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IMPROVEMENT OF SOME ESTIMATIONS RELATED TO THE REMAINDER IN GENERALIZED TAYLOR'S FORMULA

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Abstract. We prove an inequality of Grüss type and using it we strictly improve some estimations of the remainder in generalized Taylor's formula obtained via harmonic sequence of polynomials.

1. Introduction

In the recent paper [1] Matić et al. considered a generalized Taylor's formula. More precisely, the basic result from [1] is the following theorem:

THEOREM 1. *Let* $\{P_n(x)\}$ *be a harmonic sequence of polynomials, that is*

$$P_n'(x) = P_{n-1}(x),\ \textit{for } n \in \mathbf{N};\ \ P_0(x) = 1.$$

Further, let $I \subset \mathbf{R}$ *be a closed interval and* $a \in I$. *If* $f : I \to \mathbf{R}$ *is any function such that, for some* $n \in \mathbf{N}$, $f^{(n)}$ *is absolutely continuous, then for any* $x \in I$

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f; a, x), \tag{1.1}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) \mathrm{d}t.$$

Formula (1.1) can be regarded as generalized Taylor's formula. Namely, if we set in (1.1)

$$P_n(t) = \frac{(t-x)^n}{n!},\ \ n \in \mathbf{N} \tag{1.2}$$

then we get the classical Taylor's formula:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n^T(f; a, x), \tag{1.3}$$

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where the remainder $R_n^T(f;a,x)$ is given as

$$R_n^T(f;a,x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\mathrm{d}t.$$

Further, for $x \neq a$ formula (1.1) can be rewritten in perturbed form as

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x), \qquad (1.4)$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\right]$$

and

$$\tilde{G}_n(f;a,x) = R_n(f;a,x) - (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right],$$

while $\left[f^{(n)};a,x\right]$ is a divided difference defined as

$$\left[f^{(n)};a,x\right] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}.$$

In the case when the polynomials P_n are given by (1.2) we have perturbed Taylor's formula

$$f(x) = f(a) + \sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right] + G_n^T(f;a,x), \qquad (1.5)$$

where

$$G_n^T(f;a,x) = R_n^T(f;a,x) - \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right].$$

The main result in [1] is the estimation of the remainder $\tilde{G}_n(f;a,x)$ in perturbed generalized Taylor's formula (1.4):

THEOREM 2. *Let $\{P_n(x)\}$ be a harmonic sequence of polynomials. Let $I \subset \mathbf{R}$ be a closed interval and $a \in I$. Suppose $f : I \to R$, is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$, $x \neq a$, the perturbed generalized Taylor's formula (1.4) is valid. Furthermore, for $x > a$, if*

$$\Gamma(x) = \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) = \inf_{t\in[a,x]} f^{(n+1)}(t),$$

then the remainder $\tilde{G}_n(f;a,x)$ satisfies the estimation

$$\left|\tilde{G}_n(f;a,x)\right| \leqslant \frac{x-a}{2}\sqrt{T(P_n,P_n)}[\Gamma(x) - \gamma(x)], \qquad (1.6)$$

where

$$T(P_n,P_n) = \frac{1}{x-a}\int_a^b P_n^2(t)\mathrm{d}t - \left(\frac{1}{x-a}\int_a^b P_n(t)\mathrm{d}t\right)^2.$$

As the corollary to the above result an estimate of the remainder $G_n^T(f;a,x)$ in perturbed Taylor's formula (1.5) was obtained [1, Corollary 2]:

$$\left|G_n^T(f;a,x)\right| \leqslant \frac{n(x-a)^{n+1}}{2[(n+1)!]\sqrt{2n+1}}[\Gamma(x)-\gamma(x)]. \tag{1.7}$$

This result was an improvement of the result from [2] where the factor $\frac{(x-a)^{n+1}}{4(n!)}$ stands in place of $\frac{n(x-a)^{n+1}}{2[(n+1)!]\sqrt{2n+1}}$.

The estimation (1.6) was obtained by direct application of the key technical result [1, Lemma 1] which is in fact an easy consequence of the well known Cauchy-Schwarz and Grüss inequalities.

