Mathematical
Inequalities
& Applications
Volume 5, Number 4 (2002), 633–636

ONCE MORE ON THE TELYAKOVSKIĬ'S CLASS S

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(*communicated by J. Pečarić*)

Abstract. We show once more the vitality and usability of the class S of numerical sequences. It is almost hopeless to give a real extension of the class S keeping the conditions $\Sigma A_n < \infty$ and $|\Delta a_n| \leqslant A_n$. Among others we give two further virtual extensions of S which are truly equivalent to S.

1. Introduction

In 1973 S. A. Telyakovskiĭ [7] redefined the class of numerical sequences introduced by S. Sidon [4], and his definition has became very useful and convenient to apply it in many problems. He denoted this class by S referring to the class defined by Sidon. Later several mathematicians have tried to extend the definition of Telyakovskiĭ (see e.g. [5], [6], [8]). In [1] and [2] we showed that some of these "extensions" are equivalent to the class S, and some others are real extensions of S, but they are identical among themselves. In [3] we already enlist three classes defined later by different authors which are identical with S, and five classes which are true extensions of S, but identical among themselves.

The definition of the class S is the following: A null-sequence $\mathbf{a} := \{a_n\}$ belongs to *the class* S, or briefly $\mathbf{a} \in S$, if there exists a monotonically decreasing sequence $\mathbf{A} := \{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leqslant A_n$ hold for all $n \in \mathbf{N}$.

The definition of *the class* S_p, which is a real extension of S if $p > 1$, is the following: A null-sequence $\mathbf{a}$ belongs to $S_p (p > 0)$ if there exists a positive monotonically decreasing sequence $\mathbf{A}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \tag{1.1}$$

and

$$\sum_{k=1}^{m} \frac{|\Delta a_k|^p}{A_k^p} = O(m) \tag{1.2}$$

Mathematics subject classification (2000): 26D15, 42A20.

Key words and phrases: Inequalities, embedding relations, sequences of rest bounded variation.

This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant # T 029080.

hold.

Recently we have defined four new classes of sequences, two of them are virtual extensions of the class S, and the other two that of S_p. Our aim by these generalizations was naturally to extend the validity domain of the theorems proved formerly for the classes of S and S_p. But having so many aborted attempts at extending these classes we have investigated whether our new classes are real extensions or only virtual ones. It has turned out that the latter case is true, that is, our new classes to be defined later, are equivalent to S or to S_p.

Thus the aim of our note is only to prove these equivalences.

Now we present some definitions.

A positive monotonically decreasing sequence $\mathbf{c} = \{c_n\}$ belongs to *the class* CM, or briefly $\mathbf{c} \in CM$, if

$$\sum_{n=1}^{\infty} c_n < \infty. \tag{1.3}$$

A positive sequence $\mathbf{c}$ belongs to *the class* CR, or $\mathbf{c} \in CR$, if (1.3) and

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leqslant K(\mathbf{c}) c_m, \quad m = 1, 2, \ldots \tag{1.4}$$

hold and $K(c)$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c}$ belongs to *the class* CAD, or $\mathbf{c} \in CAD$, if there exists a positive monotonically decreasing sequence $\{d_n\}$ and two positive constants α and β such that (1.3) and

$$\alpha d_n \leqslant c_n \leqslant \beta d_n, \quad m = 1, 2, \ldots \tag{1.5}$$

In these definitions the signs CM, CR and CAD refer to the following abbreviations: convergent and monotonic, convergent and rest bounded variation (see (1.4)), convergent and almost decreasing, respectively.

It is plain that

$$CM \subset CR \subset CAD, \tag{1.6}$$

namely if we set $d_n := \sum_{k=n}^{\infty} |c_k - c_{k+1}| \geqslant c_n$, then by (1.4), with $\alpha = K(c)^{-1}$ and $\beta = 1$, (1.5) holds.

A null-sequence $\mathbf{a} := \{a_n\}$ $(a_n \to 0)$ belongs to *the classes* $S(CM)$, $S(CR)$ and $S(CAD)$ if there exists a sequence $\mathbf{A} := \{A_n\}$ such that

$$|\Delta a_n| \leqslant A_n \tag{1.7}$$

holds for all n, and $\mathbf{A}$ belongs to CM, CR and CAD, respectively.

