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EQUIVALENCE OF $\ell^{\{p_n\}}$ NORMS AND SHIFT OPERATORS

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(*communicated by L. Pick*)

Abstract. Given bounded mappings $p,q:\mathbb{Z}\to[1,\infty)$ (shortly $p=\{p_n\}$, $q=\{q_n\}$) we can consider Banach function spaces $\ell^{\{p_n\}}$ and $\ell^{\{q_n\}}$ with variable exponents. The necessary and sufficient condition to the p, q for the equivalence of norms in Banach spaces $\ell^{\{p_n\}}$ and $\ell^{\{q_n\}}$ is given. Moreover, considering shift operators S_k given by $(S_k a)_n = a_{n-k}$, $n\in\mathbb{Z}$, we prove that the norms $\|S_k\|_{\ell^{\{p_n\}}\to\ell^{\{p_n\}}}$, $k\in\mathbb{Z}$ are uniformly bounded with respect to k if and only if the norm in $\ell^{\{p_n\}}$ is equivalent to a norm of a classical ℓ^r with some constant exponent r.

1. Introduction

The generalized Lebesgue space $\ell^{\{p_n\}}$, $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have attracted more and more interest in recent years. We refer to [5] for the establishment of the fundamental properties of these spaces, to [1] for some properties of the norm on $L^{p(x)}$, to [3] and [8] for the density of smooth functions in $W^{1,p(x)}$ and to [4] for inequalities of Sobolev type. Further motivation for the study of these spaces is provided in [6, 7] by means of mathematical models of electrorheological fluids which involve nonlinear systems of partial differential equations with coefficients of variable rate of growth.

A crucial difference between $L^{p(x)}$ and the classical Lebesgue spaces is that $L^{p(x)}$ is not, in general, invariant under translation (see [5], Ex. 2.9). Moreover, (see [5], Theorem 2.10) there is a function $f\in L^{p(x)}$ which is not $p(x)-$mean continuous provided p is continuous and non-constant.

Consider a discrete analogue $\ell^{\{p_n\}}$ of $L^{p(x)}$. In [2] it is proved that under certain assumptions on $\{p_n\}$ the norms of shift operators given by

$$S_k a = \{(S_k a)_n\}, (S_k a)_n = a_{n-k}, a=\{a_n\},$$

are uniformly bounded on $\ell^{\{p_n\}}$. Recall that $\{p_n\}$ need not be constant. As an immediate consequence it is shown that the norms of averaging operators given by

$$(T_k a)_n = \frac{1}{k}(a_n + a_{n+1} + \ldots + a_{n+k-1}), a=\{a_n\}\in\ell^{\{p_n\}},$$

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are uniformly bounded on $\ell^{\{p_n\}}$, too.

In this paper we prove the following assertion: The norms of S_k are uniformly bounded on $\ell^{\{p_n\}}$ for a bounded $\{p_n\}$ if and only if there exists r, $1 \leqslant r < \infty$, such that the norms in $\ell^{\{p_n\}}$ and the classical space ℓ^r are equivalent.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers and let $\mathcal{M}$ denote the set of all mappings $a : \mathbb{Z} \to \mathbb{R}$. We will also denote elements of $\mathcal{M}$ by $a = \{a_n\}$. Let

$$\mathcal{E} = \{p \in \mathcal{M}; 1 \leqslant p_n \text{ for all } n \in \mathbb{Z}\}.$$

Denote by $p^* = \sup\{p_n; n \in \mathbb{Z}\}$ for any $p \in \mathcal{E}$ and

$$\mathcal{B} = \{p \in \mathcal{E}; p^* < \infty\}.$$

Let the symbol χ^k stand for the characteristic function of the set $\{n \in \mathbb{Z}; -k \leqslant n \leqslant k\}$. Let $a^k, a \in \mathcal{M}$. Say that $a \geqslant 0$ if $a_n \geqslant 0$ for each $n \in \mathbb{Z}$ and $a^k \nearrow a$ if $(a^k)_n \nearrow a_n$ for each $n \in \mathbb{Z}$.

We recall the definition of a Banach function space.

