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AN INEQUALITY FOR A POSITIVE REAL FUNCTION

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Abstract. In this paper, using a suitable mapping, we show that the result of H. Alzer can be extended and open problem is proposed.

In [1–6], two inequalities were proved using the mathematical induction and other techniques, which can be expressed as

$$\frac{n}{n+1} \leqslant \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ n \sum_{i=1}^{n+1} i^r \right)^{1/r} \leqslant \frac{\sqrt[n]{n!}}{\sqrt[(n+1)]{(n+1)}},$$

where $r > 0$ and $n \in \mathbb{N}$.

In this paper, using a suitable mapping, we show that the result of Alzer [1] can be extended and open problem is proposed.

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

LEMMA 1. *Let a, b, c and d be real numbers satisfying*

$$1 < a, c, \qquad 0 < b, d < 1, \qquad 0 < ab \leqslant 1,$$

$$1 < \frac{1}{2}(c+d) \quad \text{and} \quad 1 \leqslant a^{c-1}(ab)^d.$$

Then

$$1 < (c-1)a^x + d(ab)^x \ \text{ for all } \ x \in [0, \infty).$$

Proof. Define a function $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = (c-1)a^x + d(ab)^x \ \text{ for all } \ x \in [0, \infty).$$

After some elementary computations, this leads to

$$f'(x) > 0 \ \text{ for all } \ x \in [0, \infty).$$

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Hence we have

$$f(x) > f(0) = c - 1 + d \geqslant 1 \text{ for all } x \in [0, \infty),$$

which completes the proof of the Lemma 1.

From Lemma 1 we have the following lemma.

LEMMA 2. *Let* $\varphi : (0,\infty) \to (0,\infty)$ *be a function such that*

$$\varphi \text{ is strictly increasing on } (0,\infty), \tag{1}$$

$$\varphi'(x) \text{ exists for } x \in (0,\infty), \tag{2}$$

$$\varphi' \text{ is strictly increasing on } (0,\infty), \tag{3}$$

$$\frac{\varphi(x)}{\varphi(x+1)} \leqslant \frac{\varphi(x+1)}{\varphi(x+2)} \tag{4}$$

for all $x \in (0,\infty)$,

$$1 \leqslant \left[\frac{\varphi(u+2)}{\varphi(u+1)}\right]^{\frac{\varphi(v+2)}{\varphi(v+1)}-1} \cdot \left[\frac{\varphi(u)}{\varphi(u+1)} \cdot \frac{\varphi(u+2)}{\varphi(u+1)}\right]^{\frac{\varphi(v)}{\varphi(v+1)}} \tag{5}$$

for all $u,\ v \in (0,\infty)$.
Then

$$\varphi(v+1) < [\varphi(v+2) - \varphi(v+1)] \left\{\frac{\varphi(u+2)}{\varphi(u+1)}\right\}^r + \varphi(v) \left\{\frac{\varphi(u)}{\varphi(u+1)} \cdot \frac{\varphi(u+2)}{\varphi(u+1)}\right\}^r$$

for all $u,\ v \in (0,\infty)$ *and* $r > 0$.

THEOREM 3. *Let* $\varphi : (0,\infty) \to (0,\infty)$ *be a function satisfying* (1), (2), (3), (4) *and*

$$1 \leqslant \left\{\frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)}\right\}^{\frac{\varphi(n+m+2)}{\varphi(n+m+1)}-1} \cdot \left\{\frac{\varphi(n+m+k)}{\varphi(n+m+k+1)} \cdot \frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)}\right\}^{\frac{\varphi(n+m)}{\varphi(n+m+1)}}$$

for all $n \in \mathbb{N},\ m,\ k \in \mathbb{N} \cup \{0\}$ *and* $2 \leqslant 2\varphi(1) \leqslant \varphi(2)$.
Then

$$\frac{\varphi(n+k)}{\varphi(n+m+k)} < \left\{\frac{1}{\varphi(n)} \sum_{i=k+1}^{n+k} [\varphi(i)]^r \Big/ \frac{1}{\varphi(n+m)} \sum_{i=k+1}^{n+m+k} [\varphi(i)]^r\right\}^{\frac{1}{r}} \tag{6}$$

for all $n,\ m \in \mathbb{N},\ k \in \mathbb{N} \cup \{0\}$ *and* $r > 0$.

