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A LAGRANGIAN DUAL METHOD FOR SOLVING VARIATIONAL INEQUALITIES

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Abstract. In this paper we consider a variational inequality problem (VIP) defined by a maximal monotone operator and a feasible set defined by convex inequality constraints and bounds on the variables. A Lagrangian dual method for solving this problem is presented and its convergence is proved.

1. Introduction

Variational inequalities arise in different mathematical problems, for example, in nonlinear optimization; they are connected with operator theory, especially monotone operators, etc.

Given an operator T, point to set in general, and a closed convex subset X of $\mathbf{R}^n$. The *variational inequality problem* VIP (T, X) consists in finding a pair $x^* \in X$ and $g^* \in T(x^*)$ such that

$$\langle g^*, x - x^* \rangle \geqslant 0 \quad \forall x \in X, \tag{1}$$

where $\langle .,. \rangle$ denotes the usual inner product of $\mathbf{R}^n$.

When T is single-valued, VIP (T, X) consists in finding $x \in X$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0 \quad \forall x \in X, \tag{2}$$

where $T : \mathbf{R}^n \to \mathbf{R}^n$ and X is a nonempty, closed and convex set in $\mathbf{R}^n$.

If the constraint set X is the nonnegative orthant $\mathbf{R}^n_+ \equiv \{x \in \mathbf{R}^n : x \geqslant 0\}$ of $\mathbf{R}^n$, then the VIP reduces to the complementarity problem (CP).

Recall that the *nonlinear complementarity problem* NCP (F) is to find a point $x \in \mathbf{R}^n$ such that

$$x \geqslant 0, \quad F(x) \geqslant 0, \quad \langle x, F(x) \rangle = 0,$$

where $F : \mathbf{R}^n \to \mathbf{R}^n$.

Variational inequalities have been studied in many works.

The monograph of Kinderlehrer and Stampacchia [11] is a complete introduction in this topic.

Equivalence of variational inequality problems to unconstrained optimization problems is studied in [Peng 16].

Unconstrained optimization reformulations of variational inequality problems are proposed in [Yamashita, Taji, and Fukushima 26]. Reformulations of variational inequalities are also considered in [Andreani and Martínez 1].

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Newton-type methods for solving variational inequalities are suggested, e.g., in [Marcotte and Dussault 14], [Qi 17], [Qi and Sun 18], [Taji, Fukushima, and Ibaraki 25], etc.

A hybrid projection-proximal point algorithm is proposed in [Solodov and Svaiter 20].

Nesterov and Vial ([15]) introduced a homogeneous analytic center cutting plane method (HACCPM) which solves monotone VIPs in a conic setting and pseudopolynomial-time complexity. ACCPM is considered, for example, in [Sonnevend 22]. An ACCPM for pseudomonotone variational inequalities and a complexity bound was derived in [Goffin, Marcotte, and Zhu 10].

An analytic center quadratic cut method for strongly monotone variational inequality problems is suggested in [Lüthi and Büeler 12].

Conditions ensuring applicability of cutting plane methods for solving variational inequalities are derived in [Crouzeix, Marcotte, and Zhu 5].

Descent methods for asymmetric variational inequality problems are suggested in [Fukushima 8].

Characterization of strong regularity for variational inequalities over polyhedral sets is considered in [Dontchev and Rockafellar 6].

Complementarity problems are considered, e.g., in [Facchinei and Kanzow 7], [Gabriel and Moré 9], [Mangasarian and Solodov 13], [Solodov and Svaiter 21], etc.

The VIP and the CP can be reformulated as equivalent unconstrained optimization problems by using the $D-$ gap function (for the VIP) and the implicit Lagrangian (for the CP). The implicit Lagrangian was proposed by Mangasarian and Solodov ([13]) for the CP, and Peng ([16]) extended the implicit Lagrangian approach to the VIP and showed that the implicit Lagrangian can be expressed as the difference of two regularized gap functions proposed by Fukushima ([8]). Yamashita, Taji and Fukushima ([26]) extended the results of Peng and studied properties of the *$D-$ gap function* $g_{\alpha\beta}(x) \stackrel{\text{def}}{=} f_\alpha(x) - f_\beta(x)$, where α and β are arbitrary parameters with $\beta > \alpha > 0$ and f_α is the following regularized gap function $f_\alpha(x) \stackrel{\text{def}}{=} \max_{y \in X} \langle F(x), x - y \rangle - \dfrac{\alpha}{2} \|y - x\|^2 = \langle F(x), x - y_\alpha(x) \rangle - \dfrac{\alpha}{2} \|y_\alpha(x) - x\|^2$, $y_\alpha(x) \stackrel{\text{def}}{=} \Pi_X\left(x - \frac{1}{\alpha} F(x)\right)$ and $\Pi_X(.)$ is the projection operator onto the constraint set X.

The implicit Lagrangian is a particular case of the $D-$ gap function with $\beta = \frac{1}{\alpha}$.

