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# NEW GENERALIZATIONS OF INEQUALITIES OF HARDY AND LEVIN—COCHRAN—LEE TYPE FOR MULTIDIMENSIONAL BALLS

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*Abstract.* This paper deals with some new sharp generalizations of inequalities of Hardy and Levin–Cochran–Lee type for $n$-dimensional balls.

## 1. Introduction

Let $B(r) = B(\mathbf{O}, r)$ be the ball in $\mathbf{R}^n$ centered at the origin and of radius $r > 0$. Further, let $S^{n-1}$ denote the surface of the unit ball B(1) and let $|S^{n-1}|$ be its area. Using polar coordinates in $\mathbf{R}^n$, the volume of the ball $B(r)$, $|B(r)|$, is then $|B(r)| = \int_{B(r)} d\mathbf{x} = \int_{|\mathbf{x}|<r} d\mathbf{x} = \int_0^r t^{n-1}\left(\int_{S^{n-1}} d\mathbf{S}\right) dt = \int_{S^{n-1}}\left(\int_0^r t^{n-1} dt\right) d\mathbf{S} = \frac{r^n|S^{n-1}|}{n}$, where $|\mathbf{x}|$ denotes the Euclidean norm of the vector $\mathbf{x} \in \mathbf{R}^n$. Consequently, $|B(1)| = \frac{|S^{n-1}|}{n}$, $B(r) = r^n|B(1)|$, and for $s > 0$ we have

$$\int_{B(r)} |B(|\mathbf{x}|)|^{s-1}\, d\mathbf{x} = \frac{|B(r)|^s}{s} \tag{1}$$

and

$$\int_{B(r)} |B(|\mathbf{x}|)|^{s-1} \ln|B(|\mathbf{x}|)|\, d\mathbf{x} = \frac{|B(r)|^s}{s}\left(\ln|B(r)| - \frac{1}{s}\right). \tag{2}$$

Following the results of M. Christ and L. Grafakos, [1], and those of P. Drábek, H. P. Heinig, and A. Kufner, [6], in the paper [3] a natural generalization of the classical Hardy integral inequality (cf. [7]) to balls in $\mathbf{R}^n$ centered at the origin was obtained. That result is given in:

THEOREM A. *Let $p > 1$ and $k \neq 1$ be real numbers. If $f$ is a non-negative measurable function such that $|B(|\mathbf{x}|)|^{1-\frac{k}{p}} f \in L^p(\mathbf{R}^n)$, and the function $F$ is defined on $\mathbf{R}^n$ by*

$$F(\mathbf{x}) = \begin{cases} \int_{B(|\mathbf{x}|)} f(\mathbf{y})\, d\mathbf{y}, & k > 1, \\ \int_{\mathbf{R}^n \setminus B(|\mathbf{x}|)} f(\mathbf{y})\, d\mathbf{y}, & k < 1, \end{cases} \tag{3}$$

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*then*

$$\int_{\mathbf{R}^n} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} \leqslant \left(\frac{p}{|k-1|}\right)^p \int_{\mathbf{R}^n} |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}. \tag{4}$$

*The constant* $\left(\frac{p}{|k-1|}\right)^p$ *is the best possible.*

In the same paper the corresponding exponential inequality was also proved. It is called the Levin–Cochran–Lee inequality for balls and we state it by the following theorem.

THEOREM B. *Let* $\alpha, \gamma \in \mathbf{R}$, $\alpha \neq 0$. *If* $f$ *is a positive measurable function such that* $\int_{\mathbf{R}^n} |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x} < \infty$, *and the function* $G$ *is defined on* $\mathbf{R}^n$ *by*

$$G(\mathbf{x}) = \begin{cases} \exp\left[\frac{\alpha}{|B(|\mathbf{x}|)|^{\alpha}} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} \ln f(\mathbf{y})\,d\mathbf{y}\right], & \alpha > 0, \\ \exp\left[-\frac{\alpha}{|B(|\mathbf{x}|)|^{\alpha}} \int_{\mathbf{R}^n \setminus B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} \ln f(\mathbf{y})\,d\mathbf{y}\right], & \alpha < 0, \end{cases} \tag{5}$$

*then the inequality*

$$\int_{\mathbf{R}^n} |B(|\mathbf{x}|)|^{\gamma-1} G(\mathbf{x})\,d\mathbf{x} \leqslant e^{\frac{\gamma}{\alpha}} \int_{\mathbf{R}^n} |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x} \tag{6}$$

*holds. The constant* $e^{\frac{\gamma}{\alpha}}$ *is the best possible.*

Observe that $G$ is the weighted geometric mean of $f$ over the ball $B(|\mathbf{x}|)$, or over its complement $\mathbf{R}^n \setminus B(|\mathbf{x}|)$, with the radial power weight $w(\mathbf{y}) = |B(|\mathbf{y}|)|^{\alpha-1}$.

