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ON SHARPNESS OF SOME INTEGRAL INEQUALITIES AND AN INTEGRAL EQUATION OF VOLTERRA TYPE

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Abstract. The sharpness of some recent integral inequalities is discussed and the corresponding extremal functions are pointed out. It is also proved that the cases of equality can equivalently be obtained by solving an integral equation of Volterra type with discontinous kernel $\chi_A(t)$. This integral equation of independent interest is solved for every measurable set A on (a,b), $-\infty < a < b < \infty$.

1. Introduction

In this paper we consider nonnegative real valued functions f, g defined on an interval (a,b), $-\infty < a < b \leqslant \infty$. First we recall the inequality

$$\left(\int_a^b f^q(x)(x-a)^{q-1}\,dx\right)^{\frac{p}{q}} \leqslant pq^{-\frac{p}{q}} \int_a^b f^p(x)(x-a)^{p-1}\,dx, 0 < p \leqslant q < \infty \quad (1)$$

which holds for every decreasing function f. Here and in the sequel decreasing means non-increasing and increasing means non-decreasing.

The inequality (1) is sharp and equality occurs for every function of the type $f(x) = A\chi_{(a,t)}(x)$, $a \leqslant t \leqslant b$ (χ denotes the characteristic function and A any positive constant). Moreover, (1) holds in the reversed direction if f is increasing. For the case $q = 1$ the inequality was probably first discovered by Lorentz [9, p.39], c.f. also [7, p.100]. Moreover, various proofs and extensions can be found in [2], [3], [4], [8], [10], [11], [12], [13] and [14]. Moreover, also the analogous inequality

$$\left(\int_a^b f^q(x)(b-x)^{q-1}\,dx\right)^{\frac{p}{q}} \leqslant pq^{-\frac{p}{q}} \int_a^b f^p(x)(b-x)^{p-1}\,dx, 0 < p \leqslant q < \infty \quad (2)$$

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holds for every increasing function f. Also (2) is sharp and equality occurs for $f(x) = A\chi_{(t,b)}(x), a \leqslant t \leqslant b$, and (2) holds in the reversed direction if f is decreasing. See e.g. [5], [6], [8], [11], [12], [13] and [15] for some different proofs and extensions.

In this paper we first consider the generalizations of (1) and (2) recently proved in [13] (see Theorem 1 and Remark 1 below). We say that f is C-decreasing [C-increasing], $C > 0$, if $f(t) \leqslant Cf(s)$ [$f(s) \leqslant Cf(t)$] whenever $s \leqslant t, s,t \in (a,b)$. If g is absolutely continuous, increasing and $g(a+0) = 0$, then we say that f is C-decreasing in mean relatively to g if, for all $x \in (a,b)$,

$$f(x)g(x) \leqslant C\int_a^x f(t)dg(t),$$

and if g is absolutely continuous, decreasing and $g(b-0) = 0$, we say that f is C-increasing relatively to g if, for all $x \in (a,b)$,

$$f(x)g(x) \leqslant C\int_x^b f(t)d\left[-g(t)\right].$$

Some concrete illustrations of these concepts can be found in our Example 1.

In the sequel we assume that $-\infty < a < b < \infty$ and g is absolutely continuous on $[a,b]$. Also, if ψ is concave (convex) and $x \in \mathbf{R}$ is a point where ψ is nondifferentiable then we assume that $\psi'(x) \in [\psi'(x+0), \psi'(x-0)]$ ($\psi'(x) \in [\psi'(x-0), \psi'(x+0)]$). Inequalities (such as in Theorem 1) are interpreted to mean that if the right hand side is finite so is the left hand side. The Lebesgue-Stieltjes integral is assumed.

THEOREM 1 [13]. *Let* $\psi : [0,\infty) \to \mathbf{R}$ *be a continuous, concave increasing function such that* $\psi(0) = 0$.

1. *If f is C-decreasing in mean relatively to g, where g is increasing and $g(a) = 0$, then*
$$\psi\left(C\int_a^b f(t)dg(t)\right) \leqslant C\int_a^b \psi'\left(f(t)g(t)\right)f(t)dg(t). \tag{3}$$

2. *If f is C-increasing in mean relatively to g, where g is decreasing and $g(b) = 0$, then*
$$\psi\left(C\int_a^b f(t)d\left[-g(t)\right]\right) \leqslant C\int_a^b \psi'\left(f(t)g(t)\right)f(t)d\left[-g(t)\right]. \tag{4}$$

If the condition "ψ is concave" is replaced by "ψ is convex", then the inequalities (3) and (4) hold in the reversed direction.