Consider the box constrained variational inequality problem: find $x^* \in X$ such that $\langle F(x^*), x - x^* \rangle \geqslant 0 \quad \forall x \in X$, with $F : \mathbf{R}^n \to \mathbf{R}^n$ a continuously differentiable function and $X \stackrel{\text{def}}{=} \{x \in \mathbf{R}^n : \ a \leqslant x \leqslant b\}$, $a \in \{\mathbf{R} \cup \{-\infty\}\}^n$, $b \in \{\mathbf{R} \cup \{\infty\}\}^n$, $a < b$. If $a = 0$, $b = \infty$, then VIP becomes the well-known nonlinear complementarity problem (NCP). This VIP is also called the *mixed complementarity problem.*

As mentioned at the beginning, variational inequality problems are connected with the nonlinear/convex programming (complementary slackness conditions in the Theorem of John, in Karush–Kuhn–Tucker (KKT) theorem), complementarity problems, etc.

For example, the differential version of *KKT theorem* for problem

$$\min f(x)$$

subject to

$$g_i(x) \leqslant 0,\ i \in I_1; \quad l_i(x) \leqslant 0,\ i \in I_2; \quad l_i(x) = 0,\ i \in I_3; \quad x_j \geqslant 0,\ j \in J_1,$$

where $I_1 \cup I_2 \cup I_3 = I \equiv \{1, \dots, m\}; J_1 \subseteq J \equiv \{1, \dots, n\}$ can be formulated as follows.

Let f and g_i be differentiable convex functions, l_i be affine functions and Slater's constraint qualification be satisfied. A necessary and sufficient condition for x^* to be an optimal solution to the convex program is that there exists $\lambda^* \in \Lambda = \{\lambda = (\lambda_1, \dots, \lambda_m) : \lambda_i \geqslant 0,\ i \in I_1 \cup I_2\}$ such that

$$\frac{\partial L}{\partial x_j}(x^*, \lambda^*) \geqslant 0,\ j \in J_1 \qquad \frac{\partial L}{\partial \lambda_i}(x^*, \lambda^*) \leqslant 0,\ i \in I_1 \cup I_2$$

$$\frac{\partial L}{\partial x_j}(x^*, \lambda^*) = 0,\ j \in J \setminus J_1 \qquad \frac{\partial L}{\partial \lambda_i}(x^*, \lambda^*) = 0,\ i \in I_3$$

$$x_j^* \frac{\partial L}{\partial x_j}(x^*, \lambda^*) = 0,\ j \in J_1 \qquad \lambda_i^* \frac{\partial L}{\partial \lambda_i}(x^*, \lambda^*) = 0,\ i \in I_1 \cup I_2$$

$$x_j^* \geqslant 0, j \in J_1 \qquad \lambda_i^* \geqslant 0, i \in I_1 \cup I_2,$$

where L is the Lagrangian associated with the convex program.

Third type conditions in the above system are the complementary slackness conditions.

Convex separable minimization problems subject to bounded variables are studied, for example, in [23], [24], etc. As mentioned above, complementary slackness conditions for these problems, which are among the KKT optimality conditions, are connected with complementarity problems and variational inequalities.

In this section we also give some notation used in the paper.

Notation

0	zero or zero vector of appropriate dimension
$\|x\|$	the Euclidean norm of $x \in \mathbf{R}^n$
$y^T x = \langle x, y \rangle$	the inner (scalar) product of $x, y \in \mathbf{R}^n$
$\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$	the extended real numbers
$\mathbf{R}^n_+$	the nonnegative orthant $\{x \in \mathbf{R}^n : x \geqslant 0\}$ of $\mathbf{R}^n$
$\mathbf{R}^n_{++}$	the positive orthant $\{x \in \mathbf{R}^n : x > 0\}$ of $\mathbf{R}^n$
I	the unit matrix of appropriate dimension, the identity mapping
$\nabla F(x) = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)^T$	the gradient of $F : \mathbf{R}^n \to \mathbf{R}$ at $x \in \mathbf{R}^n$
$F'(x)$	the Jacobian of $F : \mathbf{R}^m \to \mathbf{R}^n$ at $x \in \mathbf{R}^n$
$\nabla F(x) = (F'(x))^T$	the transposed Jacobian of $F : \mathbf{R}^m \to \mathbf{R}^n$ at $x \in \mathbf{R}^n$

where $F(x) = (F_1(x), \dots, F_m(x))$ and

$$\nabla F(x) = \begin{pmatrix} \nabla F_1(x)^T \\ \vdots \\ \nabla F_m(x)^T \end{pmatrix}_{m \times n}.$$

2. Preliminaries on monotone operators and normal cones

Recall that a *point to set valued map* (or *multifunction*) $A : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is an operator which associates with each point $x \in \mathbf{R}^n$ a set (possibly empty) $A(x) \subseteq \mathbf{R}^n$. The inverse operator is denoted by $A^{-1}(y) = \{x \in \mathbf{R}^n : y \in A(x)\}$ and we have $(A^{-1})^{-1} = A$. The domain and range of A are defined by

$$\operatorname{dom} A = \{x : A(x) \neq \emptyset\},$$

$$\operatorname{rg} A = \operatorname{dom} A^{-1} = \{y : \exists\, x \text{ with } y \in A(x)\}.$$

When A is a single-valued map (that is, a function) we can write $A(x) = \{y\}$ or $A(x) = y$.

DEFINITION 1. The multivalued mapping $F : X \rightrightarrows \mathbf{R}^n$, where X is a nonempty convex subset of $\mathbf{R}^n$, is said to be:

- *monotone on* X if for all $x_1, x_2 \in X$

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geqslant 0 \quad \text{whenever} \quad v_1 \in F(x_1),\ v_2 \in F(x_2);$$

- *strictly monotone on* X if for all x_1, $x_2 \in X$, $x_1 \neq x_2$

$$\langle v_1 - v_2, x_1 - x_2 \rangle > 0 \quad \text{whenever} \quad v_1 \in F(x_1),\ v_2 \in F(x_2);$$

- *strongly monotone with modulus* $m > 0$ *on* X if for all $x_1, x_2 \in X$

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geqslant m\|x_1 - x_2\|^2 \quad \text{whenever} \quad v_1 \in F(x_1),\ v_2 \in F(x_2).$$

When F is single-valued, the monotonicity property takes the form

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geqslant 0 \quad \forall x_1, x_2 \in X;$$

the strict and strong monotonicity cases are modified similarly.

Recall that a matrix A is called

- *positive semidefinite* if $\langle y, Ay \rangle \geqslant 0$ for all $y \in \mathbf{R}^n$;
- *positive definite* if $\langle y, Ay \rangle > 0$ for all $y \in \mathbf{R}^n,\ y \neq 0$; and
- *uniformly positive definite with constant* m if

$$\langle x - y, A(x - y) \rangle \geqslant m\|x - y\|^2 \quad \forall x, y \in \mathbf{R}^n.$$

When the mapping F is differentiable, F is

- *monotone on* X if and only if $\nabla F(x)$ is positive semidefinite for all $x \in X$;
- *strictly monotone on* X if $\nabla F(x)$ is positive definite for all $x \in X$; and
- *strongly monotone with modulus* m *on* X if and only if $\nabla F(x)$ is uniformly positive definite with constant m.

When the mapping F is affine such that $F(x) = Ax + b$ with A being an $n \times n$ matrix and b being an $n-$ vector, there is no difference between the strict monotonicity and the strong monotonicity of F. More specifically, matrix A is positive definite if and only if F is strongly monotone as well as strictly monotone. Moreover, A is positive semidefinite if and only if F is monotone. In particular, the identity mapping I is strictly monotone.

DEFINITION 2. A monotone mapping F is said to be *maximal* if its graph is not properly contained in the graph of any other monotone mapping, that is, if

$$\langle y - y', x - x' \rangle \geqslant 0 \quad \forall x' \in \mathrm{dom} F, \forall y' \in F(x') \quad \text{implies} \quad y \in F(x).$$

The following properties of maximal monotone operators hold (for details see, e.g., [Rockafellar and Wets 19, Chapter 12]).

PROPOSITION 1. i) A^{-1} *is maximal monotone if and only if* A *is maximal monotone.*

ii) *Let* A_1, A_2 *be maximal monotone. Then* $A_1 + A_2$ *is also maximal monotone if either one of the following conditions is satisfied:*

a) int dom $A_1 \cap$ dom $A_2 \neq \emptyset$,

b) ri (dom $A_1) \cap$ ri (dom $A_2) \neq \emptyset$, *where* ri A *denotes the relative interior of* A.

For a nonempty closed convex set $X \in \mathbf{R}^n$ denote by X_∞ the recession cone of X. For a closed and proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, the *recession function* f_∞ of f is defined by epi $(f_\infty) = (\text{epi } f)_\infty$, where epi $f = \{(x, r) \in \mathbf{R}^n \times \mathbf{R} : f(x) \leqslant r\}$ is the epigraph of f.

Recall that a vector $\widehat{f}(x_0)$ is said to be a *subgradient* of (a convex function) f at x_0 if the inequality

$$f(x) - f(x_0) \geqslant \langle \widehat{f}(x_0), x - x_0 \rangle$$

holds for each $x \in \mathbf{R}^n$.

The set containing all subgradients of f at the point x_0 is called the *subdifferential* of f at x_0 and is denoted by $\partial f(x_0)$. If $\partial f(x_0) \neq \emptyset$, then f is said to be *subdifferentiable* at x_0. The subdifferential of f is defined as the multivalued mapping $\partial f : x \rightrightarrows \partial f(x)$.

DEFINITION 3. The *normal cone operator* associated with a closed convex set X is defined by

$$N_X(x) = \begin{cases} \{y : \langle y, v - x \rangle \leqslant 0 \; \forall v \in X\}, & \text{if } x \in X \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have dom $N_X = X$, and $N_X(x) = \{0\}$ when $X = \mathbf{R}^n$ or $x \in$ int X.

N_X is a maximal monotone operator on $\mathbf{R}^n$. Moreover, $N_X = \partial\delta(.|X)$ where $\delta(.|X)$ is a closed proper convex function (indicator function) defined as follows

$$\delta(x|X) = \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{otherwise.} \end{cases}$$

In terms of N_X, we can rewrite the VIP (1) as the one of finding the zero of the generalized equation

$$0 \in T(x) + N_X(x). \tag{3}$$

Problem (3) can be considered as another equivalent *primal* formulation of the VIP (1). Problem (3) is also called *variational condition* for any set $X \subset \mathbf{R}^n$ and any mapping $T : X \to \mathbf{R}^n$. When X is convex, (3) can be written equivalently in the form (1) or (2).

The multivalued mapping $F : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ defined by $F(x) \stackrel{\text{def}}{=} T(x) + N_X(x)$ and $F(x) = \emptyset$ when $x \notin X$, is strictly monotone when T is strictly monotone relative to X. Then the solution set has at most one element.

3. Lagrangian duality for box constrained VIPs

3.1. Introduction. Main results

It is known that x^* is a solution of the VIP (1) if and only if

$$x^* \in \arg\min\{\langle g^*, x - x^* \rangle \; : \; x \in X\}, \tag{4}$$

where $g^* \in T(x^*)$. Consider the case where T is a maximal monotone mapping from $\mathbf{R}^n$ into itself, the constraint set X is explicitly defined by $X = \{x : \; f_i(x) \leqslant 0, \; i = 1, \dots, m, \quad a_j \leqslant x_j \leqslant b_j, \; j = 1, \dots, n\}$ and $f_i : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ are closed proper convex functions. Without loss of generality suppose that at least one f_i is continuous. Denote $F(x) = (f_1(x), \dots, f_m(x))^T$.

Let $\Phi \stackrel{\text{def}}{=} \cap_{i=1}^m \operatorname{dom} f_i$ be an open set.

With the convex optimization problem

$$\min\{\langle g^*, x - x^* \rangle \; : \; f_i(x) \leqslant 0, \; i = 1, \dots, m, \quad a_j \leqslant x_j \leqslant b_j, \; j = 1, \dots, n\} \tag{5}$$

we can associate a Lagrangian defined by $L : \mathbf{R}^n \times \mathbf{R}^{m+2n} \to \overline{\mathbf{R}}$,

$$L(x, \mu; x^*) =$$

$$= \begin{cases} \langle g^*, x - x^* \rangle + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{n} [u_j(a_j - x_j) + v_j(x_j - b_j)], & \text{if } x \in \Phi \cap C, \; \mu \in \mathbf{R}_+^{m+2n} \\ -\infty, & \text{if } \mu \notin \mathbf{R}_+^{m+2n} \\ +\infty, & \text{otherwise,} \end{cases} \tag{6}$$

where $\mu = (\lambda, u, v) \in \mathbf{R}_+^{m+2n}$ is the dual variables vector and $C = \{a_j \leqslant x_j \leqslant b_j, \; j = 1, \dots, n\}$.

The dual problem is then

$$\sup_{\mu \geqslant 0} \inf\{L(x, \mu; x^*) : \; x \in \Phi \cap C\}. \tag{7}$$

It is known that $(x^*, \mu^*) \in (\Phi \cap C) \times \mathbf{R}_+^{m+2n}$ is a saddle point of L if and only if x^* $(\in \Phi \cap C)$ and μ^* $(\geqslant 0)$ are optimal solutions to the primal and dual problem, respectively, with no duality gap, that is, with equal optimal values of the primal and dual problems.

THEOREM 1. *Let Slater's constraint qualification be satisfied for the constraint set X. $x^* \in \mathbf{R}^n$ solves (3) if and only if there exists $\mu^* \in \mathbf{R}_+^{m+2n}$ such that (x^*, μ^*) solves the problem*

$$(0, 0) \in W(x^*, \mu^*), \tag{8}$$

where

$$W(x,\mu) \equiv \Big\{(y,w) \in \mathbf{R}^n \times \mathbf{R}_+^{m+2n} :$$
$$y \in T(x) + \sum_{i=1}^{m} \lambda_i \partial f_i(x) - u + v,\ w \in \Big(-F(x) + N_{\mathbf{R}_+^m}(\mu), x-a, b-x)\Big)\Big\}, \tag{9}$$

if $(x,\mu) \in$ dom $W = (\text{dom}\, T \cap \Phi \cap C) \times \mathbf{R}_+^{m+2n} \neq \emptyset$, *and* $W(x,\mu) = \emptyset$, *otherwise.*

Proof. We have

$$\partial(\langle g^*, x-x^*\rangle + \delta(x|\Phi\cap C))_{|x=x^*} = \partial(\langle g^*, x-x^*\rangle + \delta(x|\Phi)_{|x=x^*} + \delta(x|C))_{|x=x^*}$$
$$= g^* + N_\Phi(x^*) + N_C(x^*),$$

and since Φ is open then $N_\Phi(x^*) = \{0\}$. Therefore for the primal and dual problem we get

$$0 \in g^* + \sum_{i=1}^{m} \lambda_i^* \partial f_i(x^*) - u^* + v^* \tag{10}$$

and

$$0 \in \big(-F(x^*) + N_{\mathbf{R}_+^m}(\lambda^*), x^*-a, b-x^*\big), \tag{11}$$

respectively, where $g^* \in T(x^*)$. Since $\langle g^*, x-x^*\rangle \geqslant 0\ \forall x \in X$ then the optimal value of problem (5) is 0.

Since Slater's constraint qualification is satisfied, that is, there exists $x \in X$: $f_i(x) < 0,\ i = 1,\dots,m$ then there exists a KKT vector μ. The relations (10) and (11) are the KKT necessary and sufficient optimality conditions for problem (5).

Thus μ^* can be interpreted as the solution of the Lagrangian dual VIP:

$$\text{find } \mu^* \in \mathbf{R}_+^{m+2n},\ d^* \in G(\mu^*) \times A(x^*) \times B(x^*) :\ \langle d^*, \mu-\mu^*\rangle \geqslant 0 \quad \forall \mu \in \mathbf{R}_+^{m+2n}, \tag{12}$$

where

$$G(\mu) \equiv \{-F(x) :\ x \in M(\mu)\}, \tag{13}$$

$$M(\mu) \equiv \Big\{x \in \mathbf{R}^n :\ 0 \in T(x) + \sum_{i=1}^{m} \lambda_i \partial f_i(x) - u + v\Big\}, \tag{14}$$

$$A(x) = \{x-a\},\ B(x) = \{b-x\}. \tag{15}$$

The dual problem (12) can also be written as

$$0 \in T_D(\mu^*)$$

where

$$T_D(\mu) \equiv \Big\{G(\mu) + N_{\mathbf{R}_+^m}(\lambda)\Big\} \times A(x) \times B(x).$$

Using (10) and (11), we obtain the *primal-dual formulation* (8) of VIP where $W(x,\mu)$ is defined through (9). □

Thus, (3), (8) and (12) are equivalent primal, primal-dual and dual formulation of VIP (1), respectively.

The following operators

$T_P := T + N_X$ (primal)
$T_{PD} := W$ (primal-dual)
$T_D := G + N_{\mathbf{R}^m_+}$ (dual)

are associated with these three formulations, respectively.

Denote by $X^* = (T + N_X)^{-1}(0), Z^* = W^{-1}(0,0), M^*$ the set of solutions to problems (3), (8) and (12), respectively.

From Proposition 1, ii) a) it follows that T_P is maximal monotone when dom $T \cap$ int $X \neq \emptyset$.

THEOREM 2. *Let* $T : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ *be maximal monotone such that* dom $T \cap \Phi \cap C \neq \emptyset$. *Then the primal-dual operator* W *(9) is also maximal monotone.*

Proof. Let

$$D(x,\mu) = \begin{cases} T(x) \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \{\mathbf{0}\}, & \text{if } x \in \text{ dom } T \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$E(x,\mu) =$$

$$= \begin{cases} \Big[\sum_{i=1}^{m} \lambda_i \partial f_i(x)\Big] \times \big[-F(x) + N_{\mathbf{R}^m_+}(\lambda)\big] \times (x-a) \times (b-x), & \text{if } x \in \Phi \cap C, \lambda \in \mathbf{R}^m_+ \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then the operator W can be decomposed as $W = D + E$. Since T is maximal monotone, D is also maximal monotone. If we define $h : (\Phi \cap C) \times \mathbf{R}^m_+ \to \mathbf{R}$ by $h(x,\lambda) = \sum_{i=1}^m \lambda_i f_i(x)$, then $E(x,\mu) = \partial_x h(x,\lambda) \times \big(-\widehat{\partial}_\lambda h(x,\lambda)\big) \times (x-a) \times (b-x)$, where $\widehat{\partial}$ is the upper subdifferential. (Recall that the upper subdifferential $\widehat{\partial} f(\overline{x})$ is defined by $\widehat{\partial} f(\overline{x}) = \{\zeta \in \mathbf{R}^n : f(x) - f(\overline{x}) \leqslant \langle \zeta, x - \overline{x} \rangle \;\; \forall x \in \mathbf{R}^n\}$.)

Therefore E is maximal monotone, whence applying Proposition 1, ii) a) to operators D and E yields W is maximal monotone. $\square$

3.2. Auxiliary consideration

Let function φ be defined as follows

$$\varphi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \rho(t - \log t - 1), & \text{if } t > 0 \\ +\infty, & \text{otherwise,} \end{cases} \tag{16}$$

where $\nu > \rho > 0$ are given fixed parameters.

Define for any $v \in \mathbf{R}^p_{++}$ the function $d(u,v)$ associated with φ

$$d(u,v) = \begin{cases} \sum_{i=1}^{p} \frac{\nu}{2}(u_i - v_i)^2 + \rho\Big(v_i^2 \log \frac{v_i}{u_i} + u_i v_i - v_i^2\Big), & \text{if } v \in \mathbf{R}^p_{++} \\ +\infty, & \text{otherwise.} \end{cases} \tag{17}$$

The following properties of φ hold true, see [Auslender, Teboulle, and Ben-Tiba 3, 4].

PROPOSITION 2. *Let φ be defined by (16). Then*

1. φ *is a differentiable strongly convex function on* $\mathbf{R}_{++}$ *with modulus* $\nu > 0$.
2. $\lim_{t\to 0} \varphi'(t) = -\infty$.
3. *The conjugate of* φ *is*
$$\varphi^*(s) = \frac{\nu}{2}t^2(s) + \rho \log t(s) - \frac{\nu}{2},$$
where
$$t(s) = \frac{1}{2\nu}\left[\nu - \rho + s + \sqrt{(\nu-\rho+s)^2 + 4\rho\nu}\right] = (\varphi^*)'(s).$$
4. dom $\varphi^* = \mathbf{R}$, $\quad \varphi^* \in C^\infty(\mathbf{R})$.
5. $(\varphi^*)'(s) = (\varphi')^{-1}(s)$ *is Lipschitz for all* $s \in \mathbf{R}$ *with Lipschitz constant* $\frac{1}{\nu}$.
6. φ^* *is a strictly convex and increasing function on* $\mathbf{R}$.
7. $(\varphi^*)''(s) < \frac{1}{\nu}$ *for all* $s \in \mathbf{R}$.
8. $(\varphi^*)_\infty(-1) = 0$ *and* $(\varphi^*)_\infty(1) = +\infty$ *where* $(\varphi^*)_\infty$ *is the recession function of* φ^*.

Suppose that following assumptions are satisfied.

Assumptions

1. T is a maximal monotone operator with $\Phi = \cap_{i=1}^m$ dom f_i an open subset of int dom T.
2. The solution set of VIP (1) is nonempty and compact.
3. Slater's constraint qualification is satisfied for some $x \in$ dom T.

For $\gamma > 0$, $\mu > 0$ and φ (16) consider the multifunction

$$H(x,\mu,\gamma) = \begin{cases} T(x) + \sum_{i=1}^m \lambda_i(\varphi_i^*)'(\gamma f_i(x)/\lambda_i)\partial f_i(x) - u + v, & \text{if } x \in \Phi \cap C \\ \emptyset, & \text{otherwise.} \end{cases}$$

THEOREM 3. *Let φ be defined by (16) and Assumptions 1 – 3 be satisfied. Then for every $\gamma > 0$, for every $\mu \in \mathbf{R}^m_{++}$, the operator $H(.,\mu,\gamma)$ is maximal monotone on* $\mathbf{R}^n$.

Proof. From Proposition 2, 8. it follows that $(\varphi^*)_\infty(-1) = 0$, $(\varphi^*)_\infty(1) = +\infty$. Let $\gamma > 0$ and $\mu > 0$ be fixed. From Proposition 2.1 [2] it follows that the function

$$g(x) \stackrel{\text{def}}{=} \frac{1}{\gamma}\sum_{i=1}^m \lambda_i \varphi^*(\gamma f_i(x)/\lambda_i)$$

is closed, proper and convex with dom $g = \cap_{i=1}^m$ dom $f_i \neq \emptyset$. Moreover,

$$g_\infty(d) = \begin{cases} 0, & \text{if } (f_i)_\infty(d) \leqslant 0 \,\forall\, i \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $\Phi = \cap_{i=1}^m$ dom f_i is open, then $H = T + \partial g - u + v$. Applying Assumption 1 ($\Phi \subset$ int dom T) and Proposition 1, ii) a) yields H is maximal monotone. □

It can be proved that the solution set $H^{-1}(0,\mu,\gamma)$ of the generalized equation $0 \in H(x,\mu,\gamma)$ is nonempty.

3.3. The primal-dual method (PDM)

The primal-dual method is based on solving VIP (1) by solving the equivalent primal-dual problem (8):

$$0 \in W(x, \mu), \tag{8'}$$

where W is defined by (9), assuming that W is maximal monotone (for example, under assumptions of Theorem 2) and Slater's constraint qualification holds true for $x \in$ dom T.

Consider the following distance-like functional

$$D((x,\mu),(y,w)) \stackrel{\text{def}}{=} \frac{1}{2}\|x-y\|^2 + d(\mu, w),$$

where $d(u,v)$ is defined by (17).

Beginning with an initial guess $(x^0, \mu^0) \in \mathbf{R}^n \times \mathbf{R}^{m+2n}_{++}$, generate a sequence $\{(x^k, \mu^k)\} \subset \mathbf{R}^n \times \mathbf{R}^{m+2n}_{++}$ satisfying

$$0 \in W(x^k, \mu^k) + \frac{1}{\gamma_k}\nabla_{(x,\mu)} D((x^k, \mu^k),(x^{k-1}, \mu^{k-1})), \tag{18}$$

where $\gamma_k \geqslant \gamma > 0$, $\mu^k = (\lambda^k, u^k, v^k)$.

THEOREM 4. *Let W be the maximal monotone operator defined by (9). Then*

(i) *There exists a unique $(x^k, \mu^k) \in \mathbf{R}^n \times \mathbf{R}^{m+2n}_{++}$ satisfying (18) for all $\gamma_k > 0$, $\mu^{k-1} > 0$.*

(ii) *If the solution set of problem (8) is nonempty, then the sequence $\{(x^k, \mu^k)\}$ generated by (18) converges to a solution $(x^*, \mu^*) \in X^* \times M^*$.*

The proof of Theorem 4 is similar to that of a theorem from [3].

The iterative process (18) can be written as follows

$$0 \in T(x^k) + \sum_{i=1}^{m} \lambda_i^k \partial f_i(x^k) - u^k + v^k + \frac{x^k - x^{k-1}}{\gamma_k}, \tag{19}$$

$$0 \in F(x^k) + \gamma_k\left[\lambda_1^k \varphi'\left(\frac{\lambda_1^{k-1}}{\lambda_1^k}\right), \ldots, \lambda_m^k \varphi'\left(\frac{\lambda_m^{k-1}}{\lambda_m^k}\right)\right] + N_{\mathbf{R}^m_+}(\lambda^k). \tag{20}$$

This consideration leads to the so-called *primal-dual method*.

Let φ be defined by (16), $(x^0, \mu^0) \in \mathbf{R}^n \times \mathbf{R}^{m+2n}_{++}$ and $\gamma_k \geqslant \gamma > 0$ $\forall k \geqslant 1$.

Generate the sequence $\{(x^k, \mu^k)\}$ through

$$0 \in H(x^k, \mu^{k-1}, \gamma_k) + \frac{x^k - x^{k-1}}{\gamma_k}, \tag{21}$$

$$\mu_i^k = \mu_i^{k-1}(\varphi^*)'(\gamma_k f_i(x^k)/\mu_i^{k-1}), \quad i = 1, \ldots, m+2n. \tag{22}$$

For fixed $\mu^{k-1} > 0, \gamma_k > 0$ define the multifunction

$$H_k(x) \stackrel{\text{def}}{=} H(x, \mu^{k-1}, \gamma_k) + \frac{x - x^{k-1}}{\gamma_k}.$$

Using this notation, (21) can be written as

$$0 \in H_k(x^k). \tag{21'}$$

THEOREM 5. *Let φ be defined by (16), T be a maximal monotone mapping and Slater's constraint qualification be satisfied for some $x \in \operatorname{dom} T$. Then operator H_k is maximal monotone and strongly monotone with modulus $\frac{1}{\gamma_k}$, that is,*

$$\langle y - y', x - x' \rangle \geqslant \frac{1}{\gamma_k} \|x - x'\|^2 \quad \forall y \in H_k(x),\ y' \in H_k(x').$$

Proof. Using definition of H and that Slater's constraint qualification is satisfied for some $x \in \operatorname{dom} T$, we get $H = T + \partial g - u + v$ and H is maximal monotone. Define $R_k(x) = \frac{1}{2\gamma_k}\|x - x^{k-1}\|^2$. By definition of H_k, $H_k = H + \nabla R_k$, and since ∇R_k is strongly monotone then H_k is also strongly monotone. $\square$

Since the multifunction $H_k(x)$ in (21) is maximal monotone and strongly monotone then the existence and uniqueness of the sequence $\{x^k\}$ in (21) is guaranteed.

THEOREM 6. *Let T be a maximal monotone operator on $\mathbf{R}^n$, W be also maximal monotone, Slater's constraint qualification hold for $x \in \operatorname{dom} T$, and the solution set of (8) be nonempty. Then the primal-dual sequence $\{(x^k, \mu^k)\}$, generated by PDM, converges to a primal-dual solution $(x^*, \mu^*) \in X^* \times M^*$ of (8), that is, to a solution of VIP (1) and dual problem (12), respectively.*

Proof. Since the sequence $\{(x^k, \mu^k)\}$, generated by PDM, is given by (18), then Theorem 4 implies that $\{(x^k, \mu^k)\}$ converges to a solution of problem (8). $\square$

4. Extensions and concluding remarks

PDM can be modified to solve the standard nonlinear complementarity problem with

$T : \mathbf{R}^n \to \mathbf{R}^n$, single-valued and continuous,
$T(x) = (T_1(x), \ldots, T_n(x))^T$, $X = \mathbf{R}^n_+$

as follows

$$T_i(x^k) - \lambda_i^k (\varphi^*)'(-\gamma_k x^k / \lambda_i^{k-1}) - u_i^k + v_i^k + \frac{x_i^k - x_i^{k-1}}{\gamma_k} = 0, \quad i = 1, \ldots, n \tag{23}$$

$$\mu_i^k = \mu_i^{k-1} (\varphi^*)'(-\gamma_k x^k / \mu_i^{k-1}), \quad i = 1, \ldots, m + 2n. \tag{24}$$

Since (24) is a system of equations, it can be solved via a Newton-type method.