Both stated results were obtained by using an original approach via mixed-means inequalities related to integral means of arbitrary real order, with power weights (cf. [3] and [4]).

It is evident from the proof of Theorem A that (4) holds even if $\int_{\mathbf{R}^n}$ is replaced with $\int_{B(R)}$ in the case when $k > 1$, or with $\int_{\mathbf{R}^n \setminus B(R)}$ when $k < 1$. Similarly, the proof of Theorem B shows that (6) will stay preserved if we put $\int_{B(R)}$ when $\alpha > 0$, or $\int_{\mathbf{R}^n \setminus B(R)}$ when $\alpha < 0$, instead of $\int_{\mathbf{R}^n}$.

Our objective in this paper is to refine these "reduced" inequalities. By making a careful analysis of the proofs from [3], we provide smaller upper bounds for their left-hand sides, dependent on $R$. Moreover, we prove that the constant factors involved in the right-hand sides of the obtained inequalities are the best possible, that is, they cannot be replaced with smaller ones.

A similar idea has already been applied to series and one-dimensional integrals (cf. [2] and [5]).

## 2. The results

First, we give a generalization of Theorem A.

THEOREM 1. *Let* $p, k, R \in \mathbf{R}$ *be such that* $p > 1$, $k \neq 1$, *and* $R > 0$. *Suppose* $f$ *is a non-negative measurable function and the function* $F$ *is defined by* (3).

(i) *If* $k > 1$ *and* $0 < \int_{B(R)} |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x} < \infty$, *then*

$$\int_{B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} < \left(\frac{p}{k-1}\right)^p \int_{B(R)} \left[1-\left(\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right)^{\frac{k-1}{p}}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}. \tag{7}$$

(ii) *If* $k < 1$ *and* $0 < \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x} < \infty$, *then*

$$\int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} < \left(\frac{p}{1-k}\right)^p \int_{\mathbf{R}^n \setminus B(R)} \left[1-\left(\frac{|B(R)|}{|B(|\mathbf{x}|)|}\right)^{\frac{1-k}{p}}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}. \tag{8}$$

*The constant* $\left(\frac{p}{|k-1|}\right)^p$ *is the best possible for both inequalities.*

*Proof.* To prove the theorem, we use the following two relations, obtained in [3] (Theorem 5, relations (7) and (8)), considering a non-negative measurable function $f$ and parameters $r, s, R, \alpha, \gamma \in \mathbf{R}$, such that $r, s \neq 0$, $r < s$, and $R > 0$ ($f$ positive in the case $r < 0$):

$$\left\{\frac{1}{|B(R)|^\gamma} \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} \left[\frac{1}{|B(|\mathbf{x}|)|^\alpha} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} f^r(\mathbf{y})\,d\mathbf{y}\right]^{\frac{s}{r}} d\mathbf{x}\right\}^{\frac{1}{s}} \leqslant \left\{\frac{1}{|B(R)|^\alpha} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \left[\frac{1}{|B(|\mathbf{x}|)|^\gamma} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f^s(\mathbf{y})\,d\mathbf{y}\right]^{\frac{r}{s}} d\mathbf{x}\right\}^{\frac{1}{r}} \tag{9}$$

and

$$\left\{\frac{1}{|B(R)|^\gamma} \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\gamma-1} \left[\frac{1}{|B(|\mathbf{x}|)|^\alpha} \int_{\mathbf{R}^n \setminus B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} f^r(\mathbf{y})\,d\mathbf{y}\right]^{\frac{s}{r}} d\mathbf{x}\right\}^{\frac{1}{s}} \leqslant \left\{\frac{1}{|B(R)|^\alpha} \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \left[\frac{1}{|B(|\mathbf{x}|)|^\gamma} \int_{\mathbf{R}^n \setminus B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f^s(\mathbf{y})\,d\mathbf{y}\right]^{\frac{r}{s}} d\mathbf{x}\right\}^{\frac{1}{r}}. \tag{10}$$