REMARK 1. We note that if f is C-increasing [C-decreasing], then f is C-increasing [C-decreasing] in mean with respect to any g of the considered type. The reversed implication is of course not true. In particular this means that Theorem 1. holds if we replace "C-decreasing [C-increasing] in mean with respect to g" by only "C-decreasing [C-increasing]".

In the first part of this paper we study the sharpness of (3)-(4). In Section 2 we make a reformulation of the cases of equality in (3)-(4) as some *integral equations* (see Proposition 1), study the special case when f is just C-decreasing [C-increasing] (c.f. Remark 1) and find, in particular, that in this case the inequalities (3)-(4) are sharp only when $C = 1$ (see Theorem 2). For the general case studied in Theorem 1 we obtain a much more interesting theory, which, in particular, shows that now each of the inequalities (3) and (4) are sharp for *every* $C > 0$ (see Theorem 3). In Section 3 we study the fact that the integral equations which characterize the cases of equality in (3) and (4) are of independent interest and we give necessary and sufficient conditions for the existence of solutions (see Theorem 4). Finally, Section 4 is reserved for some concluding remarks and examples.

2. Sharpness of Theorem 1

In this section we derive a reformulation of the cases of equality in (3) and (4) in terms of some integral equations. This result is crucial for our further investigations. In the sequel we assume that $g' \neq 0$ on (a,b) and that all equalities are assumed to hold almost everywhere.

PROPOSITION 1. *Let $\psi : [0,\infty) \to \mathbf{R}$ be a concave [convex] continuous function, such that $\psi(0) = 0$ and ψ' is strictly decreasing [increasing] on $(0,\infty)$.*

1. *Suppose that f is C-decreasing in mean relatively to g, where $g(a) = 0$ and g increasing. Then we have equality in (3) if and only if for $x \in (a,b)$*

$$C\int_a^x f(t)dg(t) = f(x)g(x) \quad or \quad f(x) = 0. \tag{5}$$

2. *Suppose that f is C-increasing in mean relatively to g, where $g(b) = 0$ and g decreasing. Then we have equality in (4) if and only if for $x \in (a,b)$*

$$C\int_x^b f(t)d\left[-g(t)\right] = f(x)g(x) \quad or \quad f(x) = 0. \tag{6}$$

Proof. Assume that we have equality in (3). Consider the function

$$F(x) = \psi\left(C\int_a^x f(t)dg(t)\right) - C\int_a^x \psi'\left(f(t)g(t)\right)f(t)dg(t).$$

We see that $F(a) = 0$ and from derivative formula for $F(x)$ we see that $F'(x) \leqslant 0$ for $x \in (a,b)$. Moreover, equality in (3) means that $F(b) = 0$ and it follows that, for $x \in [a,b]$, $F(x) = 0$, i.e.,

$$\psi\left(C\int_a^x f(t)dg(t)\right) = C\int_a^x \psi'(f(t)g(t))f(t)dg(t). \tag{7}$$

By differentiating (7) we find that

$$f(x)g'(x)\left[\psi'\left(C\int_a^x f(t)dg(t)\right) - \psi'\left(f(x)g(x)\right)\right] = 0 \tag{8}$$

and we conclude that (5) holds. On the other hand, assume that (5) holds. Then, according to the assumptions, we find that (8) yields. Therefore, by integrating (8), we see that (7) is satisfied, which, in particular, means that we have equality in (3).

The proof of the second case is similar. □

REMARK 2. We note that for every measurable function f the set $f^{-1}(0,\infty)$ is also measurable. Therefore, according to Proposition 1, we find that the problem to study equality in inequality (3) is equivalent to the problem of constructing a function f, such that, for a given measurable subset A of $[a,b]$, it yields that

$$(P) \qquad C\int_a^x f(t)dg(t) = f(x)g(x) \quad \text{for} \quad x \in A \quad \text{and} \quad f(x) = 0 \quad \text{on} \quad A^c.$$

Similarly, the problem to study equality in the inequality (4) can be described in the obvious analogous way.

