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ABOUT VON NEUMMANN'S INEQUALITY FOR d–CONTRACTIONS

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1. Introduction

A d-contraction is a d-tuple $(T_1, \ldots, T_d)$ of mutually commuting operators acting on a common Hilbert space H satisfying:

$$\|T_1\xi_1 + \ldots + T_d\xi_d\|^2 \leqslant \|\xi_1\|^2 + \ldots + \|\xi_d\|^2$$

for every $\xi_1, \ldots, \xi_d \in H$. These are the higher dimensional counterparts of contraction.

A distinguished d-contraction, which acts on a new H^2 space associated with the unit ball B_d in complex d-space, is the d-shift $(S_1, \ldots, S_d)$. This spaces is a natural generalization of the familiar Hardy space of the unit disk. We will show, briefly, the way of defining this space.

$\mathscr{P}$ will denote the algebra of all complex holomorphic polynomials f in the variable $z = (z_1, \ldots, z_d)$. For each $n = 1, 2, \ldots$ we write E^n for the symmetric tensor product of n copies of $E = \mathbf{C}^d \cdot E^0$ is defined as the one dimensional vector space $\mathbf{C}$ with its usual inner product.

It is shown that: every polynomial $f : E \to \mathbf{C}$ takes the form:

$$f(z) = \sum_{k=0}^{n} < z^k, \, \xi_k >_{E^k}, \; z \in E$$

where: $z^k = \underbrace{z \oplus \ldots \oplus z}_{k} \in E^k, \; z \in E$.

We define a Hilbert seminorm on $\mathscr{P}$ as follows:

$$\|f\|^2 = \|\xi_0\|^2 + \|\xi_1\|^2 + \cdots + \|\xi_n\|^2 \tag{1.1}$$

H^2_d is defined as the Hilbert space obtained by completing $\mathscr{P}$ in the norm (1.1).

When there is no possibility of confusion concerning the dimension we will abbreviate H^2_d with the simpler H^2.

More information can be found in the paper [1] of W. Arverson.

By a multiplier of H^2 we mean a complex-valued function $f : B_d \mapsto \mathbf{C}$ with the property:

$$f \cdot H^2 \subset H^2.$$

The algebra of all multipliers is denoted $\mathscr{M}$.

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Every $f \in \mathscr{M}$ defines a unique bounded operator M_f on H^2 by way of:

$$M_f : g \in H^2 \mapsto f \cdot g \in H^2.$$

The natural norm in $\mathscr{M}$:

$$\|f\|_{\mathscr{M}} = \sup\{\|f \cdot g\| : g \in H^2, \|g\| \leqslant 1\}$$

satisfies:

$$\|f\|_{\mathscr{M}} = \|M_f\|.$$

We turn now to the definition of the d-dimensional analogue of the unilateral shift.

Let $e_1, e_2, \dots, e_d$ be an orthonormal basis for $E = \mathbf{C}^d$, and define $z_1, z_2, \dots, z_d \in \mathscr{P}$ by:

$$z_k(z) =< z, e_k >, \ z \in \mathbf{C}^d.$$

Such a d-tuple of linear functionals will be called a system of coordinate functions.

Let $z_1, z_2, \dots, z_d$ be a system of cordinate functions for $\mathbf{C}^d$ and let $S_k = M_{z_k}$, $k = 1, 2, \dots d$.

The d-tuple of operators:

$$\overline{S} = (S_1, S_2, \dots, S_d)$$

is called the d-dimensional shift or, briefly, the d-shift.

Perhaps the most natural generalization of von Neumann's inequality for d-dimensional operator theory would make the following assertion.

Let $\overline{T} = (T_1, \dots, T_d)$ be a d-contraction and let $f = f(z_1, \dots, z_d)$ be a polynomial in d-complex variables $z_1, \dots, z_d$. Then:

$$\|f(T_1, \dots, T_d)\| \leqslant \sup_{\|z\| \leqslant 1} |f(z_1, \dots, z_d)\|.$$

In [1], W. Arveson show that this inequality fails rather spectacuralrly for the d-shift, in that there is no constant k for wich:

$$\|f(S_1, \dots, S_d)\| \leqslant \sup_{\|z\| \leqslant 1} |f(z_1, \dots, z_d)|,$$

holds for all polynomials f.

W. Arverson in [1] establishes an appropiate version for von Neumann's inequality for dimension $d \geqslant 2$.

THEOREM 1.1. *Let $\overline{T} = T_1, \dots, T_d$ be an arbitrary d-contraction acting on a Hilbert space H. Then for every polynomial f in d complex variables we have:*

$$\|f(T_1, \dots, T_d)\| \leqslant \|f\|_{\mathscr{M}},$$

$\|f\|_{\mathscr{M}}$ being the norm of f in the multiplier algebra $\mathscr{M}$ of H^2.

Proof. The proof of this theorem given by W. Arveson in [1] is based on the notion of A-morphism. In this paper we will give a more simple proof, without using the notion of A-morphism, based on dilations, as in the case of proving von Neumann's inequality for a single contraction.