First, let $k > 1$. Setting $r = 1$, $s = p > 1$, $\alpha = 1$ and $\gamma = p - k + 1$, (9) becomes

$$\int_{B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\, d\mathbf{x} \leqslant |B(R)|^{1-k} \left\{ \int_{B(R)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} \left[ \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y})\, d\mathbf{y} \right]^{\frac{1}{p}} d\mathbf{x} \right\}^p . \tag{11}$$

Denote $I_R = \int_{B(R)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} d\mathbf{x}$. Using (1), the right-hand side of (11) is further equal to

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p \left\{ \frac{1}{I_R} \int_{B(R)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} \left[ \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y})\, d\mathbf{y} \right]^{\frac{1}{p}} d\mathbf{x} \right\}^p \\
&\quad < \left(\frac{p}{k-1}\right)^p \frac{1}{I_R} \int_{B(R)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&\quad = \left(\frac{p}{k-1}\right)^{p-1} |B(R)|^{\frac{1-k}{p}} \int_{B(R)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&\quad = \left(\frac{p}{k-1}\right)^{p-1} |B(R)|^{\frac{1-k}{p}} \int_{B(R)} |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y}) \int_{B(R)\setminus B(|\mathbf{y}|)} |B(|\mathbf{x}|)|^{\frac{k-1}{p}-1} d\mathbf{x}\, d\mathbf{y} \\
&\quad = \left(\frac{p}{k-1}\right)^{p} \int_{B(R)} \left[ 1 - \left( \frac{|B(|\mathbf{y}|)|}{|B(R)|} \right)^{\frac{k-1}{p}} \right] |B(|\mathbf{y}|)|^{p-k} f^p(\mathbf{y})\, d\mathbf{y},
\end{aligned} \tag{12}$$

so (7) is proved. The sequence of inequalities yielding the last line of (12) is obtained by applying Jensen's inequality to the convex function $t \mapsto t^p$, Fubini's theorem and the relation (1). Note that the inequality sign in the second line of (12) is strict since $f$ fulfills the conditions from the statement of Theorem 1.

For the case $k < 1$, inequality (8) is proved analogously, but this time by starting from (10), rewritten with $r = 1$, $s = p > 1$, $\alpha = 1$ and $\gamma = p - k + 1$ as parameters, and then applying Jensen's inequality and Fubini's theorem.

The proof that $\left(\frac{p}{|k-1|}\right)^p$ is the best possible constant for the inequalities (7) and (8) follows. For any $\varepsilon > 0$ and the function $f_\varepsilon : B(R) \to \mathbf{R}$ defined by $f_\varepsilon(\mathbf{x}) = |B(|\mathbf{x}|)|^{\frac{k-1+\varepsilon}{p}-1}$, the left-hand side of (7) is equal to

$$\begin{aligned}
L_\varepsilon &= \int_{B(R)} |B(|\mathbf{x}|)|^{-k} \left( \int_{B(|\mathbf{x}|)} f_\varepsilon(\mathbf{y})\, d\mathbf{y} \right)^p d\mathbf{x} \\
&= \left( \frac{p}{k-1+\varepsilon} \right)^p \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1} d\mathbf{x} = \left( \frac{p}{k-1+\varepsilon} \right)^p \frac{|B(R)|^\varepsilon}{\varepsilon},
\end{aligned}$$

while on the right-hand side we have

$$\begin{aligned} R_\varepsilon &= \left(\frac{p}{k-1}\right)^p \int_{B(R)} \left[1-\left(\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right)^{\frac{k-1}{p}}\right] |B(|\mathbf{x}|)|^{p-k} f_\varepsilon^p(\mathbf{x})\,d\mathbf{x} \\ &\leqslant \left(\frac{p}{k-1}\right)^p \int_{B(R)} |B(|\mathbf{x}|)|^{p-k} f_\varepsilon^p(\mathbf{x})\,d\mathbf{x} \\ &= \left(\frac{p}{k-1}\right)^p \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1} d\mathbf{x} = \left(\frac{p}{k-1}\right)^p \frac{|B(R)|^\varepsilon}{\varepsilon}. \end{aligned}$$

