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MULTILINEAR DIRECT AND REVERSE STOLARSKY INEQUALITIES

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(*communicated by J. Pečarić*)

Abstract. For any nonnegative measurable function $f:[0,1]\to\mathbb{R}$ and any $a>0$, let $Q(f,a)$ denote the Stolarsky transform of f, equal to $\int_0^1 f\left(x^{1/a}\right)dx$. Let S_n stand for the set of all permutations of the set $\{1,\dots,n\}$. It is shown that the function

$$(0,\infty)^n \ni \mathbf{a}=(a_1,\dots,a_n)\longmapsto \mathcal{Q}(\mathbf{a}):=\sum_{\sigma\in S_n}\prod_{i=1}^n Q\left(f_{\sigma(i)},a_i\right)$$

is Schur-convex if the functions $f_1,\dots,f_n$ are nonnegative and nondecreasing and Schur-concave if $f_1,\dots,f_n$ are nonnegative and nonincreasing. Necessary and sufficient conditions for the strict Schur convexity and concavity are given.

Similar results are obtained for certain "direct" and "reverse" extensions of the Stolarsky transform to measures.

1. Introduction, statement of results, and discussion

Let f be any nonnegative measurable function defined on $[0,1]$ and let a and b be any positive real numbers. Let

$$Q(f,a):=\int_0^1 f\left(x^{1/a}\right)dx=\frac{\int_0^1 f(u)\,u^{a-1}\,du}{\int_0^1 u^{a-1}\,du}=\int_0^1 f(u)\,a\,u^{a-1}\,du \tag{1.1}$$

denote the Stolarsky transform (or mean, or quotient) of f. The Stolarsky inequality [5] says that, if the function f is nonnegative, nonincreasing, and bounded from above by 1, then

$$Q(f,a+b)\geqslant Q(f,a)\,Q(f,b).$$

Pečarić [4] gave an equivalent formulation: if f is nonnegative and nonincreasing (and not necessarily bounded from above by 1), then

$$f(0)\,Q(f,a+b)\geqslant Q(f,a)\,Q(f,b). \tag{1.2}$$

Another equivalent formulation may be obtained if one replaces $f(0)$ in (1.2) by $f(0+):=\lim_{t\downarrow 0}f(t)$; indeed, one may replace any monotonic function f in (1.2) by its right-continuous regularization, $f_+\colon u\mapsto f(u+)$, without loss of generality (w.l.o.g.),

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because f_+ differs from f only on a countable set, at the most, so that $f_+ = f$ almost everywhere, and so, $Q(f_+, a) = Q(f, a)$ for any $a > 0$. It is not difficult to see that

$$\lim_{a \downarrow 0} Q(f, a) = f(0+)$$

if f is (say) monotonic on $[0,1]$; see Lemma 2.1 in the next section. Therefore, it makes sense to set

$$Q(f, 0) := f(0+). \tag{1.3}$$

Hence, one has yet another equivalent formulation of the Stolarsky inequality: if f is nonnegative and nonincreasing, then

$$Q(f,0)\, Q(f, a+b) \geqslant Q(f,a)\; Q(f,b). \tag{1.4}$$

Pečarić [4] showed that, if f is nonnegative and nondecreasing (rather than nonincreasing), then an inequality reverse to (1.2) takes place; in view of the above discussion, it can be written as

$$Q(f,0)\, Q(f, a+b) \leqslant Q(f,a)\; Q(f,b). \tag{1.5}$$

Formulations (1.4) and (1.5) suggest that both the direct and reverse Stolarsky inequalities may be generalized in two directions, as in Theorems 1.1 and 1.4 below, where one has two functions, f and g, and three numbers, a, b, and c. Moreover, as in the original paper [5] by Stolarsky and the subsequent letter [1] by Luecking, we also treat the question of when the inequalities are strict.

Consider the following two classes of functions $f\colon [0,1] \to \mathbb{R}$:

$$\mathcal{F}_\downarrow := \{f : f \text{ is nonincreasing, nonnegative, and left-continuous on } (0,1]; \\ \text{and right-continuous at } 0\}$$

— in contexts of direct Stolarsky-type inequalities, and

$$\mathcal{F}_\uparrow := \{f : f \text{ is nondecreasing, nonnegative, and right-continuous on } [0,1); \\ \text{and left-continuous at } 1\}$$

— in contexts of reverse Stolarsky-type inequalities.

The continuity conditions in the definitions of $\mathcal{F}_\downarrow$ and $\mathcal{F}_\uparrow$ are not quite indispensable. Yet, in order to avoid cumbersome formulations, we assume these conditions as well; in the view of above discussion, this does not diminish generality.

THEOREM 1.1. *For any f and g in $\mathcal{F}_\downarrow$ and any a, b, and c in $[0,\infty)$*

$$\begin{aligned} &Q(f,c)\, Q(g, a+b+c) + Q(g,c)\, Q(f, a+b+c) \\ &> Q(f, a+c)\; Q(g, b+c) + Q(g, a+c)\; Q(f, b+c) \end{aligned} \tag{1.6 }$$

unless at least one of the following three exceptional cases takes place, when (1.6) turns into the equality:

1. *$a = 0$ or $b = 0$;*
2. *$f = 0$ on $[0,1]$* or *$g = 0$ on $[0,1]$;*

3. $f = k_1 \cdot \mathrm{I}_{[0,t]}$ and $g = k_2 \cdot \mathrm{I}_{[0,t]}$, *for some* $t \in (0,1]$, $k_1 \in (0,\infty)$, *and* $k_2 \in (0,\infty)$.

As usual, I_A stands for the indicator function of a set A, whose values are 1 on the set A and 0 on its complement.

The proof of Theorem 1.1, as well as the other comparatively long proofs, will be given in the next section.

The Stolarsky inequality (1.4) follows from Theorem 1.1 by setting $g = f$ and $c = 0$. More generally, taking $g = f$ but an arbitrary $c \geqslant 0$, one has the following.

COROLLARY 1.2. *For any* $f \in \mathcal{F}_{\downarrow}$, *the function* $[0,\infty) \ni a \mapsto Q(f,a)$ *is strictly log-convex unless* $f = k \cdot \mathrm{I}_{[0,t]}$ *for some* $t \in (0,1]$ *and* $k \in [0,\infty)$, *in which case* $\ln Q(f,a)$ *is linear in* a *(cf. Corollary 1.9 below).*

Taking $g \equiv 1$ in (1.6), one has the following *additive* version of the Stolarsky inequality (1.4).

COROLLARY 1.3. *For any* $f \in \mathcal{F}_{\downarrow}$ *and any* a, b, *and* c *in* $[0,\infty)$

$$Q(f,c) + Q(f,a+b+c) > Q(f,a+c) + Q(f,b+c) \tag{1.7}$$

unless at least one of the following cases takes place, when (1.7) turns into the equality:

1. $a = 0$ *or* $b = 0$*;*
2. $f = k \cdot \mathrm{I}_{[0,t]}$, *for some* $t \in (0,1]$ *and* $k \in [0,\infty)$.

Thus, Corollary 1.3 states that, for any $f \in \mathcal{F}_{\downarrow}$, $Q(f,a)$ is strictly convex in a, unless $f = k \cdot \mathrm{I}_{[0,t]}$ for some $t \in (0,1]$ and $k \in [0,\infty)$.

The reverse Stolarsky inequality (1.5), too, may be extended in the same two directions, so that one has the following analogue of Theorem 1.1.

THEOREM 1.4. *For any* f *and* g *in* $\mathcal{F}_{\uparrow}$ *and any* a, b, *and* c *in* $[0,\infty)$

$$\begin{aligned} &Q(f,c)\,Q(g,a+b+c) + Q(g,c)\,Q(f,a+b+c) \\ &\quad < Q(f,a+c)\ Q(g,b+c) + Q(g,a+c)\ Q(f,b+c) \end{aligned} \tag{1.8}$$

unless at least one of the following three cases takes place, when (1.8) turns into the equality:

1. $a = 0$ *or* $b = 0$*;*
2. $f = 0$ *on* $[0,1]$ or $g = 0$ *on* $[0,1]$*;*
3. both f *and* g *are (possibly different) nonzero constants on* $[0,1]$ *(cf. the exceptional case 3 in Theorem 1.1).*

The following is similar to Corollary 1.2 and immediate from Theorem 1.4.

COROLLARY 1.5. *For any* $f \in \mathcal{F}_{\uparrow}$, *the function* $[0,\infty) \ni a \mapsto Q(f,a)$ *is strictly log-concave unless* f *is constant on* $[0,1]$, *in which case* $\ln Q(f,a)$ *is constant in* a.

Similarly to Corollary 1.3 and immediately from Theorem 1.4, one has the following additive version of the reverse Stolarsky inequality.

COROLLARY 1.6. *For any* f *in* $\mathcal{F}_{\uparrow}$ *and any* a, b, *and* c *in* $[0,\infty)$

$$Q(f,c) + Q(f,a+b+c) < Q(f,a+c) + Q(f,b+c) \tag{1.9}$$

unless at least one of the following cases takes place, when (1.9) turns into the equality:

1. *$a = 0$ or $b = 0$;*
2. *f is constant on $[0,1]$.*

Thus, Corollary 1.6 states that, for any $f \in \mathcal{F}_\uparrow$, $Q(f,a)$ is strictly concave in a, unless f is constant on $[0,1]$.

Note that both sides of the inequalities (1.6) and (1.8) are bilinear in f and g. This suggests the multilinear generalizations given by Theorems 1.7 and 1.10 below; to state them, we need the following definitions.