DEFINITION 2.1. A linear space X, $X \subset \mathcal{M}$, is called a Banach function space if there exists a functional $\|.\|_X : \mathcal{M} \to [0,\infty]$ with the norm property satisfying:

(i) $a \in X$ if and only if $\|a\|_X < \infty$;

(ii) $\|a\|_X = \|\, |a|\, \|_X$ for all $a \in \mathcal{M}$;

(iii) if $0 \leqslant a^k \nearrow a$ then $\|a^k\|_X \nearrow \|a\|_X$;

(iv) $\|a\chi^k\|_X < \infty$ for any $k \in \mathbb{N}$;

(v) for any $k \in \mathbb{N}$ there is a positive constant c_k such that
$$\sum_{|n| \leqslant k} |a_n| \leqslant c_k \|a\|_X \text{ for all } a \in X.$$

DEFINITION 2.2. Let $p \in \mathcal{E}$. Denote for $a \in \mathcal{M}$ the Luxemburg norm by

$$\|a\|_{\{p_n\}} = \inf\{\lambda > 0; \sum_{n \in \mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \leqslant 1\}.$$

Define the space $\ell^{\{p_n\}}$ by

$$\ell^{\{p_n\}} = \{a; \|a\|_{\{p_n\}} < \infty\}.$$

Remark that we will use the usual symbols ℓ^r and $\|a\|_r$ in the case of constant mapping $r \in \mathcal{E}$. Recall that $\|a\|_r = (\sum_{n \in \mathbb{Z}} |a_n|^r)^{1/r}$ in this case.

In [2], the following lemma was proved.

LEMMA 2.3. *The space* $\ell^{\{p_n\}}$ *is a Banach function space.*

DEFINITION 2.4. Let p, $q \in \mathcal{E}$ and let T be a linear mapping on $\mathcal{M}$. We will say that T is bounded from $\ell^{\{p_n\}}$ into $\ell^{\{q_n\}}$ if

$$\|T\|_{\{p_n \to q_n\}} := \sup\{\|Ta\|_{\{q_n\}}; \|a\|_{\{p_n\}} \leqslant 1\} < \infty.$$

It is not difficult to prove the next lemma.

LEMMA 2.5. *Let $p \in \mathcal{B}$. Then*

$$\ell^{\{p_n\}} = \{a; \sum_{n \in \mathbb{Z}} |a_n|^{p_n} < \infty\}.$$

Let us prove some lemmas on linear operators.

LEMMA 2.6. *Let p, $q \in \mathcal{B}$ and let T be a linear mapping which maps $\mathcal{M}$ into itself. Let c be a positive constant such that*

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1 \Longrightarrow \sum_{n \in \mathbb{Z}} |(Ta)_n|^{q_n} \leqslant c.$$

Then

$$\|T\|_{\{p_n \to q_n\}} \leqslant \max(1, c).$$

Proof. Assume $\|a\|_{\{p_n\}} \leqslant 1$. Then it is easy to verify that $\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1$ and according to the assumptions we have

$$\sum_{n \in \mathbb{Z}} |(Ta)_n|^{q_n} \leqslant \max(1, c).$$

Then

$$\sum_{n \in \mathbb{Z}} \left| \left(T\left(\frac{a}{\max(1, c)} \right) \right)_n \right|^{q_n} \leqslant \sum_{n \in \mathbb{Z}} \left| \frac{(Ta)_n}{\max(1, c)} \right|^{q_n}$$
$$\leqslant \frac{1}{\max(1, c)} \sum_{n \in \mathbb{Z}} |(Ta)_n|^{q_n} \leqslant \frac{c}{\max(1, c)} \leqslant 1.$$

This gives $\|T\|_{\{p_n \to q_n\}} \leqslant \max(1, c)$ and the lemma follows.