Similarly we can define *the classes* $S_p(CM)$, $S_p(CR)$ and $S_p(CAD)$ if the sequence $\{A_n\}$ satisfies (1.2) instead of (1.7).

It is clear that

$$S(CM) \equiv S \quad \text{and} \quad S_p(CM) \equiv S_p.$$

Furthermore, by (1.6), we clearly have that

$$S \equiv S(CM) \subset S(CR) \subset S(CAD) \tag{1.8}$$

and

$$S_p \equiv S_p(CM) \subset S_p(CR) \subset S_p(CAD) \tag{1.9}$$

hold.

If we analyze what is the reason that the awaited real extensions have been unsuccessful, in my view, is that the special conditions given for the sequence $\{A_n\}$ and the conditions (1.1) always implied the existence of a monotonically decreasing sequence $\{A_n^*\}$ with the properties:

$$A_n \leqslant KA_n^* \quad \text{and} \quad \sum_{n=1}^{\infty} A_n^* < \infty. \tag{1.10}$$

Hence, the sequence $\{A_n^*\}$ is clearly a CM-sequence, whence $|\Delta a_n| \leqslant KA_n^*$ also follows, and thus $\{a_n\} \in S$, that is, the class in question is equivalent to S, or S_p if $\{A_n\}$ satisfied (1.2).

Our theorem will be proved also in this way.

2. Theorem and proof

We have the next result.

THEOREM. *The following embedding relations*

$$S \equiv S(CR) \equiv S(CAD) \equiv S(CAD) \subsetneqq S_p \equiv S_p(CR) \equiv S_p(CAD)$$

are valid for any $p > 0$.

REMARK. In [3] we verified that if $p > 1$ then the class S_p is identical with four further classes of numerical sequences, and S with three others.

Proof. By the embedding relations (1.8), (1.9) and the known result

$$S \subsetneqq S_p$$

(see [3]), it is enough to prove that if $\{A_n\} \in CAD$ then there exists a positive monotonically decreasing sequence $\{A_n^*\}$ with the properties (1.10), that is, to prove that $\{A_n^*\} \in CM$.

Since the assumption $\{A_n\} \in CAD$ implies that there exists a positive monotonically decreasing sequence $\{d_n\}$ such that

$$\alpha\, d_n \leqslant A_n \leqslant \beta\, d_n,$$

thus setting $A_n^* := d_n$ we get that

$$A_n \leqslant \beta\, A_n^* \quad \text{and} \quad \alpha \sum_{n=1}^{\infty} A_n^* \leqslant \sum_{n=1}^{\infty} A_n < \infty,$$

that is, the conditions (1.10) hold.

Hence, as explained, the embedding relations

$$S(CAD) \subset S \quad \text{and} \quad S_p(CAD) \subset S_p$$

hold.

This completes the proof.

REFERENCES

[1] L. LEINDLER, *On the equivalence of classes of Fourier coefficients*, Math. Ineq. & Applications **3** (2000), 45–50.

[2] L. LEINDLER, *On the equivalence of classes of numerical sequences*, Analysis Math. **26** (2000), 227–234.

[3] L. LEINDLER, *A note on some classes of real sequences*, Math. Ineq. & Applications **4** (2001), 53–58.

[4] S. SIDON, *Hinreichende Bedingungen für den Fourier-charakter einer trigonometrischen Reihe*, J. London Math. Soc. **14** (1939), 158–160.

[5] N. SING AND K. M. SHARMA, *Integrability of trigonometric series*, J. Indian Math. Soc. **49** (1985), 31–38.

[6] Č. V. STANEJEVIČ, AND V. B. STANOJEVIČ, *Generalizations of Sidon-Telyakovskiĭ, theorem*, Proc. Amer. Math. Soc. **101** (1987), 679–684.

[7] S. A. TELYAKOVSKII, *On a sufficient condition of Sidon for integrability of trigonometric series*, Math. Zametki (Russian) **14** (1973), 317–328.

[8] S. Z. A. ZENEI, *Integrability of trigonometric series*, Tamkang J. Math. **21** (1990), 295–301.

(Received February 6, 2001)

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