Our main result for the C-monotone case reads:

THEOREM 2. *Let $\psi : [0,\infty) \to \mathbf{R}$ be a concave [convex] continuous function, such that $\psi(0) = 0$ and ψ' is strictly decreasing [increasing].*

1. *We have equality in (3) for a non-trivial C-decreasing function f if and only if $C = 1$ and $f = A\chi_{[a,c]}$, where $A > 0$ and $c \in (a,b)$.*
2. *We have equality in (4) for a non-trivial C-increasing function f if and only if $C = 1$ and $f = A\chi_{[c,b]}$, where $A > 0$ and $c \in (a,b)$.*

Proof. We prove the first case. It is easy to see that we have equality in (3) for the functions of the form $f = A\chi_{(a,c)}$. On the other hand, assume that we have equality in (3) for the C-decreasing function f. We define $c = \operatorname{essinf}\{x \in [a,b] : f(x) = 0\}$ and note that since $f(x) \leqslant Cf(t)$ for $t \leqslant x$ we have that if $f(x_1) = 0$, then $f(x) = 0$ for all $x \geqslant x_1$. Moreover, according to Proposition 1, it yields that, for every $x \in [a,c]$,

$$0 = C\int_a^x f(t)dg(t) - f(x)g(x) = \int_a^x (Cf(t) - f(x))\, dg(t),$$

and since the integrand in the last integral is nonnegative we conclude that $Cf(t) = f(x)$ for $t \leqslant x$. Now, if $a < x_1 < x_2 < x_3 < c$, then $Cf(x_1) = f(x_2), Cf(x_2) = f(x_3)$ and $Cf(x_1) = f(x_3)$ and it follows that $Cf(x_1) = C^2 f(x_1)$ so we must have $C = 1$ and $f(x_1) = f(x_2) = f(x_3)$. Therefore f is of the form $f = A\chi_{(a,c)}$.

The second case can be proved in analogous way. □

Next we will prove the interesting fact that in the general (monotone in mean) case it is possible to find non-trivial functions such that we have equality in (3) and (4) not only for $C = 1$, as in the previous theorem, but also for *every* $C > 0$. We will mainly treat the case (1) considered in Theorem 1 but this is no essential restriction because the other cases can be treated in a similar way or even be derived from this case. Also, we consider now the case of simple measurable sets (for general case see Section 3) i.e. open sets which are union of the open intervals such that the family of intervals is without accumulation points. (An accumulation point of a family of sets is a point such that every neighbrohood of that point intersects infinitely many sets in this family.)

THEOREM 3. *Theorem 1 is sharp, i.e., there exist functions f such that we can have equality in each of the inequalities (3) and (4). More generally the following yields: Let $K > 0$ and let g and ψ be as in Theorem 1.*

1. *Let $a = a_1 < b_1 \leqslant a_2 \leqslant b_2 \leqslant \ldots \leqslant a_n \leqslant b_n \leqslant \ldots \leqslant b$ and let f be defined by*

$$f(x) = \begin{cases} Kg^{C-1}(x), & a < x < b_1 \\ K\prod_{i=1}^{n-1} \frac{g(b_i)}{g(a_{i+1})} g^{C-1}(x), & a_n < x < b_n \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Then f is C-decreasing in mean with respect to g and we have equality in (3). On the other hand, if ψ' is strictly decreasing, f is C-decreasing in mean with respect to g, $f^{-1}(0,\infty)$ is a union of disjoint open intervals on (a,b) such that the family of these intervals has no accumulation points in $[a,b)$ and we have equality in (3), then f has the form (9).

2. *Let $b = b_1 > a_1 \geqslant b_2 \geqslant a_2 \geqslant \ldots \geqslant b_n \geqslant a_n \geqslant \ldots \geqslant a$ and let f be defined by*

$$f(x) = \begin{cases} Kg^{C-1}(x), & a_1 < x < b_1 \\ K\prod_{i=1}^{n-1} \frac{g(a_i)}{g(b_{i+1})} g^{C-1}(x), & a_n < x < b_n \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Then f is C-increasing with respect to g and we have equality in inequality (4). On the other hand, if ψ' is strictly decreasing, f is C-increasing in mean with respect to g, $f^{-1}(0,\infty)$ is a union of disjoint open intervals on (a,b) such that the family of these intervals has no accumulation points on $(a,b]$ and we have equality in (4), then f has the form (10).

