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A GENERAL FORM PRESERVING THE OPERATOR ORDER FOR CONVEX FUNCTIONS

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Abstract. We show a general form preserving the operator order for convex functions based on the Mond-Pečarić method as follows: Let A and B be positive operators on a Hilbert space H satisfying $M1_H \geqslant B \geqslant m1_H > 0$. Let $f(t)$ be a continuous convex function on $[m,M]$. If $g(t)$ is a continuous increasing convex function on $[m,M] \cup \mathrm{Sp}(A)$, then for a given $\alpha > 0$

$$A \geqslant B \geqslant 0 \quad \text{implies} \quad \alpha g(A) + \beta I_H \geqslant f(B)$$

where $\beta = \max_{m \leqslant t \leqslant M}\{f(m) + [(f(M) - f(m))/(M-m)](t-m) - \alpha g(t)\}$. We extend Kantorovich type operator inequalities via the Ky Fan-Furuta constant as applications. Among others, we show the following inequality: If $A \geqslant B > 0$ and $M1_H \geqslant B \geqslant m1_H > 0$, then

$$\frac{M^{p-1}}{m^{q-1}}A^q \geqslant \frac{(q-1)^{q-1}}{q^q}\frac{(M^p - m^p)^q}{(M-m)(mM^p - Mm^p)^{q-1}}A^q \geqslant B^p$$

holds for all $p > 1$ and $q > 1$.

1. Introduction

The Löwner-Heinz theorem asserts that the function $f(t) = t^p$ is operator monotone only for $1 \geqslant p > 0$ though it is monotone increasing for $p > 0$. Furuta [2] showed several extensions of the Kantorovich inequality and applied them to show the following form preserving the operator order.

THEOREM A (Furuta). *Let A and B be positive operators on a Hilbert space H satisfying $M\,1_H \geqslant B \geqslant m\,1_H > 0$, where $M > m > 0$. If $A \geqslant B \geqslant 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geqslant C(m,M;p)A^p \geqslant B^p \qquad \text{for all } p \geqslant 1,$$

where the Ky Fan-Furuta constant $C(m,M;p)$ is defined as

$$C(m,M;p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

Izumino and Nakamoto [3] discussed this extension for any strictly positive convex differentiable real valued function f.

Moreover, Yamazaki [7] showed the following form preserving the operator order.

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THEOREM B (Yamazaki). *Let A and B be positive operators on a Hilbert space H satisfying $M\,1_H \geqslant B \geqslant m\,1_H > 0$, where $M > m > 0$. If $A \geqslant B > 0$, then*

$$A^p + K(m,M;p)\ 1_H \geqslant B^p \qquad \text{for all } p > 1,$$

where

$$K(m,M;p) = \frac{mM^p - Mm^p}{M-m}\{C(m,M;p)^{\frac{1}{p-1}} - 1\}.$$

The object of this paper is to pursue further the study of a form preserving the operator order under general setting, based on inequalities for convex functions due to Mond-Pečarić [6, 5]. As applications, we shall present two variable versions of the operator order by Furuta, Izumino-Nakamoto and Yamazaki.

2. General results

For convenience, we define $\mu_f = \frac{f(M)-f(m)}{M-m}$ for a real valued function f on the interval $[m,M]$. Let $\mathscr{C}([m,M])$ denote the set of a real valued continuous function on $[m,M]$. Also, we introduce the following Furuta constant with two parameters:

$$C_f(m,M;q) = \frac{f(m)-\mu_f m}{1-q}\left(\frac{1-q}{q}\frac{\mu_f}{f(m)-\mu_f m}\right)^q,$$

where q is a real number such that $\frac{1-q}{q}\frac{\mu_f}{f(m)-\mu_f m} > 0$. Particularly, $C_{t^p}(m,M;p) = C(m,M;p)$ for the power function $f(t) = t^p$.

THEOREM 1. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$, where $M > m > 0$. Let $f \in \mathscr{C}([m,M])$ be a convex function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathsf{Sp}(A)$. Suppose that either of the following conditions holds:* (a) *g is increasing convex on U, or* (b) *g is decreasing concave on U. If $A \geqslant B > 0$, then for a given $\alpha \in \mathbf{R}_+$ in the case* (a) *or $\alpha \in \mathbf{R}_-$ in the case* (b)

$$\alpha g(A) + \beta\,1_H \geqslant f(B) \tag{1}$$

holds for $\beta = \max_{m \leqslant t \leqslant M}\{f(m) + \mu_f(t-m) - \alpha g(t)\}$.