Let S_n stand for the set of all permutations

$$\sigma\colon \{1,\dots,n\} \longrightarrow \{1,\dots,n\}.$$

For any given n-tuple $\mathbf{f} = (f_1,\dots,f_n)$ of nonnegative measurable functions, defined on $[0,1]$, introduce the function

$$[0,\infty)^n \ni \mathbf{a} = (a_1,\dots,a_n) \longmapsto \mathcal{Q}_n(\mathbf{f},\mathbf{a}) := \sum_{\sigma\in S_n}\prod_{i=1}^{n} Q\left(f_{\sigma(i)},a_i\right), \tag{1.10}$$

which is obviously multilinear in $f_1,\dots,f_n$.

Let $E \subseteq \mathbb{R}^n$. Recall that a function $\mathcal{Q}\colon E \to \mathbb{R}$ is referred to as Schur-convex if it preserves the Schur majorization: for any $\mathbf{a}$ and $\mathbf{b}$ in E such that $\mathbf{a} \succeq \mathbf{b}$, one has $\mathcal{Q}(\mathbf{a}) \geqslant \mathcal{Q}(\mathbf{b})$; similarly, $\mathcal{Q}\colon E \to \mathbb{R}$ is referred to as Schur-concave if it reverses the Schur majorization: for any $\mathbf{a}$ and $\mathbf{b}$ in E such that $\mathbf{a} \succeq \mathbf{b}$, one has $\mathcal{Q}(\mathbf{a}) \leqslant \mathcal{Q}(\mathbf{b})$. Recall also the definition of the Schur majorizarion: for $\mathbf{a} := (a_1,\dots,a_n)$ and $\mathbf{b} := (b_1,\dots,b_n)$ in $\mathbb{R}^n$, $\mathbf{a} \succeq \mathbf{b}$ means that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $a_{[1]} + \cdots + a_{[j]} \geqslant b_{[1]} + \cdots + b_{[j]}$ for all $j \in \{1,\dots,n\}$, where $a_{[1]} \geqslant \cdots \geqslant a_{[n]}$ are the ordered numbers $a_1,\dots,a_n$, from the largest to the smallest. As usual, let $\mathbf{a}_\downarrow := \left(a_{[1]},\dots,a_{[n]}\right)$. If $\mathbf{a} \succeq \mathbf{b}$ and $\mathbf{a}_\downarrow \neq \mathbf{b}_\downarrow$, let us write $\mathbf{a} \succ \mathbf{b}$.

Let n be any natural number, and let

$$\mathcal{F}_\downarrow^n := \underbrace{\mathscr{F}_\downarrow \times \cdots \times \mathscr{F}_\downarrow}_{n} \quad\text{and}\quad \mathcal{F}_\uparrow^n := \underbrace{\mathscr{F}_\uparrow \times \cdots \times \mathscr{F}_\uparrow}_{n}.$$

THEOREM 1.7. *For any $\mathbf{f} = (f_1,\dots,f_n) \in \mathcal{F}_\downarrow^n$, the function $\mathcal{Q}_n$ defined by (1.10) is Schur-convex in $\mathbf{a}$ on $[0,\infty)^n$; moreover, for any $\mathbf{a}$ and $\mathbf{b}$ in $[0,\infty)^n$, the relation $\mathbf{a} \succeq \mathbf{b}$ implies*

$$\mathcal{Q}_n(\mathbf{f},\mathbf{a}) > \mathcal{Q}_n(\mathbf{f},\mathbf{b}) \tag{1.11}$$

unless at least one of the following cases takes place, when (1.11) turns into the equality:

1. *$\mathbf{a}_\downarrow = \mathbf{b}_\downarrow$;*
2. *$f_i = 0$ on $[0,1]$ for* some *$i \in \{1,\dots,n\}$;*
3. *there exist some $t \in (0,1]$ and some n-tuple $(k_1,\dots,k_n) \in (0,\infty)^n$ such that $f_i = k_i \cdot \mathrm{I}_{[0,t]}$ for* all *$i \in \{1,\dots,n\}$.*

COROLLARY 1.8. *For any $f \in \mathcal{F}_\downarrow$ such that $f(0) = 1$, any $a \in [0,\infty)$, and any natural n*

$$Q\left(f,\frac{a}{n}\right)^n > Q\left(f,\frac{a}{n+1}\right)^{n+1} \tag{1.12}$$

unless at least one of the following cases takes place, when (1.12) turns into the equality:

1. $a = 0$;
2. $f = \mathrm{I}_{[0,t]}$, *for some* $t \in (0,1]$.

Letting here $n \to \infty$, it is not very difficult to obtain the following exponential (in a) lower bound on $Q(f,a)$.

COROLLARY 1.9. *For any* $f \in \mathcal{F}_{\downarrow}$ *such that* $f(0) = 1$ *and any* $a \in [0,\infty)$

$$Q(f,a) > \exp\left(-a\int_0^1 \frac{1-f(u)}{u}\,du\right) \tag{1.13}$$

unless at least one of the following cases takes place, when (1.13) turns into the equality:

1. $a = 0$;
2. $f = \mathrm{I}_{[0,t]}$, *for some* $t \in (0,1]$.

The integral $\int_0^1 \frac{1-f(u)}{u}\,du$ may be equal to ∞; in such a case, the right-hand side of (1.13) is understood as 0 if $a > 0$ and as 1 if $a = 0$.

Let us now proceed to the reverse analogues of Theorem 1.7 and its corollaries.

THEOREM 1.10. *For any* $\mathbf{f} = (f_1,\dots,f_n) \in \mathcal{F}_{\uparrow}^n$, *the function* $\mathcal{Q}_n$ *defined by (1.10) is Schur-concave in* $\mathbf{a}$ *on* $[0,\infty)^n$; *moreover, for any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $[0,\infty)^n$, *the relation* $\mathbf{a} \succeq \mathbf{b}$ *implies*

$$\mathcal{Q}_n(\mathbf{f},\mathbf{a}) < \mathcal{Q}_n(\mathbf{f},\mathbf{b}) \tag{1.14}$$

unless at least one of the following cases takes place, when (1.14) turns into the equality:

1. $\mathbf{a}_{\downarrow} = \mathbf{b}_{\downarrow}$;
2. $f_i = 0$ *on* $[0,1]$ *for* some $i \in \{1,\dots,n\}$;
3. all *of the functions* $f_1,\dots,f_n$ *are (possibly different) nonzero constants on* $[0,1]$;
4. $n_{\mathbf{f}(\mathbf{0})} + n_{\mathbf{b}} > n$, *where*

$$n_{\mathbf{f}(\mathbf{0})} := \#\{i{:}\, f_i(0) = 0\} \quad \text{and} \quad n_{\mathbf{b}} := \#\{i{:}\, b_i = 0\}.$$

Note that for $n = 2$ the exceptional case 4 of Theorem 1.10 implies the exceptional case 1 of it (because then $n_{\mathbf{b}} \geqslant 1$), so that Theorem 1.10 is indeed a generalization of Theorem 1.4.

COROLLARY 1.11. *For any* $f \in \mathcal{F}_{\uparrow}$ *such that* $f(0) = 1$, *any* $a \in [0,\infty)$, *and any natural* n

$$Q\left(f,\frac{a}{n}\right)^n < Q\left(f,\frac{a}{n+1}\right)^{n+1} \tag{1.15}$$

unless at least one of the following cases takes place, when (1.15) turns into the equality:

1. $a = 0$;
2. $f = 1$ *on* $[0,1]$.

COROLLARY 1.12. *For any $f \in \mathcal{F}_\uparrow$ such that $f(0)=1$ and any $a \in [0,\infty)$*

$$Q(f,a) < \exp\left(a \int_0^1 \frac{f(u)-1}{u}\,du\right) \tag{1.16}$$

unless at least one of the following cases takes place, when (1.16) turns into the equality:

1. *$a=0$;*
2. *$f=1$ on $[0,1]$.*

The integral $\int_0^1 \frac{f(u)-1}{u}\,du$ may be equal to ∞; in such a case, the right-hand side of (1.16) is understood as ∞ if $a>0$ and as 1 if $a=0$.

The above results can be generalized further, as we extend below the Stolarsky transform of functions to certain transforms of measures.

For any Borel subset S of $\mathbb{R}$, let $\mathcal{M}_S$ stand for the set of all (not necessarily finite) nonnegative σ-additive Borel measures on S.

PROPOSITION 1.13. *For any $f \in \mathcal{F}_\downarrow$, one has the representation*

$$f(u) = \int_{[0,1]} \mathrm{I}_{[0,t]}(u)\,\mu_f(dt) \quad \forall u \in [0,1],$$

where the measure $\mu_f \in \mathcal{M}_{[0,1]}$ is defined by the condition that

$$\mu_f([u,1]) := f(u) \quad \forall u \in [0,1]. \tag{1.17}$$

Proposition 1.13 is obvious, since $\mathrm{I}_{[0,t]}(u) = \mathrm{I}_{[u,1]}(t)$ for all u and t in $[0,1]$.

Similar to Proposition 1.13 is

PROPOSITION 1.14. *For any $f \in \mathcal{F}_\uparrow$, one has the representation*

$$f(u) = \int_{[0,1]} \mathrm{I}_{[t,1]}(u)\,\nu_f(dt) \quad \forall u \in [0,1],$$

where the measure $\nu_f \in \mathcal{M}_{[0,1]}$ is defined by the condition that

$$\nu_f([0,u]) := f(u) \quad \forall u \in [0,1]. \tag{1.18}$$

PROPOSITION 1.15. *For all $f \in \mathcal{F}_\downarrow$ and all $a \in [0,\infty)$, one has the representation*

$$Q(f,a) = \int_{[0,1]} Q(\mathrm{I}_{[0,t]},a)\,\mu_f(dt),$$

where μ_f is the measure defined by (1.17).

