Mathematical
Inequalities
& Applications
Volume 5, Number 4 (2002), 707–710

ON A PROBLEM BY K. NIKODEM

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(*communicated by Zs. Páles*)

Abstract. Concerning a problem raised by K. Nikodem, we prove the following statement. If G is an Abelian group divisible by 2, H is a Hilbert space and ε is a nonnegative real number and a function $f : G \to H$ satisfies

$$\|f(x-y) - 2f(x) - 2f(y)\| \leqslant \|f(x+y)\| + \varepsilon \quad (x, y \in G),$$

then there exists a function $g : G \to H$ fulfilling

$$g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0 \quad (x, y \in G)$$

and

$$\|f(x) - g(x)\| \leqslant \frac{5}{2}\varepsilon \quad (x \in G).$$

Introduction

During the 38th International Symposium on Functional Equations (Noszvaj, Hungary, 2000), K. Nikodem [10] formulated the following stability problem. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality

$$|f(x) + f(y)| \leqslant |f(x+y)| + \varepsilon \quad (x, y \in \mathbb{R}) \tag{1}$$

with a fixed $\varepsilon \geqslant 0$. Then does there exist a constant c and an additive function $a : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - a(x)| \leqslant c\varepsilon \quad (x, y \in \mathbb{R})?$$

Jacek Tabor and Józef Tabor [13] answered this question in the affirmative, showing that if G is an abelian group, ε is a nonnegative real number and a function $f : G \to \mathbb{R}$ satisfies condition (1) for every $x, y \in G$, then there exists a uniquely determined additive function $a : G \to \mathbb{R}$ such that

$$|f(x) - a(x)| \leqslant 5\varepsilon \quad (x \in G).$$

In their proof they used Gy. Maksa and P. Volkmann's [9] result that the solutions of inequality (1) with $\varepsilon = 0$ are the additive functions (cf. also [14], [8] and [7]).

Mathematics subject classification (2000): Primary: 39B62, 39B82.

Key words and phrases: square-norm equation, functional inequality, stability.

Research supported by the Hungarian Scientific Research Fund (OTKA), Grants Nos. T-030082 and F-038325 and by the Hungarian High Educational Research and Development Fund (FKFP), Grant No. 0215/2001. The author is also grateful for the János Bolyai Research Scholarship of the Hungarian Academy of Sciences..

Paper MIA-05-71

In the present note we consider the so-called square-norm equation

$$g(x+y)+g(x-y)-2g(x)-2g(y)=0. \tag{2}$$

Properties of this functional equation were studied by several authors. Classical results about it can be found e.g. in [5], [6], [1] and [2, Chapter 11], some alternative equations derived from it were examined in [11] and [12]. Here we investigate Nikodem's problem above for this equation and we prove, that if a function $f : G \to H$ defined on an abelian group G divisible by 2, mapping into a Hilbert space H satisfies the inequality

$$\|f(x-y)-2f(x)-2f(y)\| \leqslant \|f(x+y)\|+\varepsilon \quad (x,y \in G) \tag{3}$$

for a nonnegative real number ε, then there exists a uniquely determined function $g : G \to H$ fulfilling (2) for $x,y \in G$ and

$$\|f(x)-g(x)\| \leqslant \frac{5}{2}\varepsilon \quad (x \in G).$$

We note that for the other thirteen inequalities derived from the square-norm equation similarly to (3) the analogous statement, even with $\varepsilon = 0$, is not valid (for some counterexamples see [3, Bemerkung 2]).

Stability theorem

THEOREM. *Let G be an Abelian group divisible by 2, H be a Hilbert space and ε be a nonnegative real number. If a function $f : G \to H$ satisfies*

$$\|f(x-y)-2f(x)-2f(y)\| \leqslant \|f(x+y)\|+\varepsilon \quad (x,y \in G), \tag{4}$$

then there exists a uniquely determined function $g : G \to H$ for which

$$g(x+y)+g(x-y)-2g(x)-2g(y)=0 \quad (x,y \in G) \tag{5}$$

and

$$\|f(x)-g(x)\| \leqslant \frac{5}{2}\varepsilon \quad (x \in G). \tag{6}$$

Proof. Writing $x=y=0$ in (4), we obtain

$$\|f(0)\| \leqslant \frac{\varepsilon}{2}$$

Replacing y by $-x$ in (4), we get

$$\|f(2x)-2f(x)-2f(-x)\| \leqslant \|f(0)\|+\varepsilon \leqslant \frac{3}{2}\varepsilon \quad (x \in G). \tag{7}$$

This inequality gives, with $-x$ instead of x,

$$\|f(-2x)-2f(-x)-2f(x)\| \leqslant \frac{3}{2}\varepsilon \quad (x \in G). \tag{8}$$

The addition of both sides of (7) and (8) implies

$$\|f(2x)-f(-2x)\| \leqslant 3\varepsilon \quad (x \in G),$$

therefore, by the divisibility of G by 2,

$$\|f(x)-f(-x)\| \leqslant 3\varepsilon \quad (x \in G).$$

This property, together with (7), yields

$$\|f(2x)-4f(x)\| \leqslant \frac{15}{2}\varepsilon \quad (x \in G).$$

Using this property and the triangle inequality, we obtain

$$\begin{aligned}
\left\|f(x)-\frac{1}{4^n}f(2^n x)\right\| &\leqslant \left\|f(x)-\frac{1}{4}f(2x)\right\| + \left\|\frac{1}{4}f(2x)-\frac{1}{4^2}f(2^2 x)\right\| \\
&\quad + \cdots + \left\|\frac{1}{4^{n-1}}f(2^{n-1}x)-\frac{1}{4^n}f(2^n x)\right\| \\
&\leqslant \left(\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^n}\right)\frac{15}{2}\varepsilon \\
&\leqslant \sum_{k=1}^{\infty}\frac{1}{4^k}\frac{15}{2}\varepsilon = \frac{5}{2}\varepsilon
\end{aligned} \tag{9}$$

for each $x \in G$ and for any positive integer n. Now, we start a so-called Hyers process (cf. [4]). We define, for positive integers, the functions $g_n : G \to H$ by

$$g_n(x) = \frac{1}{4^n}f(2^n x) \quad (x \in G).$$

Inequality (9) implies

$$\|g_n(x)-g_k(x)\| \leqslant \frac{1}{4^n}\frac{5}{2}\varepsilon$$

for $x \in G$ and for positive integers n,k such that $n \leqslant k$. Therefore, $g_n(x)$ is a Cauchy sequence for each fixed $x \in G$. Thus, by the completeness of H, we can define the function $g : G \to H$ by

$$g(x) = \lim_{n\to\infty} g_n(x) \quad (x \in G).$$

Inequality (9) yields that this function satisfies (6). Writing $2^n x$ instead of x and $2^n y$ instead of y in (4), we get

$$\|f(2^n(x-y))-2f(2^n x)-2f(2^n y)\| \leqslant \|f(2^n(x+y))\| + \varepsilon \quad (x,y \in G)$$

for any positive integer n. Dividing this inequality by 4^n and letting n approach infinity, we obtain

$$\|g(x-y)-2g(x)-2g(y)\| \leqslant \|g(x+y)\| \quad (x,y \in G).$$

It was proved in [3] that this inequality is equivalent to

$$g(x+y)+g(x-y)-2g(x)-2g(y) = 0 \quad (x,y \in G),$$

which implies the existence part of our statement. The uniqueness of g is a well-known and simple consequence of properties (5) and (6).

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(Received April 11, 2001)

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