Therefore, $1 \leqslant \frac{R_\varepsilon}{L_\varepsilon} \leqslant \left(\frac{k-1+\varepsilon}{k-1}\right)^p \to 1$, as $\varepsilon \to 0$. Hence, $\left(\frac{p}{k-1}\right)^p$ is the best possible constant for (7). The proof that $\left(\frac{p}{1-k}\right)^p$ is the best possible constant for (8) is similar, if the function $f_\varepsilon : \mathbf{R}^n \setminus B(R) \to \mathbf{R}$, $f_\varepsilon(\mathbf{x}) = |B(|\mathbf{x}|)|^{\frac{k-1-\varepsilon}{p}-1}$, is considered. □

By using a similar approach we can obtain a generalization of Theorem B of the same type as the generalization of Theorem A given in Theorem 1. That result is given in the following theorem.

THEOREM 2. *Suppose $f$ is a positive measurable function, $\alpha, \gamma, R \in \mathbf{R}$ are such that $\alpha \neq 0$, $R > 0$, and the function $G$ is defined by* (5).

(i) *If $\alpha > 0$ and $0 < \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x} < \infty$, then*

$$\begin{aligned} &\int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} G(\mathbf{x})\,d\mathbf{x} \\ &\qquad < e^{\frac{\gamma}{\alpha}} \int_{B(R)} \left[1-\left(\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right)^{\alpha}\right] |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{13}$$

(ii) *If $\alpha < 0$ and $0 < \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x} < \infty$, then*

$$\begin{aligned} &\int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\gamma-1} G(\mathbf{x})\,d\mathbf{x} \\ &\qquad < e^{\frac{\gamma}{\alpha}} \int_{\mathbf{R}^n \setminus B(R)} \left[1-\left(\frac{|B(R)|}{|B(|\mathbf{x}|)|}\right)^{-\alpha}\right] |B(|\mathbf{x}|)|^{\gamma-1} f(\mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{14}$$

*The constant $e^{\frac{\gamma}{\alpha}}$ is the best possible for both inequalities.*

*Proof.* The proof is based on two relations, obtained in [3]. Let $s > 0$ be arbitrary. If $\alpha > 0$, then the inequality

$$\left\{ \frac{1}{|B(R)|^{\gamma}} \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} G^{s}(\mathbf{x})\, d\mathbf{x} \right\}^{\frac{1}{s}} \leqslant \exp\left[ \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \ln \left( \frac{1}{|B(|\mathbf{x}|)|^{\gamma}} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f^{s}(\mathbf{y})\, d\mathbf{y} \right)^{\frac{1}{s}} d\mathbf{x} \right] \tag{15}$$

holds (cf. [3], relation (18) from Theorem 6). By setting $s = 1$ in (15) and considering (2), we have

$$\begin{aligned} \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} G(\mathbf{x})\, d\mathbf{x} &\leqslant |B(R)|^{\gamma} \exp\left[ \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \right. \\ &\qquad \left. \cdot \ln \left( \frac{1}{|B(|\mathbf{x}|)|^{\gamma}} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \right] \\ &= |B(R)|^{\gamma} \exp\left[ -\frac{\alpha\gamma}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \ln |B(|\mathbf{x}|)|\, d\mathbf{x} \right. \\ &\qquad \left. + \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \ln \left( \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \right] \\ &= e^{\frac{\gamma}{\alpha}} \exp\left[ \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \ln \left( \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \right]. \end{aligned} \tag{16}$$

By using Jensen's inequality for the convex function $t \mapsto e^{t}$, the last line of (16) is less than

$$e^{\frac{\gamma}{\alpha}} \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}. \tag{17}$$

Applying Fubini's theorem and (1), the term (17) is further equal to

$$\begin{aligned} & e^{\frac{\gamma}{\alpha}} \frac{\alpha}{|B(R)|^{\alpha}} \int_{B(R)} |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y}) \int_{B(R)\setminus B(|\mathbf{y}|)} |B(|\mathbf{x}|)|^{\alpha-1}\, d\mathbf{x}\, d\mathbf{y} \\ &= e^{\frac{\gamma}{\alpha}} \int_{B(R)} \left[ 1 - \left( \frac{|B(|\mathbf{y}|)|}{|B(R)|} \right)^{\alpha} \right] |B(|\mathbf{y}|)|^{\gamma-1} f(\mathbf{y})\, d\mathbf{y}, \end{aligned}$$

so (13) is proved. Owing to the conditions of the theorem, the sign of inequality in (13) is strict.