PROPOSITION 1.16. *For all $f \in \mathcal{F}_\uparrow$ and all $a \in [0,\infty)$, one has the representation*

$$Q(f,a) = \int_{[0,1]} Q(\mathrm{I}_{[t,1]},a)\,\nu_f(dt),$$

where ν_f is the measure defined by (1.18).

Also, it is not difficult to see that

$$\mu_f(\{0\}) = 0 \quad \forall f \in \mathcal{F}_\downarrow \quad \text{and} \quad \nu_f(\{1\}) = 0 \quad \forall f \in \mathcal{F}_\uparrow. \tag{1.19}$$

In view of these facts and the obvious identities (with $0^0 := 0$)

$$Q(\mathrm{I}_{[0,t]}, a) = t^a \quad \text{and} \quad Q(\mathrm{I}_{[t,1]}, a) = 1 - t^a,$$

it is natural to extend the notion of the Stolarsky transform to arbitrary measures $\mu \in \mathcal{M}_{(0,1]}$ and $\nu \in \mathcal{M}_{[0,1)}$ as follows:

$$Q_{\mathsf{dir}}(\mu, a) := \int_{(0,1]} t^a \, \mu(dt) \quad \text{and} \quad Q_{\mathsf{rev}}(\nu, a) := \int_{[0,1)} (1 - t^a) \, \nu(dt),$$

for all $a \in [0, \infty)$. Then, according to Propositions 1.15 and 1.16, for all $a \in [0, \infty)$

$$Q(f, a) = Q_{\mathsf{dir}}(\mu_f, a) \; \forall f \in \mathcal{F}_\downarrow \quad \text{and} \quad Q(f, a) = Q_{\mathsf{rev}}(\nu_f, a) \; \forall f \in \mathcal{F}_\uparrow.$$

Thus, one naturally has *two* different extensions of the Stolarsky transform to measures; the "direct" one takes the origin in the Stolarsky transform for nonincreasing functions, while the "reverse" one generalizes the Stolarsky transform for nondecreasing functions, defined on $[0, 1]$.

In fact, as we shall see later, there is no compelling reason to confine the measure μ in the generalized *direct* Stolarsky transform $Q_{\mathsf{dir}}(\mu, a)$ to the interval $(0, 1]$; instead, one can deal with arbitrary measures on the entire interval $(0, \infty)$. Also, it is not necessary to consider only nonnegative values of a for $Q_{\mathsf{dir}}(\mu, a)$. Thus, one comes to the following definition.

DEFINITION 1.17. For any $\mu \in \mathcal{M}_{(0,\infty)}$ and any $a \in \mathbb{R}$, define the *direct generalized Stolarsky transform* by the formula

$$Q_{\mathsf{dir}}(\mu, a) := \int_{(0,\infty)} t^a \, \mu(dt) \in [0, \infty].$$

For any $\nu \in \mathcal{M}_{[0,1)}$ and any $a \in [0, \infty)$, define the *reverse generalized Stolarsky transform* by the formula

$$Q_{\mathsf{rev}}(\nu, a) := \int_{[0,1)} (1 - t^a) \, \nu(dt) \in [0, \infty];$$

in the latter formula, it is assumed that $0^0 := 0$.

Note that $Q_{\mathsf{rev}}(\nu, a)$ is concave and non-decreasing in a. Hence,

$$\exists a \in (0, \infty) \; Q_{\mathsf{rev}}(\nu, a) = \infty \quad \text{iff} \quad \forall a \in (0, \infty) \; Q_{\mathsf{rev}}(\nu, a) = \infty. \tag{1.20}$$

Some of the statements below involve products of direct or reverse generalized Stolarsky transforms, for possibly different values of a and possibly different measures μ and ν; any such product will be assumed to be equal to 0 whenever at least one of the factors is equal to 0, whether some or all of the other factors in the product are equal to ∞ or not; in other words, for such products, the rule $0 \cdot \infty := 0$ is used.

The direct generalized Stolarsky transform has an obvious relation with the Mellin transform

$$M(g,s) := \int_0^\infty t^{s-1}\, g(t)\, dt.$$

Namely, if a measure μ is absolutely continuous with a density function g relative to the Lebesgue measure on $(0,\infty)$, so that $\mu\,(dt) = g\,(t)\; dt$, then

$$Q_{\mathsf{dir}}(\mu,a) = M(g,a+1).$$

Of course, the original Stolarsky transform $Q(f,a)$ defined for functions $f\colon [0,1] \to \mathbb{R}$ is also related to the Mellin transform via the formula

$$Q(g|_{[0,1]},a) = a\, M(g \cdot \mathrm{I}_{[0,1]},a),$$

where $g|_{[0,1]}$ is the restriction of a function $g\colon [0,\infty) \to \mathbb{R}$ to $[0,1]$.

Since we have extended the direct transform $Q_{\mathsf{dir}}(\mu,a)$ from measures μ on $(0,1]$ to those on $(0,\infty)$ and in view of (1.19), let us redefine the classes $\mathcal{F}_\downarrow$ and $\mathcal{F}_\uparrow$ in a more natural manner, as follows:

$$\mathcal{F}_\downarrow := \{f\colon f \text{ is nonincreasing, nonnegative, and left-continuous on } (0,\infty)\}$$

and

$$\mathcal{F}_\uparrow := \{g\colon g \text{ is nondecreasing, nonnegative, and right-continuous on } [0,1)\},$$

so that now the classes $\mathcal{F}_\downarrow$ and $\mathcal{F}_\uparrow$ consist of (possibly unbounded) functions defined on $(0,\infty)$ and $[0,1)$, respectively; moreover, let us allow functions in these classes to take on the value ∞.

Then the relations

$$\mu\,([u,\infty)) \triangleq f\,(u) \quad \forall u \in (0,\infty) \tag{1.21}$$

and

$$\nu\,([0,u]) \triangleq g(u) \quad \forall u \in [0,1) \tag{1.22}$$

define one-to-one correspondences

$$\mathcal{M}_{(0,\infty)} \ni \mu \longleftrightarrow f \in \mathcal{F}_\downarrow \quad \text{and} \quad \mathcal{M}_{[0,1)} \ni \nu \longleftrightarrow g \in \mathcal{F}_\uparrow.$$

For any such pair $\mu \leftrightarrow f$ in the first of these two correspondences, one has

- $f\,(u) = \int\limits_{(0,\infty)} \mathrm{I}_{(0,t]}\,(u)\;\mu\,(dt) \quad \forall u \in (0,\infty)$ (cf. Proposition 1.13 above);
- $Q_{\mathsf{dir}}\,(\mu,a) = \int\limits_{(0,\infty)} f\,(u)\; a\,u^{a-1}\,du \quad \forall a \in (0,\infty)$ (using the previous expression for f and the Fubini theorem; cf. (1.1));
- $\mu = 0$ iff $f = 0$;
- for every $t \in (0,\infty)$, $\operatorname{supp}(\mu) = \{t\}$ iff $f = k \cdot \mathrm{I}_{(0,t]}$ for some $k \in (0,\infty)$;
- $\mu\,((0,\infty)) = f\,(0+)$;

as usual, $\operatorname{supp}(\mu)$ stands for the support of measure μ.

Similarly, for any pair $\nu \leftrightarrow g$ in the other kind of correspondence, one has

- $g(u) = \int\limits_{[0,1)} \mathrm{I}_{[t,1)}(u)\ \nu(dt) \quad \forall u \in [0,1)$ (cf. Proposition 1.14 above);
- $Q_{\mathsf{rev}}(\nu, a) = \int\limits_{[0,1)} g(u)\ a\, u^{a-1}\, du \quad \forall a \in (0,\infty)$ (using the previous expression for g and the Fubini theorem; cf. (1.1));
- $\nu = 0$ iff $g = 0$;
- $\operatorname{supp}(\nu) = \{0\}$ iff g is a nonzero constant;
- $\nu(\{0\}) = g(0)$.

Note that both Q_{dir} and Q_{rev} are in agreement with the original definition (1.1) of the Stolarsky transform — but only for $a > 0$.

With these modifications and correspondences in mind and taking care of the possibility that the direct and inverse generalized Stolarsky transforms may now take on infinite values, one translates above results as follows.

THEOREM 1.18. (Cf. Theorem 1.1.) *For any measures μ_1 and μ_2 in $\mathcal{M}_{(0,\infty)}$, any $c \in \mathbb{R}$, and any a and b in $[0,\infty)$*

$$\begin{aligned} &Q_{\mathsf{dir}}(\mu_1, c)\, Q_{\mathsf{dir}}(\mu_2, a+b+c) + Q_{\mathsf{dir}}(\mu_2, c)\, Q_{\mathsf{dir}}(\mu_1, a+b+c) \\ &\quad > Q_{\mathsf{dir}}(\mu_1, a+c)\ Q_{\mathsf{dir}}(\mu_2, b+c) + Q_{\mathsf{dir}}(\mu_2, a+c)\ Q_{\mathsf{dir}}(\mu_1, b+c) \end{aligned} \tag{1.23}$$

unless at least one of the following four exceptional cases takes place, when (1.23) turns into the equality:

1. *the right-hand side of (1.23) equals to ∞;*
2. *$a = 0$ or $b = 0$;*
3. *$\mu_1 = 0$ or $\mu_2 = 0$;*
4. *$\operatorname{supp}(\mu_1) = \operatorname{supp}(\mu_2) = \{t\}$, for some $t \in (0,\infty)$.*

According to the rule $0 \cdot \infty := 0$, the exceptional case 1 of Theorem 1.18 occurs if either one of the factors $Q_{\mathsf{dir}}(\mu_1, a+c)$ and $Q_{\mathsf{dir}}(\mu_2, b+c)$ is equal to ∞ while the other one is nonzero or one of the factors $Q_{\mathsf{dir}}(\mu_2, a+c)$ and $Q_{\mathsf{dir}}(\mu_1, b+c)$ is equal to ∞ while the other one is nonzero.