For the proof of the necessity part of Theorem 3 we need to prove a lemma which was partly guided by the following simple observation: If there is an interval $(c,d) \subset [a,b]$, such that $C\int_a^x f(t)dg(t) = f(x)g(x), x \in (c,d)$, then f is differentiable on (c,d) and of the form $f(x) = Ag^{C-1}(x), A \geqslant 0$ on (c,d). In particular this means that if $f(x) = 0$ for $a < x < c$ and $f(x) > 0$ for $c < x < d$, then $C\int_c^x f(t)dg(t) = f(x)g(x), x \in (c,d)$, and this is in contradiction to the obtained local form of the function f so we must avoid this situation. A stronger statement will be given in our next lemma. In the sequel we let $|A|$ denote the Lebesgue-Stieltjes measure generated by absolutely continuous function g of the set A.

LEMMA 1. *Let f be a nonnegative, nontrivial and integrable function on (a,b) and $C > 0$. If $C\int_a^x f(t)dg(t) = f(x)g(x)$ for $x \in A$ and $f(x) = 0$ for $x \in A^c$, then $|A \cap (a, a+\varepsilon)| > 0$ for every $\varepsilon > 0$.*

Proof. Let $x_0 = \operatorname{esssup}\left\{x \in (a,b) : \int_a^x f(t)dg(t) = 0\right\}$. It is sufficient to prove that $x_0 = a$. On the contrary, we assume that $x_0 > a$. It is obvious that $\lim_{x\to x_0} f(x) = 0$ because in a neighborhood of x_0 we have either $f(x) = 0$ or $C\int_{x_0}^x f(t)dg(t) = f(x)g(x)$. Thus, there exists $x_1 > x_0$ such that $f(x) < g^{C-1}(x)$ for every $x \in (x_0,x_1)$ so that for $x \in (x_0,x_1) \cap A$ we have

$$f(x)g(x) = C\int_{x_0}^x f(t)dg(t) < g^C(x) - g^C(x_0).$$

Thus, $f(x) < g^{C-1}(x) - g^C(x_0)/g(x), x \in (x_0, x_1)$. Using this inequality we find that

$$f(x)g(x) = C\int_{x_0}^{x} f(t)dg(t) < g^C(x) - g^C(x_0) - Cg^C(x_0)\ln\left(g(x)/g(x_0)\right), x \in (x_0, x_1)$$

and inductively

$$f(x)g(x) < g^C(x) - g^C(x_0)\left[\sum_{k=0}^{n} \frac{C^k}{k!}\ln^k\left(\frac{g(x)}{g(x_0)}\right)\right], x \in (x_0, x_1)$$

which, by letting $n \to \infty$, gives $f(x)g(x) \leqslant 0, x \in (x_0, x_1)$, and this is an obvious contradiction. The proof is complete. □

Proof of Theorem 3. We prove the first case. It is easy to see that the function f is C-decreasing in mean with respect to g and that it is a solution of the problem (P) for the set $A = \cup_{n=1}^{\infty}(a_n, b_n)$. The reversed implication follows from Lemma 1 and a simple induction procedure which is possible because of the simple structure of the set $f^{-1}(0,\infty)$ (compare also with the construction in our later Lemma 3).

The proof of the second case is analogous. □

3. Further investigations of the integral equation (P)

In the previous section we proved in particular that the problem of the integral equation (P) or equivalently

$$(P) \qquad C\int_a^x \chi_A(t)f(t)dg(t) = f(x)g(x),\ A \quad \text{measurable}, \quad C > 0$$

can be solved for each simple open set A (and this, in its turn, implied that Theorem 1 is sharp). In this section we shall prove some corresponding results for the case with a general measurable set A. According to Lemma 1 we see that (P) can be satisfied only when f is built up on a set A with some restriction around $x = a$. We will prove a result (see Theorem 4) which, in particular, shows that the positiveness of the one-sided density of A at $x = a$ i.e.

$$\lim_{\varepsilon \to 0} \frac{|A \cap (a, a+\varepsilon)|}{g(\varepsilon)} > 0$$

is sufficient to ensure that (P) holds for some nontrivial function f.

THEOREM 4. *Let $C > 0$, let A be a measurable subset of $[a,b]$ and let g be a strictly increasing function satisfying $g(a) = 0$. Then there exists a nonnegative nontrivial function f such that (P) holds if and only if*

$$\mu(A) = \int_a^b \chi_A(x)\frac{dg(x)}{g(x)} = \infty. \tag{11}$$

For proving the Theorem we need the following technical lemma:

LEMMA 2. *Let* $U = \cup_{i=1}^{\infty}(a_i, b_i)$*, where* (a_i, b_i) *is a sequence of disjoint open intervals in* (a, b)*. If* g *is a strictly increasing function such that* $g(a) = 0$ *and* $\prod_{i=1}^{\infty}\frac{g(a_i)}{g(b_i)} = 0$*, then* $U \cap (a, a+\varepsilon) \neq \emptyset$ *for every* $\varepsilon > 0$.

Proof. Let $a^* = \inf\{a_i\}$. It is sufficient to prove that $a^* = a$. Assume that $a^* > a$. Make an rearrangement of $I_n = (1, 2, ..., n)$ so that $a_{i_1} < b_{i_1} < a_{i_2} < b_{i_2} < ... < a_{i_n} < b_{i_n}$. Then

$$\prod_{i=1}^{n}\frac{g(a_i)}{g(b_i)} = \prod_{j=1}^{n}\frac{g(a_{i_j})}{g(b_{i_j})} = g(a_{i_1})\prod_{j=1}^{n-1}\frac{g(a_{i_{j+1}})}{g(b_{i_j})}\frac{1}{g(b_{i_n})} \geqslant \frac{g(a_{i_1})}{g(b_{i_n})} \geqslant \frac{g(a^*)}{g(b)}(>0)$$

and this is in contradiction with assumption that $\prod_{i=1}^{\infty}\frac{g(a_i)}{g(b_i)} = 0$. □

In our next lemma we present our key construction for optimal functions in the inequality (3).

LEMMA 3. *Let* $C > 0$ *and let* $U = \cup_{i=1}^{\infty}(a_i, b_i)$*, where* (a_i, b_i) *is a sequence of disjoint open intervals in* (a, b)*. If* g *is a strictly increasing function such that* $g(a) = 0$ *and* $\prod_{i=1}^{\infty}\frac{g(a_i)}{g(b_i)} = 0$*, then there exists a nonnegative function* f *such that* $C\int_a^x f(t)dg(t) = f(x)g(x)$ *for* $x \in U$ *and* $f(x) = 0$ *for* $x \in U^c$.

Proof. Let $U_n = \cup_{i=1}^{n}(a_i, b_i)$, $L_n^k = \{i \in \{1, 2, ..., n\} : (a_i, b_i) \subset (b_k, a_1)\}$, $R_n^k = \{i \in \{1, 2, ..., n\} : (a_i, b_i) \subset (b_1, a_k)\}$ for $n, k \geqslant 2$. For $K > 0$ define the sequence (f_n) of functions in the following way:

$$f_1(x) = \begin{cases} Kg^{C-1}(x), & x \in (a_1, b_1) \\ 0, & \text{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} Kg^{C-1}(x), & x \in (a_1, b_1) \\ K\frac{g^C(a_1)}{g^C(b_k)}\prod_{L_n^k}\frac{g^C(a_i)}{g^C(b_i)}g^{C-1}(x), & x \in (a_k, b_k), b_k < a_1 \\ K\frac{g^C(b_1)}{g^C(a_k)}\prod_{R_n^k}\frac{g^C(b_i)}{g^C(a_i)}g^{C-1}(x), & x \in (a_k, b_k), b_1 < a_k \\ 0 & \text{, otherwise.} \end{cases}$$

It is easy to see that the sequence (f_n) has the following properties:

1. If $2 \leqslant k \leqslant n$, then $Kg^{C-1}(x) \leqslant f_n(x) \leqslant K\left(g^C(a_1)/g^C(b_k)\right)g^{C-1}(x)$ for $a_k < x < b_k < a_1$ and $Kg^{C-1}(x) \geqslant f_n(x) \geqslant K\left(g^C(b_1)/g^C(a_k)\right)g^{C-1}(x)$ for $b_1 < a_k < x < b_k$.
2. If $n \geqslant m, x \in U_n$, then for $x < a_1$ we have $f_n(x) \geqslant f_m(x)$ and for $x > b_1$ we have $f_n(x) \leqslant f_m(x)$.
3. If $x, y \in U_n, x < y$, then $C\int_x^y f_n(t)dg(t) = f_n(y)g(y) - f_n(x)g(x)$.

We note that, according to the properties (1) and (2), the function $f(x) := \lim_{n\to\infty} f_n(x)$ is well defined on (a, b). Let $a_n^* = \min\{a_i : i = 1, 2, ..., n\}$. We use the Fatou lemma and find that, for $x \in U$,

$$\begin{aligned} C\int_a^x f(t)dg(t) &\leqslant C\lim_{n\to\infty}\int_a^x f_n(t)dg(t) = \lim_{n\to\infty}\left(f_n(x)g(x) - f_n(a_n^*+0)\,g(a_n^*)\right) \\ &= f(x)g(x) - \lim_{n\to\infty} f_n(a_n^*+0)g(a_n^*). \end{aligned}$$

Moreover, according to the definition of the sequence (f_n), the assumptions we made and Lemma 2, we have

$$\lim_{n\to\infty} f_n(a_n^*+0)g(a_n^*) = Kg^C(a_1)\prod_L \frac{g^C(a_i)}{g^C(b_i)} = 0,$$

where $L=\{i\in N:(a_i,b_i)\subset(a,a_1)\}$, and we conclude that $C\int_a^x f(t)dg(t)\leqslant f(x)g(x)$. On the other hand, choose $\varepsilon>0$ such that $a+\varepsilon\in U$. We note that the function f is integrable and, by property 1 above, the sequence (f_n) is dominated on $(a+\varepsilon,b)$ by the integrable function $K\left(g^C(a_1)/g^C(a+\varepsilon)\right)g^{C-1}(x)$. Therefore, by using the dominated convergence theorem, we find that

$$C\int_{a+\varepsilon}^x f(t)dg(t) = C\lim_{n\to\infty}\int_{a+\varepsilon}^x f_n(t)dg(t) = f(x)g(x)-f(a+\varepsilon)g(a+\varepsilon), x\in U.$$

Moreover, in view of the assumptions, we have that $\lim_{\varepsilon\to 0} f(a+\varepsilon)g(a+\varepsilon)=0$ so that for $x\in U$ we have $C\int_a^x f(t)dg(t)=f(x)g(x)$. The proof is complete. □

The following lemma gives us the exact relation between the previous lemma and Theorem 4.

LEMMA 4. *Let* $U=\cup_{i=1}^{\infty}(a_i,b_i)$, *where* (a_i,b_i) *is a sequence of disjoint open intervals on* (a,b), *and let* g *be a strictly increasing function on* (a,b) *such that* $g(a)=0$. *Then* $\prod_{i=1}^{\infty}\frac{g(a_i)}{g(b_i)}=0$ *if and only if* $\int_a^b \chi_U(x)\frac{dg(x)}{g(x)}=\infty$.

Proof. First we note that if $\liminf_i\{g(a_i)/g(b_i)\}<1$, then both claims are obvious, so we can assume that $\liminf_i\{g(a_i)/g(b_i)\}=1$. In this case $\prod_{i=1}^{\infty}\frac{g(a_i)}{g(b_i)}=0$ is equivalent to condition $\sum_{i=1}^{\infty}\ln(g(a_i)/g(b_i))=-\infty$, which, in its turn, is obviously equivalent to the second claim. □

Proof of Theorem 4. Assume that (11) holds. Let $A=(\cap_{n=1}^{\infty}U_n)\setminus Z$, where (U_n) is a decreasing sequence of open sets in $[a,b]$ and Z is a set of measure 0. Let $x_0\in A$ be fixed and $U_n=\cup_{i=1}^{\infty}(a_{n,i},b_{n,i})$, where $((a_{n,i},b_{n,i}))$ is a sequence of disjoint open intervals and where $x_0\in(a_{n,1},b_{n,1})$ for every $n\in\mathbf{N}$. Consider now a sequence (f_n) of functions defined as in the proof of Lemma 3 for the open sets U_n, where $f_n(x)=Kg^{C-1}(x)$ for $x\in(a_{n,1},b_{n,1})$ and every $n\in\mathbf{N}$. We note that, according to Lemma 3, Lemma 4 and the fact that $\mu(U_n)\geqslant\mu(A)=\infty$ the sequence exists. Notice that in contrast with the construction in Lemma 3 the sequence $(f_n(x))$ is increasing for $x<x_0, x\in A$, and decreasing for $x>x_0$ since we delete sets as n increases (both statements follow by considering the definition of $f_n(x)$ and using the trivial fact that if $(x_1,x_2)\subset(x_3,x_4)$ then $x_1/x_2\geqslant x_3/x_4$). Moreover, it follows by using property 1 in the proof of Lemma 3 that the sequence $(f_n(x))$ is upper bounded on the interval $(a+\varepsilon,x_0)$ for every $\varepsilon>0$ and lower bounded by a strictly positive constant on (x_0,b). We conclude that the function $f(x)=\lim_{n\to\infty}f_n(x)$ is well defined on $[a,b]\setminus Z$. We complete the construction by defining $f(x)=0$ for $x\in Z$.

Now we use the Fatou lemma and the optimality of the functions $f_n(x)$ to obtain that

$$C\int_a^x f(t)dg(t) \leqslant C\lim_{n\to\infty}\int_a^x f_n(t)dg(t) = \lim_{n\to\infty} f_n(x)g(x) = f(x)g(x),$$

for every $x \in A, x \leqslant x_0$. Choose $\varepsilon > 0$ so that $a+\varepsilon \in A$. We note that $\lim_{n\to\infty}|U_n \setminus A| = 0$. Moreover, for arbitrary $\varepsilon_1 > 0$, there exists $n_0 \in \mathbf{N}$ such that, for every $n \geqslant n_0$,

$$\begin{aligned}\int_a^x f(t)dg(t) &\geqslant \int_{a+\varepsilon}^x f(t)dg(t) \geqslant \int_{(a+\varepsilon,x)\cap A} f_n(t)dg(t)\\ &= \int_{(a+\varepsilon,x)\cap U_n} f_n(t)dg(t) - \int_{[(a+\varepsilon,x)\cap U_n]\setminus A} f_n(t)dg(t)\\ &\geqslant \int_{(a+\varepsilon,x)\cap U_n} f_n(t)dg(t) - \varepsilon_1\\ &= \frac{1}{C}\left(f_n(x)g(x) - f_n(a+\varepsilon)g(a+\varepsilon)\right) - \varepsilon_1.\end{aligned}$$

We conclude that for arbitrary $\varepsilon_1 > 0$ and all sufficiently large $n \in \mathbf{N}$, and arbitrary $a+\varepsilon \in A, x \leqslant x_0, x \in A$, it yields that

$$f_n(x)g(x) - f_n(a+\varepsilon)g(a+\varepsilon) - C\varepsilon_1 \leqslant C\int_a^x f(t)dg(t) \leqslant f(x)g(x),$$

which, in its turn, implies that

$$f(x)g(x) - f_n(a+\varepsilon)g(a+\varepsilon) - C\varepsilon_1 \leqslant C\int_a^x f(t)dg(t) \leqslant f(x)g(x).$$

Moreover, as in the proof of Lemma 3, we find that $\lim_{\varepsilon\to 0} f_n(a+\varepsilon)g(a+\varepsilon) = 0$, which finally gives that

$$C\int_a^x f(t)dg(t) = f(x)g(x), \text{ for x} \in \text{A, x} \leqslant \text{x}_0.$$

For $x \in A, x \geqslant x_0$ we use the fact that the sequence (f_n) is bounded on $[x_0, b]$ and the dominated convergence theorem to see that

$$\begin{aligned}C\int_{x_0}^x f(t)dg(t) &= \lim_{n\to\infty}\int_{x_0}^x f_n(t)dg(t)\\ &= \lim_{n\to\infty}\left[f_n(x)g(x) - f_n(x_0)g(x_0)\right] = f(x)g(x) - f(x_0)g(x_0)\end{aligned}$$

which, in combination with previously proven equality (for $x \leqslant x_0$) shows that f is a solution also for $x \geqslant x_0$.

We shall now prove the reverse implication. We note that it is obvious that $\mu(A \cap (a,x)) = \int_a^x \chi_A(t)\frac{dg(t)}{g(t)}$ is either finite or infinite for every $x \in (a,b)$. Suppose that $F(x) := \mu(A \cap (a,x))$ is finite. Then F is absolutely continuous, increasing and $\lim_{x\to a} F(x) =$

0. This is of course also true for the function $g(x)f(x)$ (since f is a solution of problem (P)). Using integration by parts we have

$$\begin{aligned} g(x)f(x) &= \int_a^x \chi_A(t)f(t)dg(t) = \int_a^x g(t)f(t)\chi_A(t)\frac{dg(t)}{g(t)} \\ &= g(x)f(x)F(x) - \int_a^x F(t)d\left[g(t)f(t)\right] \end{aligned}$$

so

$$\int_a^x F(t)d\left[g(t)f(t)\right] = g(x)f(x)\left(F(x)-1\right)$$

which is impossible since the left hand side is positive for all $x > a$ but the right hand side is negative for x close enough to a. This contradiction completes the proof. □

4. Concluding examples and remarks

First we illustrate the concepts of C-decreasing [C-increasing] and C- decreasing [C-increasing] in mean functions.

EXAMPLE 1. Note first that, for every function f, if $0 < m = \inf f, \infty > M = \sup f$, then the function f is $M/m-$ decreasing and $M/m-$ increasing. The increasing function $f_1(x) = x^\alpha, \alpha > 0,$ is also $(1-\alpha/\beta)-$increasing in mean on $(0,\infty)$ with respect to $g_1(x) = x^{-\beta}$ for $\alpha < \beta$, which improves the inequality (4) for power function ψ. It is also $(\alpha+1)-$decreasing in mean on $(0,\infty)$ with respect to $g_2(x) = x$. The similar can be said about the function $f_2(x) = x^{-\gamma}, 0 < \gamma < 1$. The nonmonotonuous function $f_3(x)$ which is equal to $x^{-1/2}, x \in (0,1)$ and $x^{-1/2}+1, x \in [1,\infty)$ is $2-$decreasing and $1-$decreasing in mean with respect to g_2.

By using our results in special cases we can obtain various generalizations of our introductory inequalities (1) and (2). Here we only give the following simple but illustrative example of such a generalization, which is still sharp:

EXAMPLE 2. Let f be a nonnegative function on (a,b) satisfying

$$f(x) \leqslant \frac{1}{x-a}\int_a^x f(t)dt, a < x < b.$$

Then, according to Theorem 1 (1) applied with $\psi(t) = t^p, 0 < p < 1$ and $g(x) = x - a$ we have

$$\left(\int_a^b f(x)dx\right)^p \leqslant p\int_a^b f^p(x)(x-a)^{p-1}dx. \tag{12}$$

The inequality (12) is sharp and equality can occur not only for the functions of the type $f(x) = K\chi_{(a,c)}(x)$ but also for the functions of the following type (see Theorem 3): Let $a = a_1 < b_1 \leqslant a_2 \leqslant b_2 \leqslant \ldots \leqslant a_n \leqslant b_n \leqslant \ldots \leqslant b$ and define $f(x) = 1$ on (a_1,b_1), $f(x) = \prod_{i=1}^{n-1}(b_i - a_1)/(a_{i+1} - a_1), x \in (a_n,b_n)$ and $f(x) = 0$ elsewhere.

In our next examples we will illustrate the discussion in the beginning of Section 4, Lemmas 3 and 4 for the case $g(x) = x, C = 1$ and $(a,b) = (0,1)$.

EXAMPLE 3. Let $A = \cup_{n=1}^{\infty} (a_n, b_n)$, where $a_n = (n/(n+1))\, 2^{1-n}, b_n = 2^{1-n}$. Then $\prod_{n=1}^{\infty} a_n/b_n = \prod_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right) = 0$, $\int_0^1 \chi_A(x)\frac{dx}{x} = \sum_{n=1}^{\infty} \ln \frac{n+1}{n} = \infty$. The one-sided density at $a = 0$ is equal to 0.

EXAMPLE 4. Let $A = \cup_{n=1}^{\infty} (a_n, b_n)$, where $a_n = (n+2)/(n+1)^2, b_n = 1/n$. Then $\prod_{n=1}^{\infty} a_n/b_n = \prod_{n=1}^{\infty} \left(1 - \frac{1}{(n+1)^2}\right) > 0$, and $\int_0^1 \chi_A(x)\frac{dx}{x} = \sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n(n+2)}\right) < \infty$. The construction in Lemma 3 gives us $f(x) = K\frac{n+1}{2}$ on (a_n, b_n) and we obtain $\int_0^x f(t)dt = xf(x) - K/2, x \in A$. The one-sided density at $a = 0$ is equal to 0.

EXAMPLE 5. Let $A = \cup_{n=1}^{\infty} (a_n, b_n)$, where $a_n = 2^{1-2n}, b_n = 2^{2-2n}$ one can easily see that the construction in Lemma 3 gives $f(x) = K2^n$ on (a_n, b_n) and 0 elsewhere. Also, $\prod_{n=1}^{\infty} a_n/b_n = 0, \int_0^1 \chi_A(x)\frac{dx}{x} = \infty$ and one-sided density at $a = 0$ is positive.

REMARK 3. In [1] and [12] some multidimensional versions of (1) and (2) have recently been proved and applied. It is already proved that the inequalities in [1] are sharp and the present authors believe that it is possible to develop the techniques presented in this paper to prove that also the inequalities proved in [12] are sharp in all cases.

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