LEMMA 2.7. *Let p, $q \in \mathcal{B}$ and let T be a linear mapping which maps $\mathcal{M}$ into itself. Let $c > 1$ be a positive number such that $\|T\|_{\{p_n \to q_n\}} \geqslant c$. Then there exists an $a \in \mathcal{M}$ such that*

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1 \text{ and } \sum_{n \in \mathbb{Z}} |(Ta)_n|^{q_n} \geqslant c.$$

Proof. Since $\|T\|_{\{p_n \to q_n\}} \geqslant c$ we have an $a \in \mathcal{M}$ such that for any $\lambda < c$ it is

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1 \text{ and } \sum_{n \in \mathbb{Z}} \left| \frac{(Ta)_n}{\lambda} \right|^{q_n} > 1.$$

Considering only $1 \leqslant \lambda < c$, we can write

$$1 < \sum_{n\in\mathbb{Z}} \left|\frac{(Ta)_n}{\lambda}\right|^{q_n} \leqslant \frac{1}{\lambda} \sum_{n\in\mathbb{Z}} |(Ta)_n|^{q_n},$$

from which it follows that

$$\sum_{n\in\mathbb{Z}} |(Ta)_n|^{q_n} \geqslant c,$$

and the proof is finished.

LEMMA 2.8. *Let p, $q \in \mathcal{B}$ and let T be a linear mapping from $\mathcal{M}$ into itself. Assume that there exists a number $c > 1$ and $a \in \mathcal{M}$ such that*

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \leqslant 1 \text{ and } \sum_{n\in\mathbb{Z}} |(Ta)_n|^{q_n} \geqslant c.$$

Then $\|T\|_{\{p_n\to q_n\}} \geqslant c^{1/q^*}$.

Proof. Clearly, $\|a\|_{\{p_n\}} \leqslant 1$. Further

$$\|T\|_{\{p_n\to q_n\}} \geqslant \|Ta\|_{\{q_n\}} = \inf\{\lambda > 0; \sum_{n\in\mathbb{Z}} \left|\frac{(Ta)_n}{\lambda}\right|^{q_n} \leqslant 1\}.$$

Take $\lambda < c^{1/q^*}$. Then

$$\sum_{n\in\mathbb{Z}} \left|\frac{(Ta)_n}{\lambda}\right|^{q_n} > \sum_{n\in\mathbb{Z}} \frac{|(Ta)_n|^{q_n}}{c^{q_n/q^*}} \geqslant \sum_{n\in\mathbb{Z}} \frac{|(Ta)_n|^{q_n}}{c} \geqslant 1.$$

Consequently, $\|T\|_{\{p_n\to q_n\}} \geqslant c^{1/q^*}$.

3. Key assertions

Given $\varepsilon \in \mathcal{M}$ we adopt the notation $\mathbb{P}(\varepsilon) = \{n \in \mathbb{Z} : \varepsilon_n > 0\}$.

DEFINITION 3.1. Let $\varepsilon \in \mathcal{M}$. We say that $\varepsilon \in \mathcal{P}$ if there exists a real number $c > 0$ such that

$$\sum_{n\in\mathbb{P}(\varepsilon)} \varepsilon_n c^{1/\varepsilon_n} < \infty. \tag{1}$$

Set

$$\nu(\varepsilon) = \inf\Big\{\frac{1}{c}\big(1 + \sum_{n\in\mathbb{P}(\varepsilon)} \varepsilon_n c^{1/\varepsilon_n}\big); c > 0\Big\}.$$

REMARK 3.2. It is easy to see that $\varepsilon \in \mathcal{P}$ if and only if $\nu(\varepsilon) < \infty$ and $|\varepsilon| \in \mathcal{P}$ if and only if $\varepsilon \in \mathcal{P}$ and $-\varepsilon \in \mathcal{P}$.

LEMMA 3.3. *Let $K > 0$ and $\alpha \in \mathcal{M}$ be such that $0 < \alpha_n \leqslant K$ for $n \in \mathbb{Z}$. Let $\varepsilon \in \mathcal{P}$. Then $\alpha\varepsilon \in \mathcal{P}$.*

Proof. Let c satisfy (1). Without loss of generality we can assume $c \leqslant 1$. Set $d = c^K$. Let us estimate

$$\sum_{n\in\mathbb{P}(\alpha\varepsilon)} \alpha_n \varepsilon_n d^{1/(\alpha_n\varepsilon_n)}.$$