COROLLARY 1.19. (Cf. Corollary 1.2.) *For any $\mu \in \mathcal{M}_{(0,\infty)}$, the function $\mathbb{R} \ni a \mapsto Q_{\mathsf{dir}}(\mu, a)$ is strictly log-convex on the interval*

$$\operatorname{dom}(Q_{\mathsf{dir}}(\mu,\cdot)) := \{a \in \mathbb{R} \colon Q_{\mathsf{dir}}(\mu, a) < \infty\}$$

unless $\operatorname{supp}(\mu) \subseteq \{t\}$ for some $t \in (0,\infty)$, in which case $\ln Q_{\mathsf{dir}}(\mu, a)$ is linear in a (cf. Corollary 1.24 below).

The reverse Stolarsky inequality (1.5), too, may be extended in the same manner, so that one has the following analogue of Theorem 1.18.

THEOREM 1.20. (Cf. Theorem 1.4.) *For any ν_1 and ν_2 in $\mathcal{M}_{[0,1)}$ and any a, b, and c in $[0,\infty)$*

$$\begin{aligned} &Q_{\mathsf{rev}}(\nu_1, c)\, Q_{\mathsf{rev}}(\nu_2, a+b+c) + Q_{\mathsf{rev}}(\nu_2, c)\, Q_{\mathsf{rev}}(\nu_1, a+b+c) \\ &\quad < Q_{\mathsf{rev}}(\nu_1, a+c)\, Q_{\mathsf{rev}}(\nu_2, b+c) + Q_{\mathsf{rev}}(\nu_2, a+c)\ Q_{\mathsf{rev}}(\nu_1, b+c) \end{aligned} \tag{1.24}$$

unless at least one of the following four exceptional cases takes place, when (1.24) turns into the equality:

1. *the left-hand side of (1.24) equals to* ∞*;*
2. $a=0$ *or* $b=0$*;*
3. $\nu_1=0$ or $\nu_2=0$*;*
4. $\operatorname{supp}(\nu_1)=\operatorname{supp}(\nu_2)=\{0\}$.

The following is similar to Corollary 1.19 and immediate from Theorem 1.20 and (1.20).

COROLLARY 1.21. (Cf. Corollary 1.5.) *For any* $\nu\in\mathcal{M}_{[0,1)}$*, the function* $[0,\infty)\ni a\mapsto Q_{\mathsf{rev}}(\nu,a)$ *is strictly log-concave unless either (i)* $\operatorname{supp}(\nu)\subseteq\{0\}$*, in which case* $\ln Q_{\mathsf{rev}}(\nu,a)$ *is constant in* a*, or (ii)* $Q_{\mathsf{rev}}(\nu,a)=\infty$ *for all* $a\in(0,\infty)$.

For any given n-tuple $\boldsymbol{\mu}=(\mu_1,\dots,\mu_n)\in\underbrace{\mathcal{M}_{(0,\infty)}\times\cdots\times\mathcal{M}_{(0,\infty)}}_{n}$, introduce the function

$$\mathbb{R}^n\ni\mathbf{a}=(a_1,\dots,a_n)\longmapsto\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{a}):=\sum_{\sigma\in S_n}\prod_{i=1}^n Q_{\mathsf{dir}}\left(\mu_{\sigma(i)},a_i\right)\in[0,\infty], \tag{1.25}$$

using the rule $0\cdot\infty:=0$.

Similarly, for any given n-tuple $\boldsymbol{\nu}=(\nu_1,\dots,\nu_n)\in\underbrace{\mathcal{M}_{[0,1)}\times\cdots\times\mathcal{M}_{[0,1)}}_{n}$, introduce the function

$$[0,\infty)^n\ni\mathbf{a}=(a_1,\dots,a_n)\longmapsto\mathcal{Q}_{n,\mathsf{rev}}(\boldsymbol{\nu},\mathbf{a}):=\sum_{\sigma\in S_n}\prod_{i=1}^n Q_{\mathsf{rev}}\left(\nu_{\sigma(i)},a_i\right)\in[0,\infty], \tag{1.26}$$

using the same rule $0\cdot\infty:=0$.

THEOREM 1.22. (Cf. Theorem 1.7.) *For any* n*-tuple* $\boldsymbol{\mu}=(\mu_1,\dots,\mu_n)\in\underbrace{\mathcal{M}_{(0,\infty)}\times\cdots\times\mathcal{M}_{(0,\infty)}}_{n}$*, the function* $\mathcal{Q}_{n,\mathsf{dir}}$ *defined by (1.25) is Schur-convex in* $\mathbf{a}\in\mathbb{R}^n$*; moreover, for any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{R}^n$*, the relation* $\mathbf{a}\succeq\mathbf{b}$ *implies*

$$\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{a})>\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{b}) \tag{1.27}$$

unless at least one of the following cases takes place, when (1.27) turns into the equality:

1. $\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{b})=\infty$*;*
2. $\mathbf{a}_\downarrow=\mathbf{b}_\downarrow$*;*
3. $\mu_i=0$ *for* some $i\in\{1,\dots,n\}$*;*
4. *there exists some* $t\in(0,\infty)$ *such that* $\operatorname{supp}(\mu_i)=\{t\}$ *for* all $i\in\{1,\dots,n\}$.

COROLLARY 1.23. (Cf. Corollary 1.8.) *For any* probability *measure* $\mu \in \mathcal{M}_{(0,\infty)}$, *any* $a \in \mathbb{R}$, *and any natural* n

$$Q_{\mathsf{dir}}\left(\mu, \frac{a}{n}\right)^n > Q_{\mathsf{dir}}\left(\mu, \frac{a}{n+1}\right)^{n+1} \tag{1.28}$$

unless at least one of the following cases takes place, when (1.28) turns into the equality:

1. $Q_{\mathsf{dir}}\left(\mu, \frac{a}{n+1}\right) = \infty$;
2. $a = 0$;
3. $\operatorname{supp}(\mu) = \{t\}$, *for some* $t \in (0,\infty)$.

COROLLARY 1.24. (Cf. Corollary 1.9.) *For any* $a \in \mathbb{R}$ *and any* probability *measure* $\mu \in \mathcal{M}_{(0,\infty)}$ *such that the value* $\int_{(0,\infty)} (\ln u)\, \mu(du) \in [-\infty, \infty]$ *is defined, one has*

$$Q_{\mathsf{dir}}(\mu, a) > \exp\left(a \int_{(0,\infty)} (\ln u)\, \mu(du)\right) \tag{1.29}$$

unless at least one of the following cases takes place, when (1.29) turns into the equality:

1. $a \neq 0$ *and* $a \int\limits_{(0,\infty)} (\ln u)\, \mu(du) = \infty$;
2. $a = 0$ *(in which case the value of the right-hand side of (1.29) is assumed to be* 1*)*;
3. $\operatorname{supp}(\mu) = \{t\}$, *for some* $t \in (0,\infty)$.

Let us now proceed to reverse analogues of Theorem 1.22 and its corollaries.

THEOREM 1.25. (Cf. Theorem 1.10.) *For any* n*-tuple* $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \underbrace{\mathcal{M}_{[0,1)} \times \cdots \times \mathcal{M}_{[0,1)}}_{n}$, *the function* $\mathcal{Q}_{n,\mathsf{rev}}$ *defined by (1.26) is Schur-concave in* $\mathbf{a} \in [0,\infty)^n$; *moreover, for any* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $[0,\infty)^n$, *the relation* $\mathbf{a} \succeq \mathbf{b}$ *implies*

$$\mathcal{Q}_{n,\mathsf{rev}}(\boldsymbol{\nu}, \mathbf{a}) < \mathcal{Q}_{n,\mathsf{rev}}(\boldsymbol{\nu}, \mathbf{b}) \tag{1.30}$$

unless at least one of the following cases takes place, when (1.30) turns into the equality:

1. $\mathcal{Q}_{n,\mathsf{rev}}(\boldsymbol{\nu}, \mathbf{a}) = \infty$;
2. $\mathbf{a}_\downarrow = \mathbf{b}_\downarrow$;
3. $\nu_i = 0$ *for* some $i \in \{1, \dots, n\}$;
4. $\operatorname{supp}(\nu_i) = \{0\}$ *for* all $i \in \{1, \dots, n\}$;
5. $n_{\boldsymbol{\nu}(\{\mathbf{0}\})} + n_{\mathbf{b}} > n$, *where*

$$n_{\boldsymbol{\nu}(\{\mathbf{0}\})} := \#\{i\colon \nu_i(\{0\}) = 0\} \quad \text{and} \quad n_{\mathbf{b}} := \#\{i\colon b_i = 0\}.$$

COROLLARY 1.26. (Cf. Corollary 1.11.) *For any* $\nu \in \mathcal{M}_{[0,1)}$ *such that* $\nu(\{0\}) = 1$, *any* $a \in [0,\infty)$, *and any natural* n

$$Q_{\mathsf{rev}}\left(\nu, \frac{a}{n}\right)^n < Q_{\mathsf{rev}}\left(\nu, \frac{a}{n+1}\right)^{n+1} \tag{1.31}$$

unless at least one of the following cases takes place, when (1.31) turns into the equality:

1. $Q_{\mathsf{rev}}\left(\nu, \frac{a}{n}\right) = \infty$*;*
2. $a = 0$*;*
3. $\operatorname{supp}(\nu) = \{0\}$.