In this paper we first give an improvement of the above mentioned key result from [1]. This improvement is a result which is interesting on its own and we prove it in Section 2. After that we use it to improve the estimation (1.6). We also give a corresponding improvement of the estimation (1.7) as well as of some other estimations from [1] which were obtained there as a consequences of the estimation (1.6).

2. An inequality of Grüss type

Let $a<x$ and let $g,h:[a,x]\to\mathbf{R}$ be two integrable functions. Define

$$T(g,h)=\frac{1}{x-a}\int_a^x g(t)h(t)\mathrm{d}t-\frac{1}{(x-a)^2}\int_a^x g(t)\mathrm{d}t\int_a^x h(t)\mathrm{d}t$$

Then $T(g,g)\geqslant 0$, $T(h,h)\geqslant 0$ and the following inequality is valid [4, p. 209]

$$T^2(g,h)\leqslant T(g,g)T(h,h). \tag{2.1}$$

On the other hand, if

$$\alpha\leqslant g(t)\leqslant A,\ \ \beta\leqslant h(t)\leqslant B,\ \ \forall t\in[a,x],$$

for some constants α, A, β and B, then the well known Grüss' inequality

$$|T(g,h)|\leqslant\frac{1}{4}(A-\alpha)(B-\beta) \tag{2.2}$$

holds (see [4, p. 206]). Using the inequalities (2.1) and (2.2) the following simple result is easily obtained [1, Lemma 1]:

LEMMA 1. *Let $a<x$ and let $g,h:[a,x]\to\mathbf{R}$ be two integrable functions. If*

$$\alpha\leqslant g(t)\leqslant A,\ \ \forall t\in[a,x],$$

for some constants α and A, then

$$|T(g,h)|\leqslant\frac{A-\alpha}{2}\sqrt{T(h,h)}. \tag{2.3}$$

Now we give an improvement of this result:

THEOREM 3. *Let* $a < x$ *and let* $g, h : [a,x] \to \mathbf{R}$ *be two integrable functions such that*

$$\alpha \leqslant g(t) \leqslant A, \quad \forall t \in [a,x], \tag{2.4}$$

for some constants α *and* A. *If*

$$H = \frac{1}{x-a}\int_a^x h(s)\mathrm{d}s,$$

then

$$\begin{aligned} |T(g,h)| &\leqslant \frac{A-\alpha}{2}\cdot\frac{1}{x-a}\int_a^x |h(t)-H|\,\mathrm{d}t \\ &\leqslant \frac{A-\alpha}{2}\cdot\sqrt{T(h,h)}, \end{aligned} \tag{2.5}$$

Proof. Consider the function G defined as

$$G(t) = h(t) - H, \; t \in [a,x]$$

and corresponding positive and negative parts given by the formulas

$$G^+(t) = \frac{1}{2}\left(|G(t)| + G(t)\right), \; G^-(t) = \frac{1}{2}\left(|G(t)| - G(t)\right), \; t \in [a,x].$$

A simple properties of the functions G^+ and G^- are well known:

$$G^+(t) \geqslant 0, \; G^-(t) \geqslant 0, \; t \in [a,x] \tag{2.6}$$

and

$$G^+(t) + G^-(t) = |G(t)|, \; G^+(t) - G^-(t) = G(t), \; t \in [a,x]. \tag{2.7}$$

Obviously we have

$$\int_a^x G(t)\mathrm{d}t = 0 \tag{2.8}$$

and it is easy to check that

$$T(g,h) = \frac{1}{x-a}\int_a^x g(t)G(t)\mathrm{d}t. \tag{2.9}$$

Now, using (2.7) and (2.8) we get

$$\int_a^x G^+(t)\mathrm{d}t = \int_a^x G^-(t)\mathrm{d}t = \frac{1}{2}\int_a^x |G(t)|\,\mathrm{d}t. \tag{2.10}$$

On the other side from (2.4) and (2.6) we get

$$\alpha G^+(t) \leqslant g(t)G^+(t) \leqslant AG^+(t), \quad a \leqslant t \leqslant x$$

and

$$-AG^-(t) \leqslant -g(t)G^-(t) \leqslant -\alpha G^-(t), \; a \leqslant t \leqslant x.$$