Now, we discuss the best possible constant for (13). For any $\varepsilon > 0$, let the function $f_\varepsilon : B(R) \to \mathbf{R}$ be defined by $f_\varepsilon(\mathbf{x}) = \alpha e^{-\frac{\gamma}{\alpha}} |B(|\mathbf{x}|)|^{\alpha\varepsilon-\gamma}$. Calculating the left-hand side of (13) for $f_\varepsilon$, we obtain

$$\begin{aligned}
L_\varepsilon &= \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} \exp\left( \frac{\alpha}{|B(|\mathbf{x}|)|^\alpha} \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} \ln f_\varepsilon(\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \\
&= \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} \exp\left[ \frac{\alpha}{|B(|\mathbf{x}|)|^\alpha} \ln\left(\alpha e^{-\frac{\gamma}{\alpha}}\right) \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} d\mathbf{y} \right. \\
&\quad \left. + \frac{\alpha}{|B(|\mathbf{x}|)|^\alpha} (\alpha\varepsilon - \gamma) \int_{B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\alpha-1} \ln |B(|\mathbf{y}|)|\, d\mathbf{y} \right] d\mathbf{x} \\
&= \alpha e^{-\varepsilon} \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha\varepsilon-1} d\mathbf{x} = e^{-\varepsilon} \frac{|B(R)|^{\alpha\varepsilon}}{\varepsilon}.
\end{aligned}$$

On the other hand, the right-hand side of (13), rewritten for $f_\varepsilon$, can be estimated as follows:

$$\begin{aligned}
R_\varepsilon &= e^{\frac{\gamma}{\alpha}} \int_{B(R)} \left[ 1 - \left( \frac{|B(|\mathbf{x}|)|}{|B(R)|} \right)^\alpha \right] |B(|\mathbf{x}|)|^{\gamma-1} f_\varepsilon(\mathbf{x})\, d\mathbf{x} \\
&\leqslant e^{\frac{\gamma}{\alpha}} \int_{B(R)} |B(|\mathbf{x}|)|^{\gamma-1} f_\varepsilon(\mathbf{x})\, d\mathbf{x} = \alpha \int_{B(R)} |B(|\mathbf{x}|)|^{\alpha\varepsilon-1} d\mathbf{x} = \frac{|B(R)|^{\alpha\varepsilon}}{\varepsilon}.
\end{aligned}$$

Since $1 \leqslant \frac{R_\varepsilon}{L_\varepsilon} \leqslant e^\varepsilon \to 1$, as $\varepsilon \to 0$, the constant $e^{\frac{\gamma}{\alpha}}$ is the best possible constant factor for the inequality (13).

In the case $\alpha < 0$, the inequality (14) is a consequence of the relation

$$\begin{aligned}
&\left\{ \frac{1}{|B(R)|^\gamma} \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\gamma-1} G^s(\mathbf{x})\, d\mathbf{x} \right\}^{\frac{1}{s}} \\
&\leqslant \exp\left[ -\frac{\alpha}{|B(R)|^\alpha} \int_{\mathbf{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{\alpha-1} \ln \left( \frac{1}{|B(|\mathbf{x}|)|^\gamma} \int_{\mathbf{R}^n \setminus B(|\mathbf{x}|)} |B(|\mathbf{y}|)|^{\gamma-1} f^s(\mathbf{y})\, d\mathbf{y} \right)^{\frac{1}{s}} d\mathbf{x} \right]
\end{aligned} \tag{18}$$

([3], inequality (19) from Theorem 6), derived by the same technique as (13) from (15). The proof that $e^{\frac{\gamma}{\alpha}}$ is the best possible constant for (14) is also similar to the proof in the case $\alpha > 0$, if the function $f_\varepsilon : \mathbf{R}^n \setminus B(R) \to \mathbf{R}$, $f_\varepsilon(\mathbf{x}) = -\alpha e^{-\frac{\gamma}{\alpha}} |B(|\mathbf{x}|)|^{\alpha\varepsilon-\gamma}$, is considered. □

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