2. Proof of the theorem 1.1

REMARK 2.1. Every d-contraction $(T_1,\ldots,T_d)$ in $B(H)$ gives rise to a normal completely positive map P on $B(H)$ by way of:

$$P(A) = T_1AT_1^* + \ldots + T_dAT_d^*,\ A \in B(H).$$

Because:

$$T_1T_1^* + \ldots + T_dT_d^* \leqslant 1$$

we have $P(1) \leqslant 1$, and in fact the sequence $A_n = P^n(1)$ is decreasing: $A_0 = 1 \geqslant A_1 \geqslant A_2 \geqslant \ldots \geqslant 0$. Thus:

$$A_\infty = \lim_{n\to\infty} P^n(1)$$

exists as a limit in the strong operator topology and satisfies: $0 \leqslant A_\infty \leqslant 1$.

A d-contraction $\overline{T} = (T_1,\ldots,T_d)$ is called null if $A_\infty = 0$. Notice that if the row norm of $\overline{T}$ is less than 1, i.e., $T_1T_1^* + \ldots T_dT_d^* \leqslant r\cdot 1$ fore some $0<r<1$, then: $\|P\| = \|P(1)\| \leqslant r < 1$ and hence $\overline{T}$ is a null d-contraction.

THEOREM 2.2. *[1] Let $(T_1,\ldots,T_d)$ be a d-contraction on a Hilbert space H, define the operator*

$$\Delta = (1 - T_1T_1^* - \ldots - T_dT_d^*)^{1/2}$$

and the subspace $K = \overline{\Delta H}$. Let E be a d-dimensional Hilbert space and let:

$$\mathscr{F}_+(E) = \mathbf{C} \oplus E \oplus E^2 \oplus \ldots$$

be the symmetric Fock space over E.

Then for every orthonormal basis $e_1,\ldots,e_d$ for E there is a unique bounded operator $L : \mathscr{F}_+(E)\otimes K \mapsto H$ satisfying

$$L(1\otimes\xi) = \Delta\xi \text{ and}$$

$$L(e_{i_1}e_{i_2}\ldots e_{i_n}\otimes\xi) = T_{i_1}\ldots T_{i_n}\Delta\xi \tag{2.1}$$

for every $i_1,\ldots,i_n \in \{1,2,\ldots d\}, n = 1,2,\ldots$.

In general we have $\|L\| \leqslant 1$, and if $(T_1,\ldots,T_d)$ is a null d-tuple, then L is a coisometry.

REMARK 2.3. We may consider the d-shift $(S_1,\ldots,S_d)$ is defined on $\mathscr{F}_+(E)$ by:

$$S_k\xi = e_k\xi,\ k = 1,2,\ldots,d,$$

where $e_k\xi$ denotes the projection of $e_k\otimes\xi \in \mathscr{F}(E)$ to the symmetric subspace $\mathscr{F}_+(E)$. (2.1) implies that:

$$L(f(S_1,\ldots,S_d)\otimes 1_K) = f(T_1,\ldots,T_d)L \tag{2.2}$$

for every polynomial f in d variables.

In general, a dilation theorem is a result wich characterizes some class of maps into $B(H)$ as a compressions to H of "nicer" maps into $B(K)$, where K is a Hilbert space containing H.

REMARK 2.4. If $(T_1, \dots, T_d)$ is a null d-contraction, then L is a coisometry. In this case we may identify H with the subspace L^*H of $\mathscr{F}_+(E) \otimes K$.

With this identification, L becomes the projection of $\mathscr{F}_+(E) \otimes K$ onto H, P_H. Thus, we see that (2.2) implies:

$$f(T_1, \dots, T_d) = P_H(f(S_1, \dots, S_d) \otimes 1_K)|H, \tag{2.3}$$

for every polynommial f in d variables.

So that when $(T_1, \dots, T_d)$ is a null d-contraction, theorem 2.2[1] is a dilation result in the sense described above.

REMARK 2.5. If $(T_1, \dots T_d)$ is a null d-contraction, (2.3) implies:

$$\|f(T_1, \dots, T_d)\| \leqslant \|f\|_{\mathscr{M}} \tag{2.4}$$

for every polynomial f in d varibles.

REMARK 2.6. The general case is deduced from this by a simple device. Let $\overline{T} = (T_1, \dots, T_d)$ be any d-contraction, choose a number r so that: $0 < r < 1$, and set:

$$\overline{T}_r = (rT_1, \dots, rT_d).$$

The row norm of the d-tuple $\overline{T}_r$ is at most r, hence $\overline{T}_r$ is a null d-contraction. (2.4) implies:

$$\|f(rT_1, \dots, rT_d)\| \leqslant \|f\|_{\mathscr{M}},$$

for every polynomial f in d variables, whence obtain:

$$\|f(rT_1, \dots, rT_d)\| \leqslant \|f\|_{\mathscr{M}},$$

for every polynomial f in d variables.

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