Since $0 < \alpha_n \leqslant K$, we have $d = c^K \leqslant c^{\alpha_n}$ and using the simple fact that $\mathbb{P}(\alpha\varepsilon) = \mathbb{P}(\varepsilon)$ we obtain

$$\sum_{n\in\mathbb{P}(\alpha\varepsilon)} \alpha_n\varepsilon_n d^{1/(\alpha_n\varepsilon_n)} = \sum_{n\in\mathbb{P}(\varepsilon)} \alpha_n\varepsilon_n \left(c^K\right)^{1/(\alpha_n\varepsilon_n)}$$
$$\leqslant K \sum_{n\in\mathbb{P}(\varepsilon)} \varepsilon_n \left(c^{\alpha_n}\right)^{1/(\alpha_n\varepsilon_n)} = K \sum_{n\in\mathbb{P}(\varepsilon)} \varepsilon_n c^{1/\varepsilon_n} < \infty,$$

which finishes the proof.

LEMMA 3.4. *Let $\varepsilon \in \mathcal{P}$, $b \in \mathcal{M}$ satisfy $\varepsilon < 1$, $0 \leqslant b$. Then*

$$\sum_{n\in\mathbb{Z}} b_n \leqslant 1 \Rightarrow \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} \leqslant 1 + e^{1/e}\nu(\varepsilon).$$

Proof. Let c satisfy (1) and assume $\sum_{n\in\mathbb{Z}} b_n \leqslant 1$. Set

$$\mathbb{Z}_1 = \{n \in \mathbb{Z}; \varepsilon_n \leqslant 0\},$$
$$\mathbb{Z}_2 = \{n \in \mathbb{P}(\varepsilon); b_n > \varepsilon_n c^{1/\varepsilon_n}\},$$
$$\mathbb{Z}_3 = \{n \in \mathbb{P}(\varepsilon); b_n \leqslant \varepsilon_n c^{1/\varepsilon_n}\}.$$

Since $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3$ are pairwise disjoint and $\mathbb{Z}_1 \cup \mathbb{Z}_2 \cup \mathbb{Z}_3 = \mathbb{Z}$, we can write

$$\sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \sum_{n\in\mathbb{Z}_1} b_n^{1-\varepsilon_n} + \sum_{n\in\mathbb{Z}_2} b_n^{1-\varepsilon_n} + \sum_{n\in\mathbb{Z}_3} b_n^{1-\varepsilon_n} = I_1 + I_2 + I_3. \tag{2}$$

Note that, according to the assumptions, $b_n \leqslant 1$ for all $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}_1$. Then $1 - \varepsilon_n \geqslant 1$ and $b_n^{1-\varepsilon_n} \leqslant b_n$. Thus

$$I_1 \leqslant \sum_{n\in\mathbb{Z}_1} b_n \leqslant 1. \tag{3}$$

Let $n \in \mathbb{Z}_2$. Then $b_n > \varepsilon_n c^{1/\varepsilon_n}$ and, consequently, $b_n{}^{-\varepsilon_n} < \left(\varepsilon_n c^{1/\varepsilon_n}\right)^{-\varepsilon_n}$. Since $1 > \varepsilon_n > 0$, then $\varepsilon_n^{-\varepsilon_n} \leqslant e^{1/e}$ and $b_n^{1-\varepsilon_n} \leqslant \frac{1}{c} e^{1/e} b_n$. Thus

$$I_2 \leqslant \frac{1}{c} e^{1/e} \sum_{n\in\mathbb{Z}_2} b_n \leqslant \frac{1}{c} e^{1/e}. \tag{4}$$

Let $n \in \mathbb{Z}_3$. Then $0 \leqslant b_n \leqslant \varepsilon_n c^{1/\varepsilon_n}$, which gives $b_n^{1-\varepsilon_n} \leqslant \varepsilon_n c^{1/\varepsilon_n} \left(\varepsilon_n c^{1/\varepsilon_n}\right)^{-\varepsilon_n} \leqslant \frac{1}{c} e^{1/e} \varepsilon_n c^{1/\varepsilon_n}$ and

$$I_3 \leqslant \frac{1}{c} e^{1/e} \sum_{n\in\mathbb{Z}_3} \varepsilon_n c^{1/\varepsilon_n}.$$