COROLLARY 1.27. (Cf. Corollary 1.12.) *For any* $\nu \in \mathcal{M}_{[0,1)}$ *such that* $\nu(\{0\}) = 1$ *and any* $a \in [0,\infty)$

$$Q_{\mathsf{rev}}(\nu, a) < \exp\left(-a \int_{(0,1)} (\ln u)\ \nu(du)\right) \tag{1.32}$$

unless at least one of the following cases takes place, when (1.32) turns into the equality:
1. $Q_{\mathsf{rev}}(\nu, a) = \infty$*;*
2. $a = 0$ *(in which case the value of the right-hand side of (1.32) is assumed to be* 1*);*
3. $\operatorname{supp}(\nu) = \{0\}$.

2. Proofs

Let us begin with the following lemma to justify the definition (1.3).

LEMMA 2.1. *If a function* f *is bounded on* $[0,1]$ *and the limit* $f(0+)$ *exists and is finite, then*

$$\lim_{a\downarrow 0} Q(f,a) = f(0+).$$

Proof. Let $M := \sup_{[0,1]} |f|$, so that $M < \infty$. Take any $\varepsilon > 0$. Then there exists such a $\delta \in (0,1)$ that $|f(u) - f(0+)| \leqslant \varepsilon$ for all $u \in [0,\delta]$. Therefore,

$$\begin{aligned}|Q(f,a) - f(0+)| &\leqslant \int_0^1 |f(u) - f(0+)|\ au^{a-1}\,du \\ &\leqslant \varepsilon \cdot \int_0^\delta au^{a-1}du + 2M \cdot \int_\delta^1 au^{a-1}du \\ &= \varepsilon \cdot \delta^a + 2M\cdot(1-\delta^a) \to \varepsilon\end{aligned}$$

as $a \downarrow 0$. Hence, $\limsup_{a\downarrow 0} |Q(f,a) - f(0+)| \leqslant \varepsilon$ for any $\varepsilon > 0$. The conclusion of the lemma now follows.

Proof of Proposition 1.15 (page 676). For $a > 0$, this proposition follows from Proposition 1.13, definition (1.1), and the Fubini theorem. For $a = 0$, this follows from definitions (1.3) and (1.17); indeed,

$$\begin{aligned}\int_{[0,1]} Q(\mathrm{I}_{[0,t]}, 0)\,\mu_f(dt) &= \int_{(0,1]} \mu_f(dt) \\ &= \lim_{\delta\downarrow 0} \int_{[\delta,1]} \mu_f(dt) = \lim_{\delta\downarrow 0} f(\delta) = f(0+) = Q(f,0).\end{aligned}$$

Proof of Proposition 1.16. For $a > 0$, this follows from Proposition 1.14, definition (1.1), and the Fubini theorem. For $a = 0$, this follows from definitions (1.3) and (1.18); indeed,

$$\int_{[0,1]} Q(\mathrm{I}_{[t,1]}, 0)\, \nu_f\,(dt) = \nu_f\,(\{0\}) = f(0) = f\,(0+) = Q(f, 0).$$

LEMMA 2.2. *Let μ and ν be nonzero (nonnegative σ-additive Borel) measures on a Borel subset S of $\mathbb{R}$ such that*

$$\operatorname{supp}(\mu \otimes \nu) \subseteq D := \{(t,t) : t \in S\},$$

where, as usual, supp *denotes the support of the measure and $\mu \otimes \nu$ denotes the direct product of the measures μ and ν. Then*

$$\operatorname{supp}(\mu \otimes \nu) = \{(t,t)\}$$

for some $t \in S$.

Proof. Note that $\operatorname{supp}(\mu \otimes \nu) = (\operatorname{supp}\mu) \times (\operatorname{supp}\nu)$ for any two measures μ and ν. If $\operatorname{supp}(\mu \otimes \nu) \subseteq D$ and $\operatorname{supp}(\mu \otimes \nu)$ contains at least two distinct points, say (t,t) and (s,s), then $\operatorname{supp}(\mu \otimes \nu) = (\operatorname{supp}\mu) \times (\operatorname{supp}\nu)$ must also contain the point $(t,s) \notin D$, so that one has a contradiction.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Take any f and g in $\mathcal{F}_{\downarrow}$ and any a, b, and c in $[0,\infty)$. Consider

$$\begin{aligned} \Delta(f,g;a,b,c) := \; & Q(f,c)\,Q(g,a+b+c) + Q(g,c)\,Q(f,a+b+c) \\ & - Q(f,a+c)\;Q(g,b+c) - Q(g,a+c)\;Q(f,b+c), \end{aligned} \tag{2.1}$$

the difference between the left-hand side and the right-hand side of inequality (1.6). In view of Proposition 1.15,

$$\Delta(f,g;a,b,c) = \iint_{[0,1]^2} \delta_{a,b,c}(s,t)\;(\mu_f \otimes \mu_g)\,(ds \times dt), \tag{2.2}$$

where

$$\delta_{a,b,c}(s,t) := \Delta\left(\mathrm{I}_{[0,s]}, \mathrm{I}_{[0,t]}; a,b,c\right).$$

Since $Q\left(\mathrm{I}_{[0,s]}, a\right) = s^a$ for all $s \in [0,1]$ and $a \geqslant 0$ (with $0^0 := 0$), it is not difficult to check that

$$\delta_{a,b,c}(s,t) = s^c \cdot t^c \cdot (t^a - s^a) \cdot \left(t^b - s^b\right). \tag{2.3}$$

Now it is obvious that $\delta_{a,b,c} \geqslant 0$ on $[0,1]^2$. Hence, by (2.2), $\Delta(f,g;a,b,c) \geqslant 0$, so that the non-strict version of inequality (1.6) follows.

Next, it is easy to check that in each of the three exceptional cases in Theorem 1.1, $\Delta(f,g;a,b,c) = 0$.

Finally, suppose that (1.6) turns into the equality, i.e., $\Delta(f,g;a,b,c)=0$. It remains to show that then at least one of the three exceptional cases takes place. Since $\delta_{a,b,c}\geqslant 0$ on $[0,1]^2$, identity (2.2) implies that $\delta_{a,b,c}=0$ $(\mu_f\otimes\mu_g)$-almost everywhere on $[0,1]^2$. W.l.o.g., neither of the exceptional cases 1 or 2 takes place (otherwise, there is nothing to prove). Hence, $a>0$, $b>0$, and $\mu_f\otimes\mu_g\neq 0$ (since $\mu_f\otimes\mu_g=0$ would imply that the exceptional case 2 takes place). On the other hand, in view of (2.3), one has $\delta_{a,b,c}(s,t)=0$ only if either $s=0$ or $t=0$ or $t=s$, since $a>0$ and $b>0$. Therefore, $s=0$ or $t=0$ or $t=s$ for $(\mu_f\otimes\mu_g)$-almost all pairs $(s,t)\in[0,1]^2$. On the other hand, according to the definition (1.17),

$$\mu_f(\{0\})=\lim_{u\downarrow 0}(\mu_f([0,1])-\mu_f([u,1]))=\lim_{u\downarrow 0}(f(0)-f(u))=0,$$

because f belongs to $\mathcal{F}_\downarrow$ and hence is continuous at 0. Similarly, $\mu_g(\{0\})=0$. It follows that $s\neq 0$ and $t\neq 0$ for $(\mu_f\otimes\mu_g)$-almost all pairs $(s,t)\in[0,1]^2$. Thus, $t=s$ for $(\mu_f\otimes\mu_g)$-almost all pairs $(s,t)\in[0,1]^2$. In other words, $\operatorname{supp}(\mu_f\otimes\mu_g)\subseteq\{(t,t):t\in[0,1]\}$. Now, by Lemma 2.2, $\operatorname{supp}(\mu_f\otimes\mu_g)=\{(t_*,t_*)\}$ for some $t_*\in[0,1]$ or, equivalently, $\operatorname{supp}(\mu_f)=\operatorname{supp}(\mu_g)=\{t_*\}$; in fact, since $\mu_f(\{0\})=\mu_g(\{0\})=0$, one must have $t_*\neq 0$. By Proposition 1.13, this implies that $f=k_1\cdot\mathrm{I}_{[0,t_*]}$ and $g=k_2\cdot\mathrm{I}_{[0,t_*]}$, where $k_1:=\mu_f([0,1])$ and $k_2:=\mu_g([0,1])$.

Proof of Theorem 1.4. This proof is similar to that of Theorem 1.1. In this case, one has to deal with ν_f and ν_g in place of μ_f and μ_g, with Proposition 1.16 in place of Proposition 1.15, and with

$$\bar\delta_{a,b,c}(s,t):=\Delta\left(\mathrm{I}_{[s,1]},\mathrm{I}_{[t,1]};a,b,c\right)$$

in place of $\delta_{a,b,c}(s,t)$. Since $Q\left(\mathrm{I}_{[s,1]},a\right)=1-s^a$ for all $s\in[0,1]$ and $a\geqslant 0$ (again with $0^0:=0$), it is not difficult to check that for all $(s,t)\in[0,1]^2$

$$\begin{aligned}\bar\delta_{a,b,c}(s,t)=&\, s^c\cdot t^c\cdot(t^a-s^a)\cdot\left(t^b-s^b\right)\\&-s^c\cdot(1-s^a)\cdot\left(1-s^b\right)\\&-t^c\cdot(1-t^a)\cdot\left(1-t^b\right),\end{aligned}\tag{2.4}$$

whence

$$\begin{aligned}\bar\delta_{a,b,c}(s,t)\leqslant&\, s^c\cdot t^c\cdot\left[(t^a-s^a)\cdot\left(t^b-s^b\right)\right.\\&\left.-(1-s^a)\cdot\left(1-s^b\right)-(1-t^a)\cdot\left(1-t^b\right)\right]\\=&-s^c\cdot t^c\cdot\left[(1-s^a)\cdot\left(1-t^b\right)+\left(1-s^b\right)\cdot(1-t^a)\right]\leqslant 0.\end{aligned}\tag{2.5}$$

Therefore, $\Delta(f,g;a,b,c)\leqslant 0$, so that the non-strict version of inequality (1.8) follows.

The less trivial part of what remains to prove is that, if $\bar\delta_{a,b,c}=0$ $(\nu_f\otimes\nu_g)$-almost everywhere on $[0,1]^2$ and a, b, f, and g are nonzero, then both f and g are nonzero constants on $[0,1]$. For any $(s,t)\in[0,1]^2$, the equality $\bar\delta_{a,b,c}(s,t)=0$ implies that the nonstrict inequalities in (2.5) turn into the equalities, which in turn implies that either s or t belongs to the set $\{0,1\}$, given that a and b are nonzero. But, if $s=0$, then $\bar{\delta}_{a,b,c}(s,t)=0$ implies (in view of (2.4)) that $t\in\{0,1\}$; similarly, if $t=0$, then $\bar{\delta}_{a,b,c}(s,t)=0$ implies that $s\in\{0,1\}$. Therefore, $s=1$ or $t=1$ or $s=t=0$ for $(\nu_f\otimes\nu_g)$-almost all pairs $(s,t)\in[0,1]^2$. On the other hand, according to the definition (1.18),

$$\nu_f\,(\{1\})=\lim_{u\uparrow 1}(\nu_f\,([0,1])-\nu_f\,([0,u]))=\lim_{u\uparrow 1}(f\,(1)-f\,(u))=0,$$

because f belongs to $\mathcal{F}_\uparrow$ and hence is continuous at 1. Similarly, $\nu_g\,(\{1\})=0$. Thus, $s=t=0$ for $(\nu_f\otimes\nu_g)$-almost all pairs $(s,t)\in[0,1]^2$. In other words, $\operatorname{supp}(\nu_f)=\operatorname{supp}(\nu_g)=\{0\}$. By Proposition 1.14, this implies that $f=k_1$ on $[0,1]$ and $g=k_2$ on $[0,1]$, where $k_1:=\nu_f\,([0,1])$ and $k_2:=\nu_g\,([0,1])$.

Proof of Theorem 1.7. For $n=1$, $\mathbf{a}\succeq\mathbf{b}$ means that $\mathbf{a}=\mathbf{b}$, so that the statement of the theorem is then trivial. The case $n=2$ of Theorem 1.7 is Theorem 1.1. Assume then that $n\geqslant 3$.

For any vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ such that $\mathbf{a}\succ\mathbf{b}$, let us say that vectors $\mathbf{a}^{(0)},\mathbf{a}^{(1)},\dots,\mathbf{a}^{(m)}$ in $\mathbb{R}^n$ form an *elementary majorization chain down from* $\mathbf{a}$ *to* $\mathbf{b}$ of length $m+1$ if $\mathbf{a}=\mathbf{a}^{(0)}\succ\mathbf{a}^{(1)}\succ\cdots\succ\mathbf{a}^{(m)}=\mathbf{b}$ and for each $j\in\{1,\dots,m\}$ the vectors $\mathbf{a}^{(j-1)}=\left(a_1^{(j-1)},\dots,a_n^{(j-1)}\right)$ and $\mathbf{a}^{(j)}=\left(a_1^{(j)},\dots,a_n^{(j)}\right)$ differ exactly in two coordinates, say $p^{(j)}$ th and $q^{(j)}$ th, so that $a_i^{(j-1)}\neq a_i^{(j)}$ iff $i\in\left\{p^{(j)},q^{(j)}\right\}$.

A well-known result by Muirhead [3] (see, e.g., [2, Remark B.1.a of Chapter 2]) states that, for any vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ such that $\mathbf{a}\succ\mathbf{b}$, there exists a finite (actually of length $\leqslant n$) elementary majorization chain down from $\mathbf{a}$ to $\mathbf{b}$.

Therefore, we shall assume, w.l.o.g., that $\mathbf{a}\succ\mathbf{b}$ and the vectors $\mathbf{a}=(a_1,\dots,a_n)$ and $\mathbf{b}=(b_1,\dots,b_n)$ differ exactly in two coordinates; in other words, we shall assume that there exist two different numbers p and q in the set $\{1,\dots,n\}$ such that

$$b_i=a_i\ \forall i\in\{1,\dots,n\}\setminus\{p,q\}, \tag{2.6}$$

$$a_p>a_q,\quad b_p=a_p-\varepsilon,\quad\text{and}\quad b_q=a_q+\varepsilon \tag{2.7}$$

for some

$$\varepsilon\in(0,a_p-a_q). \tag{2.8}$$

Then

$$\begin{aligned}&\mathcal{Q}_n\,(\mathbf{f},\mathbf{a})-\mathcal{Q}_n\,(\mathbf{f},\mathbf{b})=\\ &\sum_{\sigma\in S_n}\left(\prod_{i\in\{1,\dots,n\}\setminus\{p,q\}}Q\left(f_{\sigma(i)},b_i\right)\right)\times\\ &\qquad\times\left[Q\left(f_{\sigma(p)},a_p\right)Q\left(f_{\sigma(q)},a_q\right)-Q\left(f_{\sigma(p)},a_p-\varepsilon\right)Q\left(f_{\sigma(q)},a_q+\varepsilon\right)\right].\end{aligned} \tag{2.9}$$

For the given pair of p and q, consider the one-to-one correspondence $S_n \ni \sigma \leftrightarrow \tilde{\sigma} \in S_n$ given by

$$\tilde{\sigma}(i) := \sigma(i) \ \forall i \in \{1,\dots,n\} \setminus \{p,q\},$$
$$\tilde{\sigma}(p) := \sigma(q), \quad \text{and} \quad \tilde{\sigma}(q) := \sigma(p).$$

In view of this one-to-one correspondence, $\sigma(p)$ and $\sigma(q)$ in (2.9) may be mutually interchanged, so that

$$\begin{aligned} &\mathcal{Q}_n(\mathbf{f},\mathbf{a}) - \mathcal{Q}_n(\mathbf{f},\mathbf{b}) = \\ &\sum_{\sigma\in S_n} \left(\prod_{i\in\{1,\dots,n\}\setminus\{p,q\}} Q\left(f_{\sigma(i)}, b_i\right) \right) \times \\ &\quad \times \left[Q\left(f_{\sigma(q)}, a_p\right) Q\left(f_{\sigma(p)}, a_q\right) - Q\left(f_{\sigma(q)}, a_p - \varepsilon\right) Q\left(f_{\sigma(p)}, a_q + \varepsilon\right) \right]. \end{aligned} \tag{2.10 }$$

Adding now (2.9) and (2.10), one has

$$\begin{aligned} &2\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = 2\mathcal{Q}_n(\mathbf{f},\mathbf{b}) \\ &\quad + \sum_{\sigma\in S_n} \left(\prod_{i\in\{1,\dots,n\}\setminus\{p,q\}} Q\left(f_{\sigma(i)}, b_i\right) \right) \cdot \Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right), \end{aligned} \tag{2.11 }$$

where $\Delta(f,g;a,b,c)$ is defined by (2.1),

$$a := a_p - a_q - \varepsilon, \quad b := \varepsilon, \quad \text{and} \quad c := a_q,$$

so that $a+b+c = a_p$, $a+c = a_p - \varepsilon$, and $b+c = a_q + \varepsilon$. By (2.8),

$$a > 0 \quad \text{and} \quad b > 0. \tag{2.12}$$

Now it follows from Theorem 1.1 (or, a little more directly, from its proof) that every summand on the right-hand side of (2.11) is nonnegative, and so, $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) \geqslant \mathcal{Q}_n(\mathbf{f},\mathbf{b})$.

Next, it is easy to check that in each of the three exceptional cases in Theorem 1.7, $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = \mathcal{Q}_n(\mathbf{f},\mathbf{b})$.

It remains to show that the equality $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = \mathcal{Q}_n(\mathbf{f},\mathbf{b})$ implies at least one of the three exceptional cases in Theorem 1.7. W.l.o.g., neither of the exceptional cases 1 or 2 takes place (otherwise, there is nothing to prove). Hence, none of the functions $f_1,\dots,f_n$ is zero, and so, all of the numbers $Q(f_1,\alpha),\dots,Q(f_n,\alpha)$ are nonzero, for all $\alpha \in [0,\infty)$. On the other hand, the equality $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = \mathcal{Q}_n(\mathbf{f},\mathbf{b})$ implies that every summand in (2.11) is zero, whence $\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right) = 0$ for all permutations $\sigma \in S_n$. By virtue of Theorem 1.1, it now follows that the exceptional case 3 of Theorem 1.7 does take place.

Proof of Corollary 1.8. It is easy to see that

$$\Big(\underbrace{\frac{a}{n},\dots,\frac{a}{n}}_{n},0\Big) \succ \Big(\underbrace{\frac{a}{n+1},\dots,\frac{a}{n+1}}_{n+1}\Big),$$

for all $a > 0$ and all natural n. Also, it follows from (1.3) and the conditions $f \in \mathcal{F}_\downarrow$ and $f(0) = 1$ that $Q(f,0) = 1$. Now the corollary is immediate from Theorem 1.7.