Integrating over $[a,x]$ we further get

$$\alpha \int_a^b G^+(t)\mathrm{d}t \leqslant \int_a^x g(t)G^+(t)\mathrm{d}t \leqslant A \int_a^x G^+(t)\mathrm{d}t \tag{2.11}$$

and

$$-A \int_a^b G^-(t)\mathrm{d}t \leqslant -\int_a^x g(t)G^-(t)\mathrm{d}t \leqslant -\alpha \int_a^x G^-(t)\mathrm{d}t. \tag{2.12}$$

Adding the inequalities (2.11) and (2.12) together and using the equalities (2.7) and (2.10) we finally get

$$-\frac{A-\alpha}{2}\int_a^x |G(t)|\,\mathrm{d}t \leqslant \int_a^x g(t)G(t)\mathrm{d}t \leqslant \frac{A-\alpha}{2}\int_a^x |G(t)|\,\mathrm{d}t,$$

which is by (2.9) equivalent to the first inequality in (2.5). The second inequality in (2.5) follows from the well known inequality between the arithmetic mean and the quadratic mean. Namely, it is easy to check the identity

$$\begin{aligned} T(h,h) &= \frac{1}{x-a}\int_a^x h^2(t)\mathrm{d}t - \left(\frac{1}{x-a}\int_a^x h(t)\mathrm{d}t\right)^2 \\ &= \frac{1}{x-a}\int_a^x [h(t)-H]^2\,\mathrm{d}t. \end{aligned}$$

REMARK 1. The result stated in Lemma 1 is of particular interest in the case when $\int_a^x [h(t)-H]^2\,\mathrm{d}t$ can be evaluated exactly. The same is true for the first inequality in (2.5) in the case when $\int_a^x |h(t)-H|\,\mathrm{d}t$ can be evaluated exactly. In this sense the first inequality in (2.5) is indeed an improvement of the inequality (2.3) because the equality case in the second inequality in (2.5) can occur only when $|h(t)-H|$ is a constant function almost everywhere on $[a,x]$. If $|h(t)-H|$ is not a constant function almost everywhere on $[a,x]$, then the second inequality in (2.5) is strict.

REMARK 2. Let us consider the equality case in the first inequality in (2.5). Let us denote

$$I^+ = \{t \in [a,x] : G(t) > 0\},\ I^- = \{t \in [a,x] : G(t) < 0\}.$$

By the careful inspection of the proof of Theorem 3 we conclude that the equality occurs in the first inequality in (2.5) if and only if either

$$g(t) = \alpha,\ t \in I^+ \text{ and } g(t) = A,\ t \in I^- \ \text{ (a.e. on } [a,x]),$$

or

$$g(t) = A,\ t \in I^+ \text{ and } g(t) = \alpha,\ t \in I^- \ \text{ (a.e. on } [a,x]).$$

3. Main results

Assuming the notations from previous sections we can state the main result as follows:

THEOREM 4. *Let* $\{P_n(x)\}$ *be a harmonic sequence of polynomials. Let* $I \subset \mathbf{R}$ *be a closed interval and* $a \in I$. *Suppose* $f : I \to R$, *is such that* $f^{(n)}$ *is absolutely continuous. Then for any* $x \in I$, $x \neq a$, *the perturbed generalized Taylor's formula* (1.4) *is valid. Furthermore, for* $x > a$, *if*

$$\Gamma(x) = \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) = \inf_{t\in[a,x]} f^{(n+1)}(t), \tag{3.1}$$

then the remainder $\tilde{G}_n(f;a,x)$ *satisfies the estimation*

$$\left|\tilde{G}_n(f;a,x)\right| \leqslant \frac{\Gamma(x)-\gamma(x)}{2}\int_a^x |P_n(t)-[P_{n+1};a,x]|\,\mathrm{d}t, \tag{3.2}$$

where

$$[P_{n+1};a,x] = \frac{P_{n+1}(x)-P_{n+1}(a)}{x-a}.$$

Proof. As in [1], the remainder $\tilde{G}_n(f;a,x)$ can be rewritten as

$$\tilde{G}_n(f;a,x) = (-1)^n(x-a)T(f^{(n+1)},P_n),$$

so that we have

$$\left|\tilde{G}_n(f;a,x)\right| = (x-a)\left|T(f^{(n+1)},P_n)\right|.$$

Now we can apply Theorem 3 with $g = f^{(n+1)}$ and $h = P_n$. Since $P'_{n+1}(t) = P_n(t)$ we have

$$H = \frac{1}{x-a}\int_a^x P_n(t)\mathrm{d}t = \frac{P_{n+1}(x)-P_{n+1}(a)}{x-a},$$

and therefore we get

$$\left|\tilde{G}_n(f;a,x)\right| \leqslant \frac{\Gamma(x)-\gamma(x)}{2}\cdot\int_a^x |P_n(t)-[P_{n+1};a,x]|\,\mathrm{d}t.$$

COROLLARY 1. *Under the assumptions of Theorem* 4 *the perturbed Taylor's formula* (1.5) *is valid. If* $x > a$ *and if* $\Gamma(x)$ *and* $\gamma(x)$ *are gjven by* (3.1), *then the remainder* $G_n^T(f;a,x)$ *satisfies the estimation*

$$\left|G_n^T(f;a,x)\right| \leqslant \frac{n(x-a)^{n+1}}{(n+1)!(n+1)\sqrt[n]{n+1}}\cdot[\Gamma(x)-\gamma(x)]. \tag{3.3}$$

Proof. For P_n given by (1.2) we have

$$[P_{n+1};a,x] = \frac{(a-x)^n}{(n+1)!}.$$

By the substitution $t = x + (a - x)s$ we get

$$\begin{aligned}\int_a^x |P_n(t) - [P_{n+1}; a, x]|\,\mathrm{d}t &= \int_a^x \left| \frac{(t-x)^n}{n!} - \frac{(a-x)^n}{(n+1)!} \right| \mathrm{d}t \\ &= \frac{(x-a)^{n+1}}{n!} \int_0^1 \left| s^n - \frac{1}{n+1} \right| \mathrm{d}s \\ &= \frac{2n(x-a)^{n+1}}{(n+1)!(n+1)\sqrt[n]{n+1}}\end{aligned}$$

and (3.3) follows from (3.2).

REMARK 3. By δ_n and Δ_n denote the right hand sides of (3.3) and (1.7), respectively Then we have.

$$\frac{\delta_n}{\Delta_n} = \frac{2\sqrt{2n+1}}{(n+1)\sqrt[n]{n+1}} < 1,\ n = 1, 2, \cdots$$

which shows that the estimate (3.3) is indeed better than the estimate (1.7). Moreover we have $\lim_{n\to\infty} (\delta_n/\Delta_n) = 0$.

In [1] another three special cases of perturbed generalized Taylor's formula (1.4) were considered. The first of them is obtained when we choose harmonic polynomials P_n defined as

$$P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n,\ n \in \mathbf{N}. \tag{3.4}$$

In this case (1.4) reduces to the formula

$$\begin{aligned} f(x) &= f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!}[f^{(k)}(a) - (-1)^k f^{(k)}(x)] \\ &\quad + \frac{(x-a)^{n+1}[1 + (-1)^n]}{(n+1)!2^{n+1}} \left[f^{(n)}; a, x\right] + G_n^M(f; a, x), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} G_n^M(f; a, x) &= \frac{(-1)^n}{n!}\int_a^x \left(t - \frac{a+x}{2}\right)^n f^{(n+1)}(t)\mathrm{d}t \\ &\quad - \frac{(x-a)^{n+1}[1 + (-1)^n]}{(n+1)!2^{n+1}} \left[f^{(n)}; a, x\right]. \end{aligned} \tag{3.6}$$

The second special case is obtained when harmonic polynomials P_n are defined as

$$P_n(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right),\ n \in \mathbf{N}, \tag{3.7}$$

where $B_n(\cdot)$, $n \in \mathbf{N}$ are the well known Bernoulli polynomials (for details on the properties of Bernoulli polynomials and Bernoulli numbers $B_n = B_n(0)$ see [3, Chapter

23]). When P_n are defined by (3.7), formula (1.4) reduces to

$$\begin{aligned} f(x) &= f(a) + \frac{x-a}{2}[f'(x)+f'(a)]f(a) + \frac{x-a}{2}[f'(x)+f'(a)] \\ &\quad - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k}\left[f^{(2k)}(x) - f^{(2k)}(a)\right] + G_n^B(f;a,x), \end{aligned} \tag{3.8}$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$ and

$$G_n^B(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\mathrm{d}t. \tag{3.9}$$

Finally the third special case of perturbed generalized Taylor's formula (1.4) considered in [1] is obtained when we choose

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right),\ n \in \mathbf{N}, \tag{3.10}$$

where $E_n(\cdot)$, $n \in \mathbf{N}$ are the well known Euler polynomials which are closely related to the Bernoulli polynomials (for details on the properties of Euler polynomials and Euler numbers $E_n = 2^n E_n\left(\frac{1}{2}\right)$ see [3, Chapter 23]). In this case formula (1.4) reduces to

$$\begin{aligned} f(x) &= f(a) + 2\sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}(4^k-1)}{(2k)!} B_{2k}\left[f^{(2k-1)}(x) + f^{(2k-1)}(a)\right] \\ &\quad + \frac{4(-1)^n(x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right] + G_n^E(f;a,x), \end{aligned} \tag{3.11}$$

where $\left[\frac{n+1}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n+1}{2}$ and

$$\begin{aligned} G_n^E(f;a,x) &= (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\mathrm{d}t \\ &\quad - \frac{4(-1)^n(x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right]. \end{aligned} \tag{3.12}$$

Similarly as we did in the case of perturbed Taylor's formula (1.5), we can use the inequality (3.2) to improve the estimations from [1, Corollary 3] related to the formulas (3.5), (3.8) and (3.11):

COROLLARY 2. *Let the assumptions of Theorem* 4 *be satisfied. For* $x > a$ *let* $\Gamma(x)$ *and* $\gamma(x)$ *be defined by* (3.1).
(i) *If* $P_n(\cdot)$ *are defined by* (3.4), *then* (3.5) *holds and*

$$\begin{aligned} &\left|G_n^M(f;a,x)\right| \\ &\leqslant \frac{(x-a)^{n+1}}{(n+1)!2^{n+1}}\left[\frac{1-(-1)^n}{2} + \frac{(1+(-1)^n)n}{(n+1)\sqrt[n]{n+1}}\right]\cdot[\Gamma(x)-\gamma(x)]. \end{aligned} \tag{3.13}$$

(ii) *If* $P_n(\cdot)$ *are defined by* (3.7)*, then* (3.8) *holds and*

$$\left|G_n^B(f\,;a,x)\right| \leqslant \frac{(x-a)^{n+1}}{n!}\int_0^{\frac{1}{2}} |B_n(s)|\,\mathrm{d}s\cdot[\Gamma(x)-\gamma(x)]. \tag{3.14}$$

Moreover, if $n=2r-1$, $r\geqslant 1$*, then*

$$\left|G_{2r-1}^B(f\,;a,x)\right| \leqslant \frac{(x-a)^{2r}}{(2r)!}2(1-2^{-2r})\,|B_{2r}|\cdot[\Gamma(x)-\gamma(x)]. \tag{3.15}$$

(iii) *If* $P_n(\cdot)$ *are defined by* (3.10)*, then* (3.11) *holds and*

$$\begin{aligned}&\left|G_n^E(f\,;a,x)\right|\\ &\leqslant \frac{(x-a)^{n+1}}{n!}\int_0^{\frac{1}{2}}\left|E_n(s)-\frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}\right|\mathrm{d}s\cdot[\Gamma(x)-\gamma(x)].\end{aligned} \tag{3.16}$$

Moreover, if $n=2r-1$, $r\geqslant 1$*, then*

$$\left|G_{2r-1}^E(f\,;a,x)\right| \leqslant \frac{(x-a)^{2r}}{(2r)!}2^{-2r}\,|E_{2r}|\cdot[\Gamma(x)-\gamma(x)]. \tag{3.17}$$

Proof. (i) If $P_n(\cdot)$ are defined by (3.4), then we have

$$\begin{aligned}[P_{n+1};a,x] &= \frac{1}{(n+1)!(x-a)}\left[\left(x-\frac{a+x}{2}\right)^{n+1}-\left(a-\frac{a+x}{2}\right)^{n+1}\right]\\ &= \frac{1}{(n+1)!}\left(\frac{a-x}{2}\right)^{n}\frac{1+(-1)^n}{2}.\end{aligned}$$

Therefore, using the substitution $t=\frac{a+x}{2}+\frac{a-x}{2}s$ we get

$$\begin{aligned}&\int_a^x |P_n(t)-[P_{n+1};a,x]|\,\mathrm{d}t\\ &=\frac{1}{n!}\int_a^x\left|\left(t-\frac{a+x}{2}\right)^n-\left(\frac{a-x}{2}\right)^n\frac{1+(-1)^n}{2(n+1)}\right|\mathrm{d}t\\ &=\frac{1}{n!}\left(\frac{x-a}{2}\right)^{n+1}\int_{-1}^1\left|s^n-\frac{1+(-1)^n}{2(n+1)}\right|\mathrm{d}s\\ &=\frac{(x-a)^{n+1}}{n!2^n}\int_0^1\left|s^n-\frac{1+(-1)^n}{2(n+1)}\right|\mathrm{d}s.\end{aligned}$$

Further, we have

$$\int_0^1\left|s^n-\frac{1+(-1)^n}{2(n+1)}\right|\mathrm{d}s=\begin{cases}\frac{1}{n+1}, & \text{when } n \text{ is odd,}\\ \frac{2n}{(n+1)^2\sqrt[n]{n+1}}, & \text{when } n \text{ is even,}\end{cases}$$

so that

$$\int_a^x |P_n(t)-[P_{n+1};a,x]|\,\mathrm{d}t=\frac{(x-a)^{n+1}}{(n+1)!2^n}\left[\frac{1-(-1)^n}{2}+\frac{(1+(-1)^n)n}{(n+1)\sqrt[n]{n+1}}\right]$$

and (3.13) follows from (3.2).
(ii) If $P_n(\cdot)$ are defined by (3.7), then

$$[P_{n+1};a,x] = \frac{(x-a)^{n+1}}{(n+1)!(x-a)}\left[B_{n+1}(1)-B_{n+1}(0)\right] = 0,$$

since [3, Formula 23.1.6]

$$B_k(t+1)-B_k(t) = kt^{k-1},\ k=0,1,\cdots$$

implies $B_{n+1}(1)-B_{n+1}(0)=0$. Therefore, using the substitution $t=a+(x-a)s$ we get

$$\begin{aligned}\int_a^x |P_n(t)-[P_{n+1};a,x]|\,\mathrm{d}t &= \frac{(x-a)^n}{n!}\int_a^x \left|B_n\left(\frac{t-a}{x-a}\right)\right|\mathrm{d}t\\ &= \frac{(x-a)^{n+1}}{n!}\int_a^x |B_n(s)|\,\mathrm{d}s\\ &= \frac{(x-a)^{n+1}}{n!}2\int_a^{\frac{1}{2}} |B_n(s)|\,\mathrm{d}s.\end{aligned}$$

The last equality follows from [3, Formula 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t),\ k=0,1,\cdots.$$

Now (3.14) follows from (3.2). Further, for $n=2r-1$ we have [3, Formula 23.1.14]

$$(-1)^r B_{2r-1}(t) > 0,\ 0<t<\frac{1}{2},$$

so that

$$\begin{aligned}\int_0^{\frac{1}{2}} |B_{2r-1}(s)|\,\mathrm{d}s &= \left|\int_0^{\frac{1}{2}} B_{2r-1}(s)\mathrm{d}s\right|\\ &= \frac{1}{2r}\left|B_{2r}\left(\frac{1}{2}\right)-B_{2r}(0)\right|\\ &= \frac{1}{2r}\left|-(1-2^{1-2r})B_{2r}-B_{2r}\right|\\ &= \frac{1-2^{-2r}}{r}\left|B_{2r}\right|.\end{aligned}$$