To prove Theorem 1, we need the following results [6, Theorem 1] and [5, Theorem 4]:

LEMMA. *Let A be a positive operator on a Hilbert space H satisfying $\mathsf{Sp}(A) \subseteq [m,M]$, where $M > m > 0$ and $f \in \mathscr{C}([m,M])$ be a convex function. Then*

$$(f(A)x,x) \geqslant f((Ax,x)) \tag{2}$$

holds for every unit vector $x \in H$. Moreover, let $g \in \mathscr{C}([m,M])$. Then for any number $\alpha \in \mathbf{R}$

$$\alpha g((Ax,x)) + \beta \geqslant (f(A)x,x) \tag{3}$$

holds for every unit vector $x \in H$ where $\beta = \max_{m \leqslant t \leqslant M}\{f(m) + \mu_f(t-m) - \alpha g(t)\}$.

Proof of Theorem 1. Let $x \in H$ be any unit vector. By the convexity of αg, it follows from (2) in Lemma that $\alpha(g(A)x,x) \geqslant \alpha g((Ax,x))$. By the increase of αg, we have $\alpha g((Ax,x)) \geqslant \alpha g((Bx,x))$. Therefore, combining two inequalities above and (3) in Lemma we have

$$\alpha(g(A)x,x)+\beta \geqslant \alpha g((Ax,x))+\beta \geqslant \alpha g((Bx,x))+\beta \geqslant (f(B)x,x). \quad \square$$

REMARK 2. Assume that conditions of Theorem 1 hold and let αg be strictly convex differentiable on $[m,M]$. Then β can be written explicitly as $\beta = f(m) + \mu_f(t_0-m) - \alpha g(t_0)$, where t_0 is defined as the unique solution of $g'(t) = \frac{\mu_f}{\alpha}$ when $\alpha g'(m) \leqslant \mu_f \leqslant \alpha g'(M)$, otherwise t_0 is defined as M or m according as $\mu_f > \alpha g'(M)$ or $\alpha g'(m) > \mu_f$.

As a matter of fact, put $h(t) = f(m)+\mu_f(t-m)-\alpha g(t)$ and $\beta = \max_{m\leqslant t\leqslant M} h(t)$. If αg is strictly convex differentiable on $[m,M]$, then $\alpha g'$ is strictly increasing, so $h'(t)$ is strictly decreasing. Hence if $\alpha g'(m) \leqslant \mu_f \leqslant \alpha g'(M)$, then the equation $h'(t)=0$ has exactly one solution t_0 in the interval $[m,M]$. Since the maximum value of $h(t)$ on $[m,M]$ is attained for t_0, we have $\max_{m\leqslant t\leqslant M} h(t) = h(t_0) = \beta$.

Next, if $\mu_f > \alpha g'(M)$, then $h(t)$ is increasing on $[m,M]$ since $h'(t)>0$. Therefore, $\max_{m\leqslant t\leqslant M} h(t) = h(M) = \beta$. Similarly, if $\alpha g'(m) > \mu_f$, then $\max_{m\leqslant t\leqslant M} h(t) = h(m) = \beta$.

Let us consider a complementary result to Theorem 1.

THEOREM 3. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(A) \subseteq [m,M]$, where $M > m > 0$. Let $f \in \mathscr{C}([m,M])$ be a concave function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathsf{Sp}(B)$. Suppose that either of the following conditions holds:* (a) *g is increasing concave on U, or* (b) *g is decreasing convex on U. If $A \geqslant B > 0$, then for a given $\alpha \in \mathbf{R}_+$ in the case* (a) *or $\alpha \in \mathbf{R}_-$ in the case* (b)

$$f(A) \geqslant \alpha g(B) + \beta\, 1_H$$

holds for $\beta = \min_{m\leqslant t\leqslant M}\{f(m)+\mu_f(t-m)-\alpha g(t)\}$.