This yields with (2), (3) and (4)

$$\sum_{n\in\mathbb{Z}} {b_n}^{1-\varepsilon_n} \leqslant 1 + \frac{1}{c} e^{1/e} (1 + \sum_{n\in\mathbb{Z}_3} \varepsilon_n c^{1/\varepsilon_n}).$$

Consequently,

$$\sum_{n\in\mathbb{Z}} {b_n}^{1-\varepsilon_n} \leqslant 1 + e^{1/e} \nu(\varepsilon).$$

LEMMA 3.5. *Let $\varepsilon \notin \mathcal{P}$, $\varepsilon < 1$. Then there exists $b \in \mathcal{M}$, $0 \leqslant b$, such that*

$$\sum_{n\in\mathbb{Z}} b_n \leqslant 1 \ \text{ and } \ \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \infty.$$

Proof. Assume first

$$0 < \varepsilon_n < 1 \ \text{ for all } \ n \in \mathbb{Z}. \tag{5}$$

Set $N_0 = -1$. We will construct sequences $\{N_k\}_{k\in\mathbb{N}}$, $N_k \in \mathbb{N}$, and $\{c_k\}_{k\in\mathbb{N}}$, $c_k \in (0,\infty)$, satisfying for any $k \in \mathbb{N}$

$$0 < c_k \leqslant \frac{1}{2^k} \quad \text{and} \quad \sum_{N_{k-1}\leqslant|n|\leqslant N_k} \varepsilon_n {c_k}^{1/\varepsilon_n} = 1. \tag{6}$$

According to the assumption on $\{\varepsilon_n\}$, we have

$$\sum_{n\in\mathbb{Z}} \varepsilon_n c^{1/\varepsilon_n} = \infty \qquad \text{for all } c > 0. \tag{7}$$

Thus, we can find $N_1 \in \mathbb{N}$ such that $\sum\limits_{|n|\leqslant N_1} \varepsilon_n (\frac{1}{2})^{1/\varepsilon_n} \geqslant 1$. Then there exists a number $0 < c_1 \leqslant \frac{1}{2}$ such that

$$\sum_{|n|\leqslant N_1} \varepsilon_n {c_1}^{1/\varepsilon_n} = \sum_{N_0<|n|\leqslant N_1} \varepsilon_n {c_1}^{1/\varepsilon_n} = 1.$$

Assume that we have constructed positive integers $N_1 < N_2 < \cdots < N_k$ and real numbers $c_1, c_2, \ldots, c_k$ such that

$$0 < c_r \leqslant \frac{1}{2^r} \ \text{ and } \sum_{N_{r-1}<|n|\leqslant N_r} \varepsilon_n {c_r}^{1/\varepsilon_n} = 1.$$

for $r = 1, 2, \ldots, k$. According to (7), we can find N_{k+1} such that

$$\sum_{N_k<|n|\leqslant N_{k+1}} \varepsilon_n \Big(\frac{1}{2^{k+1}}\Big)^{1/\varepsilon_n} \geqslant 1.$$

Then we can take c_{k+1} such that

$$0 < c_{k+1} \leqslant \frac{1}{2^{k+1}} \ \text{ and } \sum_{N_k<|n|\leqslant N_{k+1}} \varepsilon_n (c_{k+1})^{1/\varepsilon_n} = 1$$

which proves (6).

Define $b \in \mathcal{M}$ by

$$b_n = (\varepsilon_n {c_k}^{1/\varepsilon_n})^{1/(1-\varepsilon_n)} \ \text{ if } \ N_{k-1} < |n| \leqslant N_k.$$

Using (6) we have

$$\sum_{n\in\mathbb{Z}} {b_n}^{1-\varepsilon_n} = \sum_{k=1}^{\infty} \sum_{N_{k-1}<|n|\leqslant N_k} \varepsilon_n {c_k}^{1/\varepsilon_n} = \sum_{k=1}^{\infty} 1 = \infty.$$

Let us estimate $\sum\limits_{n\in\mathbb{Z}} b_n$. Clearly, by (5) it is $0 < \varepsilon_n {c_k}^{1/\varepsilon_n} \leqslant 1$ for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Since $1 - \varepsilon_n^2 < 1$ we obtain

$$b_n = \left(\varepsilon_n {c_k}^{1/\varepsilon_n}\right)^{1/(1-\varepsilon_n)} \leqslant \left(\varepsilon_n {c_k}^{1/\varepsilon_n}\right)^{1+\varepsilon_n}$$

which implies with (6)

$$\begin{aligned}\sum_{n\in\mathbb{Z}} b_n &\leqslant \sum_{k=1}^{\infty} \sum_{N_{k-1}<|n|\leqslant N_k} \left(\varepsilon_n {c_k}^{1/\varepsilon_n}\right)^{1+\varepsilon_n} \\ &\leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{N_{k-1}<|n|\leqslant N_k} \left(\varepsilon_n {c_k}^{1/\varepsilon_n}\right) {\varepsilon_n}^{\varepsilon_n} c_k \\ &\leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{N_{k-1}<|n|\leqslant N_k} \varepsilon_n {c_k}^{1/\varepsilon_n} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.\end{aligned}$$

Assume that (5) is not satisfied. Since $\varepsilon \notin \mathcal{P}$, the set $\mathbb{P}(\varepsilon)$ must be infinite. Then there exists a one-to-one mapping $\pi : \mathbb{P}(\varepsilon) \to \mathbb{Z}$. Set $\delta_n = \varepsilon_{\pi^{-1}(n)}$, $n \in \mathbb{Z}$. Then $\delta \notin \mathcal{P}$ and satisfies (5). Thus, there exists $a \in \mathcal{M}$, $a > 0$, such that

$$\sum_{n\in\mathbb{Z}} a_n \leqslant 1 \ \text{ and } \sum_{n\in\mathbb{Z}} (a_n)^{1-\delta_n} = \infty.$$

Define

$$b_n = \begin{cases} a_{\pi(n)} & \text{if } n \in \mathbb{P}(\varepsilon), \\ 0 & \text{if } n \notin \mathbb{P}(\varepsilon). \end{cases}$$

Now, it is easy to see that

$$\sum_{n\in\mathbb{Z}} b_n = \sum_{n\in\mathbb{P}(\varepsilon)} a_{\pi(n)} = \sum_{k\in\mathbb{Z}} a_k \leqslant 1$$

and

$$\sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \sum_{n\in\mathbb{P}(\varepsilon)} a_{\pi(n)}^{1-\delta_{\pi(n)}} = \sum_{k\in\mathbb{Z}} a_k^{1-\delta_k} = \infty.$$

Thus, b satisfies the desired properties, which completes the proof.

4. Equivalence of $\ell^{\{p_n\}}$ norms

Let us denote by Id the identity operator on $\mathcal{M}$ and by $\ell^{\{p_n\}} \hookrightarrow \ell^{\{q_n\}}$ the imbedding of $\ell^{\{p_n\}}$ into $\ell^{\{q_n\}}$. Recall that $\ell^{\{p_n\}} \hookrightarrow \ell^{\{q_n\}}$ if $\|\mathrm{Id}\|_{\{p_n \to q_n\}} < \infty$.

THEOREM 4.1. *Let $p,\ q \in \mathcal{B},\ p-q \in \mathcal{P}$. Then*

$$\ell^{\{p_n\}} \hookrightarrow \ell^{\{q_n\}}.$$

Proof. Let $\sum\limits_{n\in\mathbb{Z}} |a_n|^{p_n} \leqslant 1$. For each $n \in \mathbb{Z}$ set $b_n = |a_n|^{p_n}$, $\varepsilon_n = \frac{p_n - q_n}{p_n}$. Then $\sum\limits_{n\in\mathbb{Z}} b_n \leqslant 1$ and, according to Lemma 3.3, $\{\varepsilon_n\} \in \mathcal{P}$. By Lemma 3.4, we have

$$\sum_{n\in\mathbb{Z}} |a_n|^{q_n} = \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} \leqslant 1 + e^{-1/e}\nu(\varepsilon)$$

and consequently, using Lemma 2.6

$$\|\mathrm{Id}\|_{\ell^{\{p_n\}} \hookrightarrow \ell^{\{q_n\}}} \leqslant 1 + e^{-1/e}\nu(\varepsilon) < