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INTEGRAL INEQUALITIES OF THE BIHARI TYPE

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Abstract. In this paper a new integral inequality similar to Bihari's inequality and its two independent variable generalization are established. The discrete analogues of the main results are also given.

1. Introduction

I. Bihari [3] proved an integral inequality, which has played a fundamental role in the development of the theory of differential and integral equations. A great deal of attention has been given to this inequality and many papers dealing with various genaralizations, extensions and discrete analogues have appeared in the literature, see [1–10] and the references given therein. However, the well known Gronwall inequality and its nonlinear version due to Bihari are not directly applicable in certain situations. It is desirable to find a new inequality of the Bihari type containing the Gronwall type inequality, which will prove its importance to achieve a diversity of desired goals. The main purpose of this paper is to establish a new inequality similar to Bihari's inequality and its two independent variable generalization. The discrete analogues of the main results and some applications are also given.

2. Main results

In what follows, we denote by $\mathbf{R}$ the set of real numbers. Let $\mathbf{R}_+ = [0, \infty)$, $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ and $'$ denote the derivative. For any real-valued function $z(x, y)$, x, $y \in \mathbf{R}$ the first order partial derivatives with respect to x and y are denoted by $D_1 z(x, y)$ and $D_2 z(x, y)$, respectively. We use the usual convention that the empty sum is taken to be 0.

Our main result is given in the following theorem.

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THEOREM 1. *Let* $u(t)$, $f(t) \in c(\mathbf{R}_+, \mathbf{R}_+)$, $h(t,s) \in c(\mathbf{R}_+^2, \mathbf{R}_+)$, *for* $0 \leqslant s \leqslant t < \infty$ *and* $c \geqslant 0$, $p > 1$ *are real constants. Let* $g \in c(\mathbf{R}_+, \mathbf{R}_+)$ *be a nondecreasing function,* $g(u) > 0$ *for* $u > 0$ *and*

$$u^p(t) \leqslant c + \int_0^t \Big[f(s)g(u(s)) + \int_0^s h(s,\sigma)g(u(\sigma))d\sigma\Big]ds, \tag{2.1}$$

for $t \in \mathbf{R}_+$, *then for* $0 \leqslant t \leqslant t_1$,

$$u(t) \leqslant [G^{-1}[G(c) + A(t)]]^{1/p}, \tag{2.2}$$

where

$$A(t) = \int_0^t \Big[f(s) + \int_0^s h(s,\sigma)d\sigma\Big]ds, \tag{2.3}$$

$$G(r) = \int_{r_0}^r \frac{ds}{g(s^{1/p})}, \quad r > 0, \quad r_0 > 0, \tag{2.4}$$

G^{-1} *is the inverse function of* G *and* $t_1 \in \mathbf{R}_+$ *is chosen so that*

$$G(c) + A(t) \in Dom(G^{-1}),$$

for all $t \in \mathbf{R}_+$ *lying in the interval* $0 \leqslant t \leqslant t_1$ *of* $\mathbf{R}_+$.

Proof. We first assume that $c > 0$ and define a function $z(t)$ by the right hand side of (2.1). Then $z(t) > 0$, $z(0) = c$, $u(t) \leqslant (z(t))^{1/p}$ and

$$\begin{aligned} z'(t) =& f(t)g(u(t)) + \int_0^t h(t,\sigma)g(u(\sigma))d\sigma \\ \leqslant& f(t)g((z(t))^{1/p}) + \int_0^t h(t,\sigma)g((z(\sigma))^{1/p})d\sigma \\ \leqslant& g((z(t))^{1/p})\Big[f(t) + \int_0^t h(t,\sigma)d\sigma\Big]. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we have

$$\begin{aligned} \frac{d}{dt}G(z(t)) =& \frac{z'(t)}{g((z(t))^{1/p})} \\ \leqslant& \Big[f(t) + \int_0^t h(t,\sigma)d\sigma\Big]. \end{aligned} \tag{2.6}$$

By setting $t = s$ in (2.6) and integrating it from 0 to t we have

$$G(z(t)) \leqslant G(c) + A(t). \tag{2.7}$$

Since G^{-1} is incresing, from (2.7) we have

$$z(t) \leqslant G^{-1}[G(c) + A(t)]. \tag{2.8}$$

Using (2.8) in $u(t) \leqslant (z(t))^{1/p}$ we have the required inequality in (2.2). If c is nonnegative, we carry out the above procedure with $c+\varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, anb by letting $\varepsilon \to 0$, we obtain (2.2). The interval $0 \leqslant t \leqslant t_1$ is obvious.

REMARK 1. We note that the definition of the function G in (2.4) is motivated from the recent work of Medved [6, p. 298].

If $\int_{r_0}^{\infty} \dfrac{ds}{g(s^{1/p})} = \infty$, then $G(\infty) = \infty$ and the inequality in (2.2) is true for $t \in \mathbf{R}_+$.

An interesting and useful special version of Theorem 1 is given in the following

COROLLARY 1. *Let* u, f, h, c, p *be as in theorem 1. If*

$$u^p(t) \leqslant c + \int_0^t \Big[f(s)u(s) + \int_0^s h(s,\sigma)u(\sigma)d\sigma \Big] ds, \tag{2.9}$$

for $t \in \mathbf{R}_+$, *then*

$$u(t) \leqslant \Big[c^{(p-1)/p} + \Big(\frac{p-1}{p}\Big) A(t) \Big]^{1/(p-1)}, \tag{2.10}$$

for $t \in \mathbf{R}_+$, *where* $A(t)$ *is defined by (2.3).*

Proof. Let $g(u) = u$ in Theorem 1. Then (2.1) reduces to (2.9) and

$$G(r) = \frac{p}{p-1}\Big[r^{(p-1)/p} - r_0^{(p-1)/p} \Big],$$

$$G^{-1}(r) = \Big[\frac{p-1}{p} r + r_0^{(p-1)/p} \Big]^{p/(p-1)}$$

and consequently the bound obtained in (2.2) reduces to the bound in (2.10).

REMARK 2. In the special case when $p = 2$, then inequality given in corollary 1 reduces to a variant of the inequality given in [7, p. 233]. For an explict bound on a defferent version of the inequality (2.9), see, also Willett and Wong [10].

An important and useful two independent variable generalization of theorem 1 is embodied in the

THEOREM 2. *Let* $u(x,y)$, $f(x,y) \in c(\mathbf{R}_+^2, \mathbf{R}_+)$, $h(x,y,s,t) \in c(\mathbf{R}_+^2 x \mathbf{R}_+^2, \mathbf{R}_+)$ *for* $0 \leqslant s \leqslant x < \infty$, $0 \leqslant t \leqslant y < \infty$. *Let* c, p, g, G, G^{-1} *be as in Theorem 1. If*

$$u^p(x,y) \leqslant c + \int_0^x \int_0^y \Big[f(s,t)g(u(s,t)) + \int_0^s \int_0^t h(s,t,\sigma,\eta)g(u(\sigma,\eta))d\eta d\sigma \Big] dsdt \tag{2.11}$$

for x, $y \in \mathbf{R}_+$, *then for* $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$, x, x_1, y, $y_1 \in \mathbf{R}_+$,

$$u(x,y) \leqslant [G^{-1}[G(c) + A(x,y)]]^{1/p}, \tag{2.12}$$

where

$$A(x,y) = \int_0^x \int_0^y \Big[f(s,t) + \int_0^s \int_0^t h(s,t,\sigma,\eta)d\eta d\sigma \Big] dtds, \tag{2.13}$$

and x_1, $y_1 \in \mathbf{R}_+$, *are chosen so that*

$$G(c) + A(x,y) \in Dom(G^{-1}),$$

for all x, y *lying in the intervals* $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$ *of* $\mathbf{R}_+$.

Proof. Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (2.11). Then $z(0,y) = z(x,0) = c$, $u(x,y) \leqslant (z(x,y))^{1/p}$ and

$$\begin{aligned} D_1 z(x,y) = & \int_0^y \Big[f(x,t)g(u(x,t)) + \int_0^x \int_0^t h(x,t,\sigma,\eta)g(u(\sigma,\eta))d\eta d\sigma\Big]dt \\ \leqslant & \int_0^y \Big[f(x,t)g((z(x,t))^{1/p}) + \int_0^x \int_0^y h(x,t,\sigma,\eta)g((z(\sigma,\eta))^{1/p})d\eta d\eta\Big]dt \\ \leqslant & g((z(x,y))^{1/p}) \int_0^y \Big[f(x,t) + \int_0^x \int_0^t h(x,t,\sigma,\eta)d\eta d\sigma\Big]dt. \end{aligned} \quad (2.14)$$

From (2.4) and (2.14) we observe that

$$\begin{aligned} D_1 G(z(x,y)) = & \frac{D_1 z(x,y)}{g((z(x,y))^{1/p})} \\ \leqslant & \int_0^y \Big[f(x,t) + \int_0^x \int_0^t h(x,t,\sigma,\eta)d\eta d\sigma\Big]dt. \end{aligned} \quad (2.15)$$

Keeping y fixed in (2.15), setting $x = s$ and integrating with respect to s from 0 to x and using the fact that $z(0,y) = c$, we have

$$G(z(x,y)) \leqslant G(c) + A(x,y). \quad (2.16)$$

Now substituting the bound on $z(x,y)$ from (2.16) in $u(x,y) \leqslant (z(x,y))^{1/p}$ we obtain the desired bound in (2.12). The proof of the case when $c \geqslant 0$ can be completed as mentioned in the proof of Theorem 1. The domain $O \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$ is obvious.

Next, we shall give the following corollary whose proof is similar to that of corollary 1.

COROLLARY 2. *Let* u, f, c, p *be as in Theorem 2. If*

$$u^p(x,y) \leqslant c + \int_0^x \int_0^y \Big[f(s,t)u(s,t) + \int_0^s \int_0^t h(s,t,\sigma,\eta)u(\sigma,\eta)d\eta d\sigma\Big]dtds, \quad (2.17)$$

for x, $y \in \mathbf{R}_+$, *then*

$$u(x,y) \leqslant \Big[c^{(p-1)/p} + \Big(\frac{p-1}{p}\Big)A(x,y)\Big]^{1/(p-1)}, \quad (2.18)$$

for x, $y \in \mathbf{R}_+$, *where* $A(x,y)$ *is defined by (2.13).*

REMARK 3. We not that, the upper bound on the inequality (2.17) when $p = 1$ and $h = 0$ is etablished by Wendroff (see [2, p. 154]). For various generalizations of Wendroff's inequality, see [1,7].

3. Discrete analogues

The discrete analogue of the inequality given in Theorem 1 is established in the following

THEOREM 3. *Let* $u(n)$, $f(n)$, $h(n,s)$, $0 \leqslant s \leqslant n < \infty$, n, $s \in \mathbf{N}_0$ *be real-valued nonnegative functions. Let* p, c, g, G, G^{-1} *be as in Theorem 1. If*

$$u^p(n) \leqslant c + \sum_{s=0}^{n-1} \Big[f(s)g(u(s)) + \sum_{\sigma=0}^{s-1} h(s,\sigma)g(u(\sigma)) \Big], \tag{3.1}$$

for $n \in \mathbf{N}_0$, *then for* $0 \leqslant n \leqslant n_1$, n, $n_1 \in \mathbf{N}_0$,

$$u(n) \leqslant [G^{-1}[G(c) + B(n)]]^{1/p}, \tag{3.2}$$

where

$$B(n) = \sum_{s=0}^{n-1} \Big[f(s) + \sum_{\sigma=0}^{s-1} h(s,\sigma) \Big], \tag{3.3}$$

and $n_1 \in \mathbf{N}_0$ *is chosen so that*

$$G(c) + B(n) \in Dom(G^{-1}),$$

for all $n \in \mathbf{N}_0$ *lying in* $0 \leqslant n \leqslant n_1$.

Proof. First we assume that $c > 0$ and define a function $z(n)$ by the right hand side of (3.1). Then $z(n) > 0$, $z(0) = c$, $u(n) \leqslant (z(n))^{1/p}$ and

$$\begin{aligned} z(n+1) - z(n) &= f(n)g(u(n)) + \sum_{\sigma=0}^{n-1} h(n,\sigma)g(u(\sigma)). \\ &\leqslant g((z(n))^{1/p}) \Big[f(n) + \sum_{\sigma=0}^{n-1} h(n,\sigma) \Big]. \end{aligned} \tag{3.4}$$

From (2.4) and (3.4) we observe that

$$\begin{aligned} G(z(n+1)) - G(z(n)) &= \int_{z(n)}^{z(n+1)} \frac{ds}{g(s^{1/p})} \\ &\leqslant \frac{z(n+1) - z(n)}{g((z(n))^{1/p})} \\ &\leqslant f(n) + \sum_{\sigma=0}^{n-1} h(n,\sigma). \end{aligned} \tag{3.5}$$

By takin $n = s$ in (3.5) and summing up over s from 0 $n-1$, it follows that

$$G(z(n)) \leqslant G(c) + B(n). \tag{3.6}$$

From (3.6) we have

$$z(n) \leqslant G^{-1}[G(c)+B(n)] \tag{3.7}$$

Using (3.7) in $u(n) \leqslant (z(n))^{1/p}$ we have the required inequality in (3.2). The proof of the case when $c \geqslant 0$ can be completed as mentioned in the proof of Theorem 1, $0 \leqslant n \leqslant n_1$ is obvious.

COROLLARY 3. *Let u, f, h, c, p be as in Theorem 3. If*

$$u^p(n) \leqslant c + \sum_{s=0}^{n-1}\Big[f(s)u(s) + \sum_{\sigma=0}^{s-1} h(s,\sigma)u(\sigma)\Big], \tag{3.8}$$

for $n \in \mathbf{N}_0$, then

$$u(n) \leqslant \Big[c^{(p-1)/p} + \Big(\frac{p-1}{p}\Big)B(n)\Big]^{1/(p-1)}, \tag{3.9}$$

for $n \in \mathbf{N}_0$, where $B(n)$ is defined by (3.3).

The proof is similar to that of mentioned in corollary 1, and we omit it here.

REMARK 4. We note that, the discrete version of Bihari's inequality is established by Hull and Luxemberg in [4]. For other useful nonlinear discrete inequalities, see [8–10].

The following result is the discrete analogue of the inequality given in Theorem 2.

THEOREM 4. *Let $u(m,n)$, $f(m,n)$, $h(m,n,s,t)$, $0 \leqslant s \leqslant m < \infty$, $0 \leqslant t \leqslant n < \infty$, m, n, s, $t \in \mathbf{N}_0$, be real-valued nonnegativ functions. Let c, p, g, G, G^{-1} be as in Theorem 1. If*

$$u^p(m,n) \leqslant c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\Big[f(s,t)g(u(s,t)) + \sum_{\sigma=0}^{s-1}\sum_{\eta=0}^{t-1} h(s,t,\sigma,\eta)g(u(\sigma,\eta))\Big], \tag{3.10}$$

for m, $n \in \mathbf{N}_0$, then for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, m, m_1, n, $n_1 \in \mathbf{N}_0$,

$$u(m,n) \leqslant [G^{-1}[G(c)+B(m,n)]]^{1/p}, \tag{3.11}$$

where

$$B(m,n) = \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\Big[f(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\eta=0}^{t-1} h(s,t,\sigma,\eta)\Big] \tag{3.12}$$

and m_1, $n_1 \in \mathbf{N}_0$ are chosen so that

$$G(c) + B(m,n) \in Dom(G^{-1})$$

for all m, n lying in $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$.

Proof. First we assume that $c > 0$ and define a function $z(m,n)$ by the right hand side fo (3.10). Then $z(0,n) = z(m,0) = c$, $u(m,n) \leqslant (z(m,n))^{1/p}$ and

$$\begin{aligned} z(m+1,n) - z(m,n) &= \sum_{t=0}^{n-1}\Big[f(m,t)g(u(m,t)) + \sum_{\sigma=0}^{m-1}\sum_{\eta=0}^{t-1} h(m,t,\sigma,\eta)g(u(\sigma,\eta))\Big] \\ &\leqslant \sum_{t=0}^{n-1}\Big[f(m,t)g((z(m,t))^{1/p}) \\ &+ \sum_{\sigma=0}^{m-1}\sum_{\eta=0}^{t-1} h(m,t,\sigma,\eta)g((z(\sigma,\eta))^{1/p})\Big] \\ &\leqslant g((z(m,n))^{1/p})\sum_{t=0}^{n-1}\Big[f(m,t)\sum_{\sigma=0}^{m-1}\sum_{n=0}^{t-1} h(m,t,\sigma,\eta)\Big]. \end{aligned} \tag{3.13}$$

From (2.4), (3.13) we observe that

$$\begin{aligned} G(z(m+1,n)) - G(z(m,n)) &= \int_{z(m,n)}^{z(m+1,n)} \frac{ds}{g(s^{1/p})} \leqslant \frac{z(m+1,n)-z(m,n)}{g((z(m,n))^{1/p})} \\ &\leqslant \sum_{t=0}^{n-1}\Big[f(m,t) + \sum_{\sigma=0}^{m-1}\sum_{\eta=0}^{t-1} h(m,t,\sigma,\eta)\Big]. \end{aligned} \tag{3.14}$$

Keeping n fixed in (3.14), setting $m = s$ and summing over s from 0 to $m-1$ we obtain

$$G(z(m,n)) \leqslant G(c) + B(m,n). \tag{3.15}$$

Now substituting the bound on $z(m,n)$ from (3.15) in $u(m,n) \leqslant (z(m,n))^{1/p}$, we obtained required inequality in (3.11). The proof of the case when $c \geqslant 0$ can be completed as mentioned in the proof of Theorem 1. The domain $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$ is obvious.

As an immediate consequence of Theorem 4 is given in the following

COROLLARY 4. *Let u, f, h, c, p be as in Theorem 4. If*

$$u^p(m,n) \leqslant c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\Big[f(s,t)u(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\eta=0}^{t-1} h(s,t,\sigma)u(\sigma,\eta)\Big], \tag{3.16}$$

for m, $n \in \mathbf{N}_0$, then

$$u(m,n) \leqslant \Big[c^{(p-1)/p} + \Big(\frac{p-1}{p}\Big)B(m,n)\Big]^{1/(p-1)}, \tag{3.17}$$

for m, $n \in \mathbf{N}_0$, where $B(m,n)$ is defined by (3.12).

The proof is similar to that of corollary 1. We omit the details.

REMARK 5. We note that the inequalities established in Theorems 2 and 4 can be extended very easily to functions of several independent variables. The precise formulations of these results are very close to that of given above and closely looking at the results given in [7, pp. 396–409] and [9].

4. Applications

In this section, we present some direct applications of Theorems 1 and 2 to obtain the explicit bounds on the solutions of certain differential equations.

EXAMPLE 1. As a first application, we obtain a bound on the solution of the differential equation

$$x^{p-1}(t)x'(t) = h(t, x(t)), \quad x(0) = x_0, \tag{4.1}$$

where x_0, $p > 1$ are constants, $x : \mathbf{R}_+ \to \mathbf{R}$, $h : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ are continuous functions. Let $x(t)$ be a solution of (4.1), then x(t) satisfies the integral equation

$$\frac{x^p(t)}{p} - \frac{x_0^p}{p} = \int_0^t h(s, x(s,))ds. \tag{4.2}$$

We assume that the function h satisfies the condition

$$|h(t, x(t))| \leqslant f(t)g(|x(t)|), \tag{4.3}$$

where f and g are as defined in Theorem 1. From (4.2) and (4.3) we observe that

$$|x(t)|^p \leqslant |x_0|^p + \int_0^t pf(s)g(|x(s)|)ds. \tag{4.4}$$

Now a suitable application of Theorem 1 yields

$$|x(t)| \leqslant \left[G^{-1}\left[G\left(|x_0|^p\right) + \int_0^t pf(s)ds\right]\right]^{1/p}, \tag{4.5}$$

for $0 \leqslant t \leqslant t_1$, t, $t_1 \in \mathbf{R}_+$, where G, G^{-1} are defined in Theorem 1. Then right hand side of (4.5) gives the bound on the solution of (4.1) in terms of the known functions.

EXAMPLE 2. As a second application we establish the bound on the solution of the following partial differential equation

$$\frac{\partial}{\partial y}\left(u^{p-1}(x, y)\frac{\partial}{\partial x}u(x, y)\right) = F(x, y, u(x, y)), \tag{4.6}$$

with the given intial boundary conditions

$$u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y), \quad \sigma(0) = \tau(0) = 0, \tag{4.7}$$

where $p > 1$ is a real constant, $u : \mathbf{R}_+^2 \to \mathbf{R}$, $F : \mathbf{R}_+^2 \times \mathbf{R} \to \mathbf{R}$, σ, $\tau : \mathbf{R}_+ \to \mathbf{R}$ are continuous functions. Let $u(x, y) : \mathbf{R}_+^2 \to \mathbf{R}$ is a solution of (4.6)–(4.7). It is easy to observe that the problem (4.6)–(4.7) is equivalent to the integral equation

$$u^p(x, y) = \sigma^p(x) + \tau^p(y) + p\int_0^x \int_0^y F(s, t, u(s, t))dtds. \tag{4.8}$$

We assume that

$$|F(x,y,u)| \leqslant h(x,y)|u|, \tag{4.9}$$

$$|\sigma^p(x) + \tau^p(y)| \leqslant c, \tag{4.10}$$

where $h : \mathbf{R}_+^2 \to \mathbf{R}_+$ is a continuous function and $c \geqslant 0$ is a real constan. From (4.8)–(4.10) we observe that

$$|u(x,y)|^p \leqslant c + \int_0^x \int_0^y ph(s,t)|u(s,t)|dtds. \tag{4.11}$$

Now a suitable application of Theorem 2 yields

$$|u(x,y)| \leqslant \left[G^{-1}\left[G(c) + \int_0^x \int_0^y ph(s,t)dtds\right]\right]^{1/p}, \tag{4.12}$$

for $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$, x, y, x_1, $y_1 \in \mathbf{R}_+$ where G, G^{-1} are as in Theorem 2. The right hand side of (4.12) gives the bound on the solution of (4.6)–(4.7) in terms of the known functions.

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