Proof. It follows that

$$\begin{aligned}
(f(A)x,x) &\geqslant \alpha g((Ax,x))+\beta && \text{by the concavity of } f\\
&\geqslant \alpha g((Bx,x))+\beta && \text{by } A\geqslant B \text{ and the increase of } \alpha g\\
&\geqslant \alpha(g(B)x,x)+\beta && \text{by the concavity of } \alpha g
\end{aligned}$$

for every unit vector $x \in H$. $\square$

REMARK 4. Assume that conditions of Theorem 3 hold and let αg be strictly concave differentiable on $[m,M]$. Then β can be written explicitly as $\beta = f(m) + \mu_f(t_0-m) - \alpha g(t_0)$, where t_0 is defined as the unique solution of $g'(t) = \frac{\mu_f}{\alpha}$ when $\alpha g'(M) \leqslant \mu_f \leqslant \alpha g'(m)$, otherwise t_0 is defined as M or m according as $\mu_f < \alpha g'(M)$ or $\alpha g'(m) < \mu_f$.

If we put $\alpha = 1$ in Theorem 1 and Theorem 3, then we have the following corollary:

COROLLARY 5. *Let A and B be positive operators on a Hilbert space H satisfying* $\mathsf{Sp}(B) \subseteq [m,M]$ (*resp.* $\mathsf{Sp}(A) \subseteq [m,M]$), *where* $M > m > 0$. *Let* $f \in \mathscr{C}([m,M])$ *be a convex* (*resp. concave*) *function and* $g \in \mathscr{C}(U)$ *an increasing convex* (*resp. concave*) *function, where* $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. *If* $A \geqslant B > 0$, *then*

$$g(A) + \beta\, 1_H \geqslant f(B) \qquad (\textit{resp.} \quad f(A) \geqslant g(B) + \beta\, 1_H)$$

holds for $\beta = \max_{m \leqslant t \leqslant M}\{(f(m)+\mu_f(t-m))-g(t)\}$ (*resp.* $\beta = \min_{m \leqslant t \leqslant M}\{(f(m)+\mu_f(t-m))-g(t)\}$).

If we choose α such that $\beta = 0$ in Theorem 1 and Theorem 3, we have the following corollary:

COROLLARY 6. *Let A and B be positive operators on a Hilbert space H satisfying* $\mathsf{Sp}(B) \subseteq [m,M]$ (*resp.* $\mathsf{Sp}(B) \subseteq [m,M]$), *where* $M > m > 0$. *Let* $f \in \mathscr{C}([m,M])$ *be a convex* (*resp. concave*) *function and* $g \in \mathscr{C}(U)$, *where* $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. *Suppose that either of the following conditions holds:*

(i) *g is increasing convex on U, $g > 0$ on $[m,M]$ and $f(m) > 0, f(M) > 0$,*
(ii) *g is increasing convex on U, $g < 0$ on $[m,M]$ and $f(m) < 0, f(M) < 0$,*
(iii) *g is decreasing concave on U, $g > 0$ on $[m,M]$ and $f(m) < 0, f(M) < 0$,*
(iv) *g is decreasing concave on U, $g < 0$ on $[m,M]$ and $f(m) > 0, f(M) > 0$.*

If $A \geqslant B > 0$, *then*

$$\alpha_+ g(A) \geqslant f(B) \qquad (\textit{resp.} \quad f(A) \geqslant \alpha_- g(B))$$

holds for

$$\alpha_+ = \max_{m \leqslant t \leqslant M}\left\{\frac{f(m)+\mu_f(t-m)}{g(t)}\right\} \quad \left(\textit{resp.} \quad \alpha_- = \min_{m \leqslant t \leqslant M}\left\{\frac{f(m)+\mu_f(t-m)}{g(t)}\right\}\right)$$

in case (i) *and* (iii), *or*

$$\alpha_+ = \min_{m \leqslant t \leqslant M}\left\{\frac{f(m)+\mu_f(t-m)}{g(t)}\right\} \quad \left(\textit{resp.} \quad \alpha_- = \max_{m \leqslant t \leqslant M}\left\{\frac{f(m)+\mu_f(t-m)}{g(t)}\right\}\right)$$

in case (ii) *and* (iv).

