lambda B)-((1-\lambda)\log A+\lambda\log B)\geqslant 0 \tag{3.1}$$

for $A,\ B>0$ and $0\leqslant\lambda\leqslant 1$. By using Lemma 3.4, we estimate the upper bound in (3.1), in which Specht's ratio appears.

LEMMA 3.5. *Let A and B be positive invertible operators on H satisfying $MI \geqslant A, B \geqslant mI > 0$ for some scalar $M > m > 0$. Then*

$$\log M_h(1)\geqslant \log\left((1-\lambda)A+\lambda B\right)-\left((1-\lambda)\log A+\lambda\log B\right)\ \ (\geqslant 0)$$

for all $0\leqslant\lambda\leqslant 1$.

Proof. For fixed $0\leqslant\lambda\leqslant 1$ and unit vector $x\in H$, put $A_1=A$, $A_2=B$, $x_1=\sqrt{1-\lambda}x$ and $x_2=\sqrt{\lambda}x$ in Lemma 3.4. Then we have

$$\log M_h(1)\geqslant\log\left((1-\lambda)(Ax,x)+\lambda(Bx,x)\right)-\left((1-\lambda)(\log Ax,x)+\lambda(\log Bx,x)\right).$$

Hence

$$\begin{aligned}
\log M_h(1)&\geqslant \log\left(((1-\lambda)A+\lambda B)x,x\right)-\left(((1-\lambda)\log A+\lambda\log B)x,x\right)\\
&\geqslant (\log((1-\lambda)A+\lambda B)x,x)-(((1-\lambda)\log A+\lambda\log B)x,x),
\end{aligned}$$

where the second inequality is ensured by the concavity of $\log t$. $\square$

4. Golden-Thompson type inequality

Ando and Hiai [1] show that for every Hermitian matrix H and K and $0\leqslant\lambda\leqslant 1$

$$|||\{e^{tH}\ \sharp_\lambda\ e^{tK}\}^{1/t}|||\leqslant|||e^{(1-\lambda)H+\lambda K}|||$$

holds for all $t>0$ and $|||\{e^{tH}\ \sharp_\lambda\ e^{tK}\}^{1/t}|||$ increases to $|||e^{(1-\lambda)H+\lambda K}|||$ as $t\downarrow 0$ for any unitarily invariant norm $|||\cdot|||$ by using the log-majorization. Related to this, we give another proof to Lemma 2.1, i.e., For $A,\ B>0$ satisfying $MI\geqslant A,B\geqslant mI>0$,

$\log\left((1-\lambda)A^t+\lambda B^t\right)^{1/t}$ decreases to $(1-\lambda)\log A+\lambda\log B$ as $t\downarrow 0$ in the strong operator topology. As a matter of fact, since

$$\log\left((1-\lambda)A^t+\lambda B^t\right)^{1/t}\geqslant(1-\lambda)\log A+\lambda\log B$$

holds for all $t>0$, it follows from Lemma 3.5 that

$$\begin{aligned}0&\leqslant\frac{1}{t}\log\left((1-\lambda)A^t+\lambda B^t\right)-\left((1-\lambda)\log A+\lambda\log B\right)\\&\leqslant\frac{1}{t}\left(\log M_h(t)+(1-\lambda)\log A^t+\lambda\log B^t\right)-\left((1-\lambda)\log A+\lambda\log B\right)\\&=\log M_h(t)^{1/t}.\end{aligned}$$

Moreover, it is known that $M_h(t)^{1/t}\to 1$ as $t\downarrow 0$ by Yamazaki and Yanagida [16], so that we have

$$\lim_{t\downarrow 0}\log\left((1-\lambda)A^t+\lambda B^t\right)^{1/t}=(1-\lambda)\log A+\lambda\log B.$$

We now show Golden-Thompson type inequalities under the usual order in terms of Specht's ratio.

THEOREM 4.1. *Let H and K be selfadjoint operators on a Hilbert space H satisfying $MI\geqslant H,K\geqslant mI$ for some scalar $M>m$. Then*

$$M_h(1)\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}\geqslant e^{(1-\lambda)H+\lambda K}\geqslant M_h(1)^{-1}M_h(t)^{-1/t}\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}$$

holds for all $t>0$ and $0\leqslant\lambda\leqslant 1$, where $h=e^{M-m}$ and $M_h(t)$ is defined as (1.2).