REFERENCES

[1] R. ANDREANI AND J. M. MARTÍNEZ, *"Reformulation of variational inequalities on a simplex and compactification of complementarity problems"*, SIAM Journal on Optimization, **10** (2000) No. 3, 878–895.

[2] A. AUSLENDER, R. COMINETTI, AND M. HADDOU, *"Asymptotic analysis of penalty and barrier methods in convex and linear programming"*, Mathematics of Operations Research, **22** (1997) 43–62.

[3] A. AUSLENDER, M. TEBOULLE, AND S. BEN-TIBA, *"A logarithmic-quadratic proximal method for variational inequalities"*, Computational Optimization and Applications, **12** (1998) 31–40.

[4] A. AUSLENDER, M. TEBOULLE, AND S. BEN-TIBA, *"Interior proximal and multiplier methods based on second order homogeneous kernels"*, Mathematics of Operations Research, **24** (1999) 645–668.

[5] J.-P. CROUZEIX, P. MARCOTTE, AND D. ZHU, *"Conditions ensuring the applicability of cutting-plane methods for solving variational inequalities"*, Mathematical Programming, **88** (2000) No. 3, 521–539.
[6] A. L. DONTCHEV AND R. T. ROCKAFELLAR, *"Characterizations of strong regularity for variational inequalities over polyhedral sets"*, SIAM Journal on Optimization, **6** (1996) 1087–1105.
[7] F. FACCHINEI AND C. KANZOW, *"Beyond monotonicity in regularization methods for nonlinear complementarity problems"*, SIAM Journal on Control and Optimization, **37** (1999) 1150–1161.
[8] M. FUKUSHIMA, *"Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems"*, Mathematical Programming, **53** (1992) 99–110.
[9] S. A. GABRIEL, J. J. MORÉ, *"Smoothing of mixed complementarity problems"*, In: Complementarity and Variational Problems: State of the Art, M. C. Ferris and J. S. Pang, eds., SIAM, Philadelphia, 1997, pp. 105–116.
[10] J.-L. GOFFIN, P. MARCOTTE, AND D. L. ZHU, *"An analytic center cutting plane method for pseudomonotone variational inequalities"*, Operations Research Letters, **20** (1997) 1–6.
[11] D. KINDERLEHRER, G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
[12] HANS-JACOB LÜTHI AND BENNO BÜELER, *"The analytic center quadratic cut method for strongly monotone variational inequality problems"*, SIAM Journal on Optimization, **10** (2000) No. 2, 415–426.
[13] O. L. MANGASARIAN AND M. V. SOLODOV, *"Nonlinear complementarity as unconstrained and constrained minimization"*, Mathematical Programming, **62** (1993) 277–297.
[14] P. MARCOTTE AND J. P. DUSSAULT, *"A note on a globally convergent Newton method for solving monotone variational inequalities"*, Operations Research Letters, **6** (1987) 35–42.
[15] Y. NESTEROV AND J.-P. VIAL, *"Homogeneous analytic center cutting plane methods for convex problems and variational inequalities"*, SIAM Journal on Optimization, **9** (1999) 707–728.
[16] J. M. PENG, *"Equivalence of variational inequality problems to unconstrained optimization"*, Mathematical Programming, **78** (1997) 347–355.
[17] H.-D. QI, *"A regularized smoothing Newton method for box constrained variational inequality problems with P_0-functions"*, SIAM Journal on Optimizationz, **10** (2000), No. 2, 315–330.
[18] L. QI, J. SUN, *"A nonsmooth version of Newton's method"*, Mathematical Programming, **58** (1993) 691–714.
[19] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer-Verlag, Berlin Heidelberg New York, 1998.
[20] M. SOLODOV AND B. SVAITER, *"A hybrid projection-proximal point algorithm"*, Journal on Convex Analysis, **6** (1999) 59–70.
[21] M. V. SOLODOV AND B. F. SVAITER, *"A trully globally convergent Newton-type method for the monotone nonlinear complementarity problem"*, SIAM Journal on Optimization, **10** (2000) No. 2, 605–625.
[22] G. SONNEVEND, *"An "analytic center" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming"*, in: Proceedings of the 12th IFIP Conference, Budapest 1985, A. Prekopa, J. Szelezsan, and B. Strazicky, eds., Lecture Notes in Control and Information Sciences 84, Springer, Berlin, 1985, pp. 866–876.
[23] S. M. STEFANOV, *"Convex separable minimization subject to bounded variables"*, Computational Optimization and Applications. An International Journal, **18** (2001) No. 1, 27-48.
[24] S. M. STEFANOV, *Separable Programming. Theory and Methods*, Applied Optimization, vol. 53, Kluwer Academic Publishers, Boston Dordrecht London, 2001.
[25] K. TAJI, M. FUKUSHIMA, AND T. IBARAKI, *"A globally convergent Newton method for solving strongly monotone variational inequalities"*, Mathematical Programming, **58** (1993) 369–383.
[26] N. YAMASHITA, K. TAJI, AND M. FUKUSHIMA, *"Unconstrained optimization reformulations of variational inequality problems"*, Journal of Optimization Theory and Applications, **92** (1997) 439–456.

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