Proof. If we put $\beta = \max_{m \leqslant t \leqslant M}\{f(m)+\mu_f(t-m)-\alpha g(t)\} = 0$, then we have $\alpha = \frac{f(m)+\mu_f(t_0-m)}{g(t_0)}$ for some $t_0 \in [m,M]$. In fact, it follows that $\alpha = \max_{m \leqslant t \leqslant M}\left\{\frac{f(m)+\mu_f(t-m)}{g(t)}\right\}$, because $0 \geqslant f(m)+\mu_f(t-m)-\alpha g(t)$ for all $t \in [m,M]$. The remainder of the proof is the same as in the proof in Theorem 1. □

REMARK 7. Assume that conditions of Corollary 6 hold. Moreover, suppose that either of the following additional conditions holds: (a) g is strictly convex differentiable in case (i) or (ii), or (b) g is strictly concave differentiable in case (iii) or (iv). Then a value of $\alpha_\pm$ may be determined more precisely as follows:

$$\alpha_\pm = \frac{f(m)+\mu_f(t_o-m)}{g(t_o)},$$

where $t_o \in [m,M]$ is defined as the unique solution of $\mu_f g(t) = g'(t)(f(m)+\mu_f(t-m)$ if $f(m)g'(m)/g(m) \leqslant \mu_f \leqslant f(M)g'(M)/g(M)$, otherwise t_o is defined as M or m according as $\mu_f > f(M)g'(M)/g(M)$ or $f(m)g'(m)/g(m) < \mu_f$.

3. Preserving the operator order by convex functions

As applications of our general results, we show function preserving the operator order. We recall that $f \in \mathscr{C}([0,\infty\rangle)$ is operator monotone if and only if it is operator concave. Therefore, if f is a convex function, then it cannot be operator monotone. We show that a convex function preserves the order in the following sense:

THEOREM 8. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$. Let $f \in \mathscr{C}(U)$ be a strictly convex increasing differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A)$. If $A \geqslant B > 0$, then for a given $\alpha \in \mathbf{R}_+$*

$$\alpha f(A) + \beta 1_H \geqslant f(B)$$

holds for $\beta = f(m) + \mu_f(t_0 - m) - \alpha f(t_0)$ and t_0 is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(m) \leqslant \mu_f/\alpha \leqslant f'(M)$, otherwise t_0 is defined as M or m according as $f'(M) < \mu_f/\alpha$ or $\mu_f/\alpha < f'(m)$.

Proof. Put $g \equiv f$ in Theorem 1. As a matter of fact, by the convexity and the increase of f on U we have

$$(f(B)x,x) \leqslant \alpha f((Bx,x)) + \beta \leqslant \alpha f((Ax,x)) + \beta \leqslant \alpha(f(A)x,x) + \beta$$

for every unit vector $x \in H$. $\square$

Though a concave increasing function is not always operator monotone, we have the following theorem which is a complementary result to Theorem 8.

THEOREM 9. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(A) \subseteq [m,M]$. Let $f \in \mathscr{C}(U)$ be a strictly concave increasing differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(B)$. If $A \geqslant B > 0$, then for a given $\alpha \in \mathbf{R}_+$*

$$f(A) \geqslant \alpha f(B) + \beta 1_H$$

holds for $\beta = f(m) + \mu_f(t_0 - m) - \alpha f(t_0)$ and t_0 is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(M) \leqslant \mu_f/\alpha \leqslant f'(m)$, otherwise t_0 is defined as M or m according as $f'(M) > \mu_f/\alpha$ or $\mu_f/\alpha > f'(m)$.

REMARK 10. If we put $\alpha = 1$ in Theorem 8 and Theorem 9, then we have the following claim: Let $f \in \mathscr{C}([m,M])$ be a strictly convex (resp. concave) increasing differentiable function. If $M 1_H \geqslant A \geqslant B \geqslant m 1_H > 0$, where $M > m > 0$, then

$$f(A) + \beta 1_H \geqslant f(B) \quad (\text{resp.} \quad f(A) \geqslant f(B) + \beta 1_H)$$

holds for $\beta = f(m) + \mu_f(t_0 - m) - f(t_0)$ and t_0 is exactly one solution the equation $f'(t) = \mu_f$ in the interval $[m,M]$. Indeed, since f is strictly convex or concave, it follows that $f'(m) \leqslant \mu_f \leqslant f'(M)$ (resp. $f'(M) \leqslant \mu_f \leqslant f'(m)$).

If we choose α such that $\beta = 0$ in Theorem 8, we have the following corollary (cf. [3 Theorem 2.2]).