Proof. If we put $A=e^H$ and $B=e^K$ in Lemma 3.5, then we have

$$\log\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}\geqslant(1-\lambda)H+\lambda K\qquad\text{for all } t>0.$$

Since $MI\geqslant(1-\lambda)H+\lambda K\geqslant mI$, it follows from Lemma 3.2 that

$$M_h(1)\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}\geqslant e^{(1-\lambda)H+\lambda K}.$$

Next, since $e^{tM}\geqslant e^{tH},e^{tK}\geqslant e^{tm}$ for $t>0$, then it follows from Lemma 3.5 that

$$\begin{aligned}(1-\lambda)H+\lambda K&=\frac{1}{t}\left((1-\lambda)\log e^{tH}+\lambda\log e^{tK}\right)\\&\geqslant\frac{1}{t}\left(\log\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)-\log M_h(t)\right)\\&=\log\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}M_h(t)^{-1/t}.\end{aligned}$$

Hence it follows from Lemma 3.2 that

$$M_h(1)e^{(1-\lambda)H+\lambda K}\geqslant M_h(t)^{-1/t}\left((1-\lambda)e^{tH}+\lambda e^{tK}\right)^{1/t}.\quad\square$$

In particular, we obtain lower and upper bounds on e^{H+K}.

COROLLARY 4.2. *Let H and K be selfadjoint operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI$ for some scalar $M > m$. Then*

$$M_h(2)\left(\frac{e^{tH}+e^{tK}}{2}\right)^{2/t} \geqslant e^{H+K} \geqslant M_h(2)^{-1}M_h(t)^{-2/t}\left(\frac{e^{tH}+e^{tK}}{2}\right)^{2/t} \tag{4.1}$$

holds for all $t > 0$, where $h = e^{M-m}$.

By virtue of Theorem 4.1, we have an order relation between $e^{(1-\lambda)H+\lambda K}$ and $\left(e^{tH}\natural_\lambda e^{tK}\right)^{1/t}$ under the usual order via Specht's ratio.

THEOREM 4.3. *Let H and K be positive operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI > 0$ for some scalar $M > m > 0$. Then*

$$M_h(1)M_h(t)^{1/t}\left(e^{tH}\natural_\lambda e^{tK}\right)^{1/t} \geqslant e^{(1-\lambda)H+\lambda K} \geqslant M_h(1)^{-1}M_h(t)^{-1/t}\left(e^{tH}\natural_\lambda e^{tK}\right)^{1/t}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$, where $h = e^{M-m}$ and $M_h(t)$ is defined as (1.2).

In particular,

$$M_h(1)^2\left(e^{H}\natural_\lambda e^{K}\right) \geqslant e^{(1-\lambda)H+\lambda K} \geqslant M_h(1)^{-2}\left(e^{H}\natural_\lambda e^{K}\right)$$

and

$$M_h(1)^2\left(e^{2H}\natural_\lambda e^{2K}\right) \geqslant e^{H+K} \geqslant M_h(1)^{-2}\left(e^{2H}\natural_\lambda e^{2K}\right).$$

Proof. By [13], it follows that

$$M_h(t)e^{tH}\natural_\lambda e^{tK} \geqslant e^{tH}\nabla_\lambda e^{tK} \geqslant e^{tH}\natural_\lambda e^{tK} \qquad \text{for all } t > 0.$$

Therefore, we have

$$\log M_h(t)^{1/t}(e^{tH}\natural_\lambda e^{tK})^{1/t} \geqslant \log(e^{tH}\nabla_\lambda e^{tK})^{1/t} \geqslant \log(e^{tH}\natural_\lambda e^{tK})^{1/t} \qquad \text{for all } t > 0.$$

We have this Theorem from this fact and Theorem 4.1. □

We recall the arithmetic and harmonic means: $A\nabla_\lambda B = (1-\lambda)A + \lambda B$ and $A\,!_\lambda B = (A^{-1}\nabla_\lambda B^{-1})^{-1}$ for $\lambda \in [0,1]$.