We used the facts that $B_{2r}(0)=B_{2r}$ and [3, Formula 23.1.21] $B_{2r}\left(\frac{1}{2}\right) = -(1-2^{1-2r})B_{2r}$ .Now the inequality (3.15) follows from (3.14).
(iii) If $P_n(\cdot)$ are defined by (3.10), then

$$\begin{aligned}[P_{n+1};a,x] &= \frac{(x-a)^{n+1}}{(n+1)!(x-a)}\left[E_{n+1}(1)-E_{n+1}(0)\right]\\ &= \frac{(x-a)^n}{(n+2)!}4(2^{n+2}-1)B_{n+2},\end{aligned}$$

since [3, Formula 23.1.20]

$$E_k(0) = -E_k(1) = -\frac{2(2^{k+1}-1)B_{k+1}}{k+1},\ k = 1, 2, \cdots .$$

Therefore, using the substitution $t = a + (x-a)s$ we get

$$\begin{aligned}\int_a^x |P_n(t) - [P_{n+1}; a, x]|\,\mathrm{d}t &= \frac{(x-a)^n}{n!}\int_a^x \left| E_n\left(\frac{t-a}{x-a}\right) - \frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}\right| \mathrm{d}t \\ &= \frac{(x-a)^{n+1}}{n!}\int_a^x \left| E_n(s) - \frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}\right| \mathrm{d}s \\ &= \frac{(x-a)^{n+1}}{n!} 2\int_a^{\frac{1}{2}} \left| E_n(s) - \frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}\right| \mathrm{d}s\end{aligned}$$

The last equality follows from [3, Formula 23.1.8]

$$E_k(1-t) = (-1)^k E_k(t),\ k = 0, 1, \cdots$$

and the fact that $B_k = 0$ for odd $k \geqslant 3$. Now (3.16) follows from (3.2). Further, for $n = 2r-1$ we have [3, Formula 23.1.14]

$$(-1)^r E_{2r-1}(t) > 0,\ 0 < t < \frac{1}{2},$$

so that

$$\begin{aligned}\int_a^{\frac{1}{2}} \left| E_n(s) - \frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}\right| \mathrm{d}s &= \int_0^{\frac{1}{2}} |E_{2r-1}(s)|\,\mathrm{d}s \\ &= \left|\int_0^{\frac{1}{2}} E_{2r-1}(s)\mathrm{d}s\right| \\ &= \frac{1}{2r}\left| E_{2r}\left(\frac{1}{2}\right) - E_{2r}(0)\right| \\ &= \frac{1}{2r} 2^{-2r} |E_{2r}|.\end{aligned}$$

We used the facts that $B_{2r+1} = 0$, $E_{2r}(0) = 0$ and [3, Formula 23.1.21] $E_{2r}\left(\frac{1}{2}\right) = 2^{-2r}E_{2r}$. Now the inequality (3.17) follows from (3.16).

REMARK 4. By the observations given in Remark 1, we conclude that the estimations given in (3.13), (3.14) and (3.16) are strict improvements of the corresponding results from [1].

REMARK 5. Applying the observations given in Remark 2 it is not hard to see that all the inequalities proved in this section are strict except in a trivial case when $f^{(n+1)}$ is constant on I, that is when f is polynomial whose degree is at most $n+1$.

REFERENCES

[1] M. Matić, J. Pečarić and N. Ujević, On new estimation of the remainder in generalized Taylor's formula, Math. Inequal. Appl. 2 (3) (1999), 343–361.
[2] S.S. Dragomir, New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications, Math. Inequal. Appl. 2 (2) (1999), 183–193.
[3] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 4th printing, Washington, 1965
[4] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, 1992.

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