COROLLARY 11. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$ (resp. $\mathsf{Sp}(A) \subseteq [m,M]$, where $M > m > 0$. Let $f \in \mathscr{C}(U)$ be a increasing convex (resp. concave) function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. Suppose that either of the following conditions holds: (i) $f > 0$ on $[m,M]$ or (ii) $f < 0$ on $[m,M]$. If $A \geqslant B > 0$, then*

$$\alpha_+ f(A) \geqslant f(B) \quad (\textit{resp. } f(A) \geqslant \alpha_- f(B))$$

holds for

$$\alpha_+ = \max_{m \leqslant t \leqslant M} \left\{ \frac{f(m) + \mu_f(t-m)}{f(t)} \right\} \quad \left(\textit{resp. } \alpha_- = \min_{m \leqslant t \leqslant M} \left\{ \frac{f(m) + \mu_f(t-m)}{f(t)} \right\} \right)$$

in case (i)*, or*

$$\alpha_+ = \min_{m \leqslant t \leqslant M} \left\{ \frac{f(m) + \mu_f(t-m)}{f(t)} \right\} \quad \left(\textit{resp. } \alpha_- = \max_{m \leqslant t \leqslant M} \left\{ \frac{f(m) + \mu_f(t-m)}{f(t)} \right\} \right)$$

in case (ii).

We shall show a functional order version of Theorem A (Furuta).

COROLLARY 12. *Let A and B be positive invertible operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$, where $M > m > 0$. Let $f \in \mathscr{C}(U)$ be a strictly convex increasing twice differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. Let (i) $f > 0$ on $[m,M]$ or (ii) $f < 0$ on $[m,M]$. If $A \geqslant B > 0$, then*

$$\frac{f'(M)}{f'(m)} f(A) \geqslant \alpha f(A) \geqslant f(B),$$

where $\alpha = \frac{f(m)+\mu_f(t_0-m)}{f(t_0)}$ and $t_0 \in \langle m,M \rangle$ is the unique solution of $\mu_f f(t) = f'(t)(f(m) + \mu_f(t-m))$.

Proof. By the assumption of f, we have $\mu_f \leqslant f'(M)$ and $0 < f'(m) \leqslant f'(t_0)$, where $t_0 \in \langle m,M \rangle$ is such that $\mu_f f(t_0) = f'(t_0)(f(m) + \mu_f(t_0 - m))$. Then it follows that

$$0 < \alpha = \frac{f(m) + \mu_f(t_0 - m)}{f(t_0)} = \frac{\mu_f}{f'(t_0)} \leqslant \frac{f'(M)}{f'(m)}.$$

Therefore Corollary 11 implies $\frac{f'(M)}{f'(m)} f(A) \geqslant \alpha f(A) \geqslant f(B)$. $\square$

REMARK 13. If we put $f(t) = t^p$ in Corollary 12, then we have $\frac{f'(M)}{f'(m)} = \left(\frac{M}{m}\right)^{p-1}$.

If we put $f(t) = e^t$ in Theorem 8, we have the following corollary.

COROLLARY 14. *Let A and B be selfadjoint operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$, where $M > m$. If $A \geqslant B$, then for a given $\alpha \in \mathbf{R}_+$*

$$\alpha e^A + \beta 1_H \geqslant e^B$$

where

$$\beta = \begin{cases} \frac{e^M-e^m}{M-m}\log\frac{e^M-e^m}{\alpha(M-m)} + \frac{(M+1)e^m-(m+1)e^M}{M-m} & \text{if } \ m \leqslant \log\frac{e^M-e^m}{\alpha(M-m)} \leqslant M \\ (1-\alpha)e^M & \text{if } \ M < \log\frac{e^M-e^m}{\alpha(M-m)} \\ (1-\alpha)e^m & \text{if } \ \log\frac{e^M-e^m}{\alpha(M-m)} < m. \end{cases}$$

In particular,

$$\frac{e^M-e^m}{M-m}\exp\left(\frac{(M+1)e^m-(m+1)e^M}{e^M-e^m}\right)e^A \geqslant e^B,$$

$$e^A + \left(\frac{(M+1)e^m-(m+1)e^M}{M-m} + \frac{e^M-e^m}{M-m}\log\left(\frac{e^M-e^m}{M-m}\right)\right)1_H \geqslant e^B.$$

If we put $f(t) = t^p$ for $p \in \mathbf{R} \setminus [0,1\rangle$ and $g(t) = t^q$ for $q > 1$ in Theorem 1, then we have the following corollary.