COROLLARY 4.4. *Let H and K be selfadjoint operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI$ for some scalar $M > m$. Then*

$$M_h(1)\left(e^{tH}\nabla_\lambda e^{tK}\right)^{1/t} \geqslant e^{(1-\lambda)H+\lambda K} \geqslant M_h(1)^{-1}\left(e^{tH}\,!_\lambda\, e^{tK}\right)^{1/t}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$, where $h = e^{M-m}$.

It is well-known that

$$A\nabla_\lambda B \geqslant A\natural_\lambda B \geqslant A\,!_\lambda\, B$$

for $A,\ B > 0$ and $\lambda \in [0,1]$. The following corollary is easily implied by the above corollary, but it is a variant of the arithmetic-geometric mean inequality stated above. Namely it gives an estimation of the chaotically geometric mean by the arithmetic and harmonic means, in which the Specht ratio appears.

COROLLARY 4.5. *Let A and B be positive operators on a Hilbert space H satisfying $MI \geqslant A, B \geqslant mI > 0$ for some scalar $M > m > 0$ and $h = \frac{M}{m}$. Then*

$$M_h(1)\,(A^t \nabla_\lambda B^t)^{\frac{1}{t}} \geqslant A \diamondsuit_\lambda B \geqslant M_h(1)^{-1}\,(A^t\,!_\lambda\, B^t)^{\frac{1}{t}}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$ and particularly

$$M_h(1)\,(A \nabla_\lambda B) \geqslant A \diamondsuit_\lambda B \geqslant M_h(1)^{-1}\,(A\,!_\lambda\, B)\,.$$

Finally we show the following variant of Theorem 4.1 for $t \in (0,1]$ by using Ky Fan-Furuta constant ([5]). It is defined for $M > m > 0$ and $p > 1$ by

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p} \cdot \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

It appears in a complementary inequality of Hölder-McCarthy inequality as follows:

LEMMA 4.6. ([5]). *If $0 < mI \leqslant A \leqslant MI$ and $p > 1$, then*

$$(Ax,x)^p \leqslant (A^p x,x) \leqslant K_+(m,M,p)(Ax,x)^p$$

holds for every unit vector $x \in H$.

THEOREM 4.7. *Let H and K be selfadjoint operators on a Hilbert space H satisfying $MI \geqslant H, K \geqslant mI$ for some scalar $M > m$. Then*

$$M_h(1)\left((1-\lambda)e^{tH} + \lambda e^{tK}\right)^{1/t} \geqslant e^{(1-\lambda)H+\lambda K}$$
$$\geqslant K_+(e^{mt}, e^{Mt}, \frac{1}{t})M_h(t)^{-1/t}\left((1-\lambda)e^{tH} + \lambda e^{tK}\right)^{1/t}$$

holds for all $t > 0$ and $0 \leqslant \lambda \leqslant 1$, where $h = e^{M-m}$ and $M_h(t)$ is defined as (1.2).

Proof. we put $A = e^H$ and $B = e^K$, i.e., $H = \log A$ and $K = \log B$. The right hand side is proved as follows:

$$\begin{aligned}
&(e^{(1-\lambda)\log A+\lambda\log B}x,x)\\
&\quad\geqslant \exp[(((1-\lambda)\log A + \lambda \log B)x,x)]\\
&\quad= \exp[\frac{1}{t}((1-\lambda)(\log A^t x,x) + \lambda(\log B^t x,x))]\\
&\quad\geqslant \exp[\frac{1}{t}(\log(((1-\lambda)A^t + \lambda B^t)x,x) - \log M_h(t))]\\
&\quad= M_h(t)^{-\frac{1}{t}}(((1-\lambda)A^t + \lambda B^t)x,x)^{\frac{1}{t}}\\
&\quad\geqslant M_h(t)^{-\frac{1}{t}}K_+(m^t, M^t, \frac{1}{t})^{-1}(((1-\lambda)A^t + \lambda B^t)^{\frac{1}{t}}x,x).
\end{aligned}$$

Incidentally, the left hand side is shown in Theorem 4.1. □

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