Proof. From Lemma 2 and hypotheses, we obtain

$$\begin{aligned}\varphi(n+m+1) <& [\varphi(n+m+2) - \varphi(n+m+1)] \left\{\frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)}\right\}^r \\ &+ \varphi(n+m) \left\{\frac{\varphi(n+m+k) \cdot \varphi(n+m+k+2)}{\varphi(n+m+k+1) \cdot \varphi(n+m+k+1)}\right\}^r\end{aligned} \tag{7}$$

for all $n \in \mathbb{N}$, m, $k \in \mathbb{N} \cup \{0\}$ and $r > 0$. We shall deduce inequality (6) via mathematical induction. Let us denote

$$B(k+1, n+k) = \sum_{i=k+1}^{n+k} [\varphi(i)]^r,$$

where $n \in \mathbb{N}, k \in \mathbb{N} \bigcup \{0\}$ and $0 < r$. Then the inequality (6) is equivalent to

$$\frac{\varphi(n)[\varphi(n+k)]^r}{\varphi(n+m)[\varphi(n+m+k)]^r} < \frac{B(k+1, n+k)}{B(k+1, n+m+k)}, \tag{8}$$

where $n, m \in \mathbb{N}$ and $r > 0$. For $m = 1$, (8) is equivalent to

$$\frac{\varphi(n)[\varphi(n+k)]^r}{\varphi(n+1)[\varphi(n+k+1)]^r} < \frac{B(k+1, n+k)}{B(k+1, n+k+1)} \tag{9}$$

which is true for $n = 1$ by hypotheses. We assume that (9) is valid for n. To prove that (9) is valid for $n+1$, it is sufficient to show that

$$\begin{aligned} \varphi(n+1) <& [\varphi(n+2) - \varphi(n+1)] \left\{ \frac{\varphi(n+k+2)}{\varphi(n+k+1)} \right\}^r \\ &+ \varphi(n) \left\{ \frac{\varphi(n+k) \cdot \varphi(n+k+2)}{\varphi(n+k+1) \cdot \varphi(n+k+1)} \right\}^r. \end{aligned} \tag{10}$$

From (7), inequality (10) is valid. Thus (9) holds for all $n \in \mathbb{N}$. Next we assume that

$$\frac{\varphi(n)[\varphi(n+k)]^r}{\varphi(n+m)[\varphi(n+m+k)]^r} < \frac{B(k+1, n+k)}{B(k+1, n+m+k)} \tag{11}$$

for fixed $m \in \mathbb{N}$. To prove inequality substituted $m+1$ for m in inequality (11), it is sufficient to prove the following inequality:

$$\begin{aligned} \varphi(n+m+1) <& [\varphi(n+m+2) - \varphi(n+m+1)] \left\{ \frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)} \right\}^r \\ &+ \varphi(n+m) \left[\frac{\varphi(n+m+k)}{\varphi(n+m+k+1)} \right]^r \left[\frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)} \right]^r, \end{aligned}$$

which is true from (7). Therefore inequality (6) is proved by inductive method.

From Theorem 3, we have the following corollary

COROLLARY 4. *If* n, $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $r > 0$ *and* $a \geqslant 2$, *then*

$$\frac{1}{a^m} < \left\{ \frac{1}{a^n} \sum_{i=k+1}^{n+k} a^{ir} \Big/ \frac{1}{a^{n+m}} \sum_{i=k+1}^{n+m+k} a^{ir} \right\}^{\frac{1}{r}}.$$

Proof. Let $\varphi : (0, \infty) \to (0, \infty)$ be a function such that

$$\varphi(x) = a^x$$

for all $x \in (0, \infty)$. Then, φ is satisfied all conditions of Theorem 3. Thus Corollary 4 follows from Theorem 3.

The following corollary is a generalization of H. Alzer's inequality [1].

COROLLARY 5. *If* $p = 1$ *or* $p \geqslant 2$, *then*

$$\left(\frac{n+k}{n+m+k}\right)^p < \left\{ \frac{1}{n^p} \sum_{i=k+1}^{n+k} i^{pr} \Big/ \frac{1}{(n+m)^p} \sum_{i=k+1}^{n+m+k} i^{pr} \right\}^{\frac{1}{r}},$$

where $n,\ m \in \mathbb{N},\ k \in \mathbb{N} \cup \{0\}$ *and* $r > 0$.

Proof. Let $\varphi : (0,\infty) \to (0,\infty)$ be a function such that

$$\varphi(x) = x^p, \quad p = 1 \text{ or } p \geqslant 2$$

for all $x \in (0,\infty)$. Then, φ is satisfied all conditions of Theorem 3. Thus Corollary 5 follows from Theorem 3.

Open Problem. Does the inequality of Corollary 5 hold for $1 < p < 2$?

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