COROLLARY 15. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$, where $M > m > 0$. If $A \geqslant B > 0$, then for a given $\alpha \in \mathbf{R}_+$*

$$\alpha A^q + \beta\, 1_H \geqslant B^p, \qquad \textit{for all } \ p \in R \setminus [0,1\rangle \ \textit{ and } q > 1,$$

where

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\frac{M^p-m^p}{M-m}\right)^{\frac{q}{q-1}} + \frac{Mm^p-mM^p}{M-m} & \text{if } \ qm^{q-1} \leqslant \frac{M^p-m^p}{\alpha(M-m)} \leqslant qM^{q-1} \\ \max\{M^p-\alpha M^p, m^p-\alpha m^p\} & \text{otherwise.} \end{cases}$$

In particular,

$$\alpha A^p + \beta\, 1_H \geqslant B^p \qquad \textit{for all } \ p \geqslant 1$$

where

$$\beta = \begin{cases} \alpha(p-1)\left(\frac{1}{\alpha p}\frac{M^p-m^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{Mm^p-mM^p}{M-m} & \text{if } \ pm^{p-1} \leqslant \frac{M^p-m^p}{\alpha(M-m)} \leqslant pM^{p-1}, \\ \max\{M^p-\alpha M^p, m^p-\alpha m^p\} & \text{otherwise.} \end{cases}$$

REMARK 16. We have Theorem B (Yamazaki) if we put $\alpha = 1$ in the second inequality of Corollary 15.

4. Preserving the operator order associated with Kantorovich inequality

Finally, as applications of our general results, we show a form preserving the operator order associated with Kantorovich inequality via the Furuta constant. The following theorem is a two variable version of Theorem A (Furuta).

THEOREM 17. *Let A and B be positive operators on a Hilbert space H satisfying $\mathsf{Sp}(B) \subseteq [m,M]$, where $M > m > 0$. If $A \geqslant B > 0$, then*

$$\frac{M^{p-1}}{m^{q-1}}A^q \geqslant C_+(m,M;p,q)A^q \geqslant B^p \qquad \textit{for all } p > 1 \textit{ and } q > 1 \tag{4}$$

where

$$C_+(m,M;p,q) = \begin{cases} C(m,M;p,q) & \text{if } \; qm^{p-1} \leqslant \frac{M^p-m^p}{M-m} \leqslant qM^{p-1} \\ m^{p-q} & \text{if } \; \frac{M^p-m^p}{M-m} < qm^{p-1} \\ M^{p-q} & \text{if } \; qM^{p-1} < \frac{M^p-m^p}{M-m} \end{cases} \tag{5}$$

and $C(m,M;p,q) = C_{t^p}(m,M;q)$ *is the Furuta constant.*

Proof. First, we shall prove that the inequality

$$h^{p-1} \geqslant \frac{(q-1)^{q-1}}{q^q} \frac{(h^p-1)^q}{(h-1)(h^p-h)^{q-1}} \tag{6}$$

holds for all $p > 1$, $q > 1$ and $h > 1$ with $q \leqslant \frac{h^p-1}{h-1} \leqslant qh^{p-1}$. Put $l(t) = \mu t + f(m) - \mu m$, $t_0 = \frac{q}{q-1}\frac{-f(m)-\mu m}{\mu}$ and $g(t) = t^q$, where $\mu = \frac{h^p-1}{h-1}$, $f(m) - \mu m = \frac{h-h^p}{h-1}$. Since $p,q,h > 1$, it follows that $\mu \geqslant 0$ and $f(m) - \mu m \leqslant 0$. We see that the condition $q \leqslant \frac{h^p-1}{h-1} \leqslant qh^{p-1}$ is equivalent to the condition $1 \leqslant t_0 \leqslant h$, and

$$\max_{1 \leqslant t_0 \leqslant h} \left\{ \frac{l(t)}{g(t)} \right\} = \frac{\mu_f t_0 + f(m) - \mu_f m}{t_0^q} = \frac{(q-1)^{q-1}}{q^q} \frac{(h^p-1)^q}{(h-1)(h^p-h)^{q-1}}.$$

Put $l_1(t) = \frac{\mu t + f(m) - \mu m}{t}$ and $g_1(t) = t^{q-1}$. Then $l_1(t)$ and $g_1(t)$ are increasing, so we have $l_1(h) \geqslant l_1(t_0) > 0$ and $g_1(t_0) \geqslant g_1(1) > 0$. Hence it follows that

$$h^{p-1} = \frac{l_1(h)}{g_1(1)} \geqslant \frac{l_1(t_0)}{g_1(t_0)} = \frac{\mu_f t_0 + f(m) - \mu_f m}{t_0} \frac{1}{t_0^{q-1}},$$

as desired inequality (6).

Now, we prove the first inequality in (4). Put $h = \frac{M}{m} > 1$. If $qm^{p-1} \leqslant \frac{M^p-m^p}{M-m} \leqslant qM^{p-1}$, then this inequality follows from (6)

$$\frac{M^{p-1}}{m^{q-1}} = m^{q-p} h^{p-1} \geqslant m^{q-p} \frac{(q-1)^{q-1}}{q^q} \frac{(h^p-1)^q}{(h-1)(h^p-h)^{q-1}} = C_{t^p}(m,M;q).$$

Otherwise, we see that $\frac{M^{p-1}}{m^{q-1}} \geqslant \frac{m^{p-1}}{m^{q-1}}$ and $\frac{M^{p-1}}{m^{q-1}} \geqslant \frac{M^{p-1}}{M^{q-1}}$. Therefore we have the first inequality in (4). We have the second inequality in (4) if we choose α such that $\beta = 0$ in the first inequality of Corollary 15. Then it follows that α coincides with $C_+(m,M;p,q)$ in (4).

REMARK 18. If we put $q = p$ in (6), then the assumption is automatically satisfied and so the constant $C_+(m,M;p,q)$ coincides with the Ky Fan-Furuta constant $C(m,M;p)$. Therefore we have Theorem A (Furuta).

As a generalization of Theorem 17, we show the following theorem.

THEOREM 19. *Let A and B be positive operators on a Hilbert space H satisfying* $\mathsf{Sp}(B) \subseteq [m,M]$*, where $M > m > 0$. If $A \geqslant B > 0$, then*

$$C_+(m^r, M^r; \frac{p-1+r}{r}, \frac{q-1+r}{r})A^q \geqslant B^p \qquad \text{for all } p>1,\ q>1 \text{ and } r>1,$$

where $C_+(m,M;p,q)$ is defined as (5).

To prove Theorem 19, we need the following Furuta inequality [1]:

THEOREM F (the Furuta inequality). *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \qquad \text{and} \qquad \left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

Proof of Theorem 19. By Furuta inequality, it follows that $A \geqslant B$ ensures that $(B^{\frac{r}{2}}A^qB^{\frac{r}{2}})^{\frac{1+r}{q+r}} \geqslant B^{1+r}$ for all $q>1$ and $r>0$. Put $A_1 = (B^{\frac{r}{2}}A^qB^{\frac{r}{2}})^{\frac{1+r}{q+r}}$ and $B_1 = B^{1+r}$, then $A_1 \geqslant B_1 > 0$ and $M^{1+r}1_H \geqslant B_1 \geqslant m^{1+r}1_H > 0$. Applying Theorem 17 to A_1 and B_1, we have

$$C_+(m^{1+r}, M^{1+r}; p_1, q_1)A_1^{q_1} \geqslant B_1^{p_1} \qquad \text{for all } p_1>1 \text{ and } q_1>1.$$

Put $p_1 = \frac{p+r}{1+r} > 1$ and $q_1 = \frac{q+r}{1+r} > 1$, then we have

$$C_+\left(m^{1+r}, M^{1+r}; \frac{p+r}{1+r}, \frac{q+r}{1+r}\right) B^{\frac{r}{2}}A^qB^{\frac{r}{2}} \geqslant B^{p+r} \qquad \text{for all } p>1 \text{ and } q>1.$$

Multiply $B^{\frac{-r}{2}}$ on both sides and replace r by $r-1$, then it follows that

$$C_+\left(m^r, M^r; \frac{p-1+r}{r}, \frac{q-1+r}{r}\right) A^q \geqslant B^p \qquad \text{for all } p>1,\ q>1 \text{ and } r>1. \quad \square$$

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