Proof of Corollary 1.9. Note that the conditions $f \in \mathcal{F}_\downarrow$ and $f(0) = 1$ imply that $0 \leqslant f \leqslant 1$ on $[0,1]$. In the case $\int_0^1 \frac{1-f(u)}{u}\,du = \infty$, the statement is trivial. Assume therefore that $\int_0^1 \frac{1-f(u)}{u}\,du < \infty$. Then, for every $a \in [0,\infty)$,

$$Q\left(f, \frac{a}{n}\right) = 1 - \frac{a}{n}\int_0^1 \frac{1-f(u)}{u}\, u^{a/n}\,du,$$

and

$$\int_0^1 \frac{1-f(u)}{u}\, u^{a/n}\,du \longrightarrow \int_0^1 \frac{1-f(u)}{u}\,du$$

as $n \to \infty$, by the Lebesgue theorem; hence,

$$Q\left(f, \frac{a}{n}\right)^n \longrightarrow \exp\left(-a\int_0^1 \frac{1-f(u)}{u}\,du\right)$$

as $n \to \infty$. Now Corollary 1.9 follows from Corollary 1.8.

Proof of Theorem 1.10. As in the proof of Theorem 1.7, assume w.l.o.g. that $n \geqslant 3$. Also similarly to the corresponding part of the proof of Theorem 1.7, the Schur concavity follows from (2.11), using in the present case Theorem 1.4 rather than Theorem 1.1.

Note next that, for any $\mathbf{a}$ and $\mathbf{b}$ in $[0,\infty)^n$, majorization $\mathbf{a} \succeq \mathbf{b}$ implies $n_\mathbf{a} \geqslant n_\mathbf{b}$ ($n_\mathbf{b}$ is defined in the formulation of the exceptional case 4 in Theorem 1.10). Hence, for any elementary majorization chain $\mathbf{a} = \mathbf{a}^{(0)} \succ \mathbf{a}^{(1)} \succ \cdots \succ \mathbf{a}^{(m)} = \mathbf{b}$, one has $n_{\mathbf{f}(\mathbf{0})} + n_\mathbf{b} > n$ iff $\forall j \in \{1,\dots,m\}\quad n_{\mathbf{f}(\mathbf{0})} + n_{\mathbf{a}^{(j)}} > n$. Taking this into account, here too we shall assume, w.l.o.g., that $\mathbf{a} \succ \mathbf{b}$ and the vectors $\mathbf{a} = (a_1,\dots,a_n)$ and $\mathbf{b} = (b_1,\dots,b_n)$ differ exactly in two coordinates, pth and qth, so that once again one has (2.6), (2.7), (2.8), and (2.11).

Next, it is not hard to check that in each of the four exceptional cases in Theorem 1.10, $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = \mathcal{Q}_n(\mathbf{f},\mathbf{b})$. In particular, in the exceptional case 4, $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) = \mathcal{Q}_n(\mathbf{f},\mathbf{b})$ once again follows by (2.11), because then for every permutation $\sigma \in S_n$ at least one of the $n-2$ numbers $Q\left(f_{\sigma(i)}, b_i\right)$, $i \in \{1,\dots,n\} \setminus \{p,q\}$, is zero. Indeed, since the sum of the cardinalities $n_{\mathbf{f}(\mathbf{0})}$ and $n_\mathbf{b}$ of the sets

$$C_{\mathbf{f}(\mathbf{0})} := \{i : f_i(0) = 0\} \quad \text{and} \quad C_\mathbf{b} := \{i : b_i = 0\} \tag{2.13}$$

exceeds n, their intersection is non-empty; therefore, at least one of the n numbers $Q\left(f_{\sigma(i)}, b_i\right)$, $i \in \{1,\dots,n\}$, is zero, in view of (1.3) and the definition of $\mathcal{F}_\uparrow$, which implies

$$Q(f,0) = 0 \Longleftrightarrow f(0) = 0, \tag{2.14}$$

for all $f \in \mathcal{F}_{\uparrow}$. On the other hand, let us assume, w.l.o.g., that the exceptional case 2 does not take place. Then it is seen from (2.7) and (2.8) that b_p and b_q are both strictly positive, whence $Q\left(f_{\sigma(p)}, b_p\right)$ and $Q\left(f_{\sigma(q)}, b_q\right)$ are so, and one verifies that at least one of the $n-2$ numbers $Q\left(f_{\sigma(i)}, b_i\right)$, $i \in \{1, \dots, n\} \setminus \{p, q\}$, is zero.

Thus, indeed one has $\mathcal{Q}_n(\mathbf{f}, \mathbf{a}) = \mathcal{Q}_n(\mathbf{f}, \mathbf{b})$ in each of the four exceptional cases in Theorem 1.10.

The fact that b_p and b_q are both strictly positive can be expressed as

$$C_{\mathbf{b}} \cap \{p, q\} = \emptyset \tag{2.15}$$

and will be used again below.

It remains to show that $\mathcal{Q}_n(\mathbf{f}, \mathbf{a}) < \mathcal{Q}_n(\mathbf{f}, \mathbf{b})$ if none of the four exceptional cases in Theorem 1.10 takes place. Assume this and introduce one more subset of the set $\{1, \dots, n\}$:

$$C_{\mathbf{f}} := \{i : f_i = \text{const} \neq 0 \text{ on } [0,1]\}. \tag{2.16}$$

At least one of the following three cases must take place:

$$C_{\mathbf{f}(\mathbf{0})} = \emptyset \quad \text{or} \quad C_{\mathbf{f}} = \emptyset \quad \text{or} \quad \left(C_{\mathbf{f}(\mathbf{0})} \neq \emptyset \;\&\; C_{\mathbf{f}} \neq \emptyset\right).$$

Consider first the case $C_{\mathbf{f}(\mathbf{0})} = \emptyset$. Then, in view of (2.14), one has $Q(f_i, a) > 0$ for all $i \in \{1, \dots, n\}$ and all $a \geqslant 0$. Since the exceptional case 3 does not take place, one has $\overline{C_{\mathbf{f}}} \neq \emptyset$; here and in the sequel, $\overline{A} := \{1, \dots, n\} \setminus A$. Hence, there exists a permutation $\sigma \in S_n$ such that $\sigma(p) \in \overline{C_{\mathbf{f}}}$. According to Theorem 1.4 and (2.12), it follows now that $\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right) < 0$, as the exceptional case 2 of Theorem 1.10 does not take place either. Therefore, for such a permutation σ, the summand $\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right) \cdot \left(\prod_{i \in \{1, \dots, n\} \setminus \{p,q\}} Q\left(f_{\sigma(i)}, b_i\right)\right)$ in (2.11) is strictly negative, which implies $\mathcal{Q}_n(\mathbf{f}, \mathbf{a}) < \mathcal{Q}_n(\mathbf{f}, \mathbf{b})$.

Consider next the case $C_{\mathbf{f}} = \emptyset$. According to Theorem 1.4 and (2.12), it follows now that $\Delta(f_i, f_j; a, b, c) < 0$ for all i and j in $\{1, \dots, n\}$, as the exceptional case 2 of Theorem 1.10 does not take place. Since the exceptional case 4 of Theorem 1.10 does not take place either, one has $n_{\mathbf{b}} \leqslant n - n_{\mathbf{f}(\mathbf{0})}$; hence, there exists a permutation $\sigma \in S_n$ such that $\sigma(C_{\mathbf{b}}) \subseteq \overline{C_{\mathbf{f}(\mathbf{0})}}$. Then (2.14) implies that $Q(f_{\sigma(i)}, b_i) > 0$ for all $i \in \{1, \dots, n\}$. Therefore, for such a permutation σ, the summand $\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right) \cdot \left(\prod_{i \in \{1, \dots, n\} \setminus \{p,q\}} Q\left(f_{\sigma(i)}, b_i\right)\right)$ in (2.11) is strictly negative, which implies $\mathcal{Q}_n(\mathbf{f}, \mathbf{a}) < \mathcal{Q}_n(\mathbf{f}, \mathbf{b})$.

Consider finally the case $C_{\mathbf{f}(\mathbf{0})} \neq \emptyset$ & $C_{\mathbf{f}} \neq \emptyset$. According to (2.15), $\{p\} \cap C_{\mathbf{b}} = \emptyset$. Therefore and because $n_{\mathbf{b}} \leqslant n - n_{\mathbf{f}(\mathbf{0})}$, there exists a permutation $\sigma \in S_n$ such that $\sigma(p) \in C_{\mathbf{f}(\mathbf{0})}$ and $\sigma(C_{\mathbf{b}}) \subseteq \overline{C_{\mathbf{f}(\mathbf{0})}}$. But $C_{\mathbf{f}(\mathbf{0})} \subseteq \overline{C_{\mathbf{f}}}$, according to the definitions (2.13) and (2.16). Hence, $\sigma(p) \in \overline{C_{\mathbf{f}}}$, so that $\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right) < 0$, just as in the case $C_{\mathbf{f}(\mathbf{0})} = \emptyset$. On the other hand, $\sigma(C_{\mathbf{b}}) \subseteq \overline{C_{\mathbf{f}(\mathbf{0})}}$ implies that $Q\left(f_{\sigma(i)}, b_i\right) > 0$ for all $i \in \{1, \dots, n\}$, just as in the case $C_{\mathbf{f}} = \emptyset$. Thus, the summand

$\Delta\left(f_{\sigma(p)}, f_{\sigma(q)}; a, b, c\right)\cdot\left(\prod\limits_{i\in\{1,\dots,n\}\setminus\{p,q\}} Q\left(f_{\sigma(i)}, b_i\right)\right)$ in (2.11) is strictly negative, which implies $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) < \mathcal{Q}_n(\mathbf{f},\mathbf{b})$.

Thus, one has $\mathcal{Q}_n(\mathbf{f},\mathbf{a}) < \mathcal{Q}_n(\mathbf{f},\mathbf{b})$ if none of the four exceptional cases takes place. Theorem 1.10 is completely proved.

The proofs of Corollaries 1.11 and 1.12 are quite similar to those of Corollaries 1.8 and 1.9, respectively, and therefore omitted.

Since we have redefined the classes $\mathcal{F}_\downarrow$ and $\mathcal{F}_\uparrow$ (before (1.21) and (1.22)) and also allowed negative values of a in $Q_{\mathsf{dir}}(\mu, a)$, we present here some of the proofs of the extensions of the Stolarsky type inequalities to transforms $Q_{\mathsf{dir}}(\mu, a)$ and $Q_{\mathsf{rev}}(\nu, a)$ of measures, starting with Theorem 1.18, to underscore the changes to be made.

Proof of Theorem 1.18. Take any μ_1 and μ_2 in $\mathcal{M}_{(0,\infty)}$, any $c\in\mathbb{R}$, and any a and b in $[0,\infty)$. Consider

$$\Delta_{\mathsf{dir}}(\mu_1,\mu_2;a,b,c) := \iint\limits_{(0,\infty)^2} \delta_{a,b,c}\, d(\mu_1\otimes\mu_2), \tag{2.17}$$

where $\delta_{a,b,c}$ is given by (2.3), now for all $(s,t)\in(0,\infty)^2$. It is not difficult to see (cf. (2.1) and (2.2)) that

$$\begin{aligned} &Q_{\mathsf{dir}}(\mu_1,c)\,Q_{\mathsf{dir}}(\mu_2,a+b+c) + Q_{\mathsf{dir}}(\mu_2,c)\,Q_{\mathsf{dir}}(\mu_1,a+b+c)\\ &= Q_{\mathsf{dir}}(\mu_1,a+c)\;Q_{\mathsf{dir}}(\mu_2,b+c) + Q_{\mathsf{dir}}(\mu_2,a+c)\;Q_{\mathsf{dir}}(\mu_1,b+c)\\ &+\Delta_{\mathsf{dir}}(\mu_1,\mu_2;a,b,c); \end{aligned} \tag{2.18}$$

note that $\Delta_{\mathsf{dir}}(\mu_1,\mu_2;a,b,c)$ is the difference between the left-hand side and the right-hand side of inequality (1.23) provided that the right-hand side of (1.23) is finite. It is obvious that $\delta_{a,b,c}\geqslant 0$ on $(0,\infty)^2$. Hence, by (2.17), $\Delta_{\mathsf{dir}}(\mu_1,\mu_2;a,b,c)\geqslant 0$, so that the non-strict version of inequality (1.23) follows.

Next, it is easy to check that in each of the four exceptional cases in Theorem 1.18, inequality (1.23) turns into the equality.

Finally, suppose that (1.23) turns into the equality. It remains to show that then at least one of the four exceptional cases takes place. Suppose that the contrary is true: none of the four exceptional cases holds. Since the exceptional case 1 does not takes place, (2.18) implies that $\Delta_{\mathsf{dir}}(\mu_1,\mu_2;a,b,c)=0$. Since $\delta_{a,b,c}\geqslant 0$ on $(0,\infty)^2$, identity (2.17) implies that $\delta_{a,b,c}=0$ $(\mu_1\otimes\mu_2)$-almost everywhere on $(0,\infty)^2$. Since neither of the exceptional cases 2 or 3 takes place, one has $a>0$, $b>0$, and $\mu_1\otimes\mu_2\neq 0$. For any given pair $(s,t)\in(0,\infty)^2$, one has $\delta_{a,b,c}(s,t)=0$ only if $t=s$, in view of (2.3) and since $a>0$ and $b>0$. Therefore, $t=s$ for $(\mu_1\otimes\mu_2)$-almost all pairs $(s,t)\in(0,\infty)^2$. In other words, $\operatorname{supp}(\mu_1\otimes\mu_2)\subseteq\left\{(t,t):t\in(0,\infty)^2\right\}$.

Now, by Lemma 2.2, $\operatorname{supp}(\mu_1 \otimes \mu_2) = \{(t,t)\}$ for some $t \in (0,\infty)$ or, equivalently, $\operatorname{supp}(\mu_1) = \operatorname{supp}(\mu_2) = \{t\}$, so that the exceptional case 4 must take place.

Proof of Theorem 1.20. This proof is similar to that of Theorem 1.18; cf. also the proof of Theorem 1.4. In the present case, one has to deal with $\bar{\delta}_{a,b,c}$ given by (2.4), in place of $\delta_{a,b,c}$. Using (2.5), one has the non-strict version of inequality (1.24).

Also, (2.5) implies that, for any given $(s,t) \in [0,1)^2$ and any positive a and b, one has $\bar{\delta}_{a,b,c}(s,t) = 0$ only if $s = t = 0$. Hence, if $\bar{\delta}_{a,b,c} = 0$ $(\nu_1 \otimes \nu_2)$-almost everywhere on $[0,1)^2$ and a, b, ν_1, and ν_2 are nonzero, then $\operatorname{supp}(\nu_1) = \operatorname{supp}(\nu_2) = \{0\}$. Thus, if inequality (1.24) turns into the equality and none of the first three exceptional cases of Theorem 1.20 takes place, then the exceptional case 4 does.

Finally, it is obvious that in any one of four exceptional cases inequality (1.24) turns into the equality.

Proof of Theorem 1.22. This proof repeats almost word for word the proof of Theorem 1.7, with f_i, $Q(f_i,\cdot)$, and $\mathcal{Q}_n(\mathbf{f},\cdot)$ replaced by μ_i, $Q_{\mathsf{dir}}(\mu_i,\cdot)$, and $\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\cdot)$, respectively.

That here $Q_{\mathsf{dir}}(\mu_i,\cdot)$ may be infinite does not cause serious difficulties. If the exceptional case 3 of Theorem 1.22 does not take place, then $Q_{\mathsf{dir}}(\mu_i,\cdot)$ is strictly positive on $(0,\infty)$ for every i; hence, if the exceptional case 1 of Theorem 1.22 does not take place either while $\mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{a}) = \mathcal{Q}_{n,\mathsf{dir}}(\boldsymbol{\mu},\mathbf{b})$, then $Q_{\mathsf{dir}}(\mu_{\sigma(i)},a_i)$ and $Q_{\mathsf{dir}}(\mu_{\sigma(i)},b_i)$ are finite for all permutations σ and all i.

The only place in the context of the direct Stolarsky inequalities for $Q(f,a)$ where it was essentially used that the function f (and hence the measure μ_f) was defined only to the left of 1 was in Corollary 1.9, corresponding to Corollary 1.24 for $Q_{\mathsf{dir}}(\mu,a)$. Here we reason as follows. Note that

$$h(a) := \begin{cases} \dfrac{u^a-1}{a} & \text{if} \quad a \neq 0, \\ \ln u & \text{if} \quad a = 0 \end{cases}$$

is increasing in $a \in \mathbb{R}$, for each $u \in (0,\infty)$. Indeed, for each $u \in (0,\infty)$,

$$\left(a^2 h'(a)\right)' = \left(u^a\,(a \ln u - 1) + 1\right)' = a\,u^a \ln^2 u \lesseqgtr 0 \quad \text{if} \quad a \lesseqgtr 0,$$

and $a^2 h'(a) \to 0$ as $a \to 0$; hence, $h'(a) > 0$ for all $a \neq 0$ (in fact, for $a = 0$ as well), which implies that h is increasing on $\mathbb{R}$. Thus, by the Fatou lemma, for any probability measure $\mu \in \mathcal{M}_{(0,\infty)}$,

$$\frac{Q_{\mathsf{dir}}(\mu,a)-1}{a} = \int_{(0,\infty)} \frac{u^a-1}{a}\,\mu(du) \longrightarrow \int_{(0,\infty)} (\ln u)\;\mu(du)$$

as $a \to 0$. Therefore,

$$\lim_{a\to 0} Q_{\mathsf{dir}}\left(\mu,\frac{a}{n}\right)^n = \exp\left(a\int_{(0,\infty)} (\ln u)\;\mu(du)\right).$$

This and Corollary 1.23 imply Corollary 1.24.

Corollary 1.27 may be proved similarly. Indeed, there one has

$$\lim_{a\downarrow 0}\frac{Q_{\mathsf{rev}}(\nu,a)-1}{a}=\lim_{a\downarrow 0}\int_{[0,1)}\frac{1-u^a-\mathrm{I}_{\{0\}}\,(u)}{a}\,\nu\,(du)$$
$$=\lim_{a\downarrow 0}\int_{(0,1)}\frac{1-u^a}{a}\,\nu\,(du)=-\int_{(0,1)}(\ln u)\ \nu\,(du)\,.$$

Alternatively, one may use Corollary 1.12 and the Fubini theorem (i.e., integration by parts) to see that for all $g\in\mathcal{F}_{\uparrow}$ with $g(0)=1$ and the corresponding (via (1.22)) measure ν, one has

$$\int_0^1\frac{g(u)-1}{u}\,du=\int_{(0,1)}\frac{\nu\,((0,u])}{u}\,du$$
$$=\int_{(0,1)}\frac{du}{u}\int_{(0,u]}\nu\,(ds)=\int_{(0,1)}\nu\,(ds)\int_{[s,1)}\frac{du}{u}$$
$$=-\int_{(0,1)}(\ln s)\ \nu\,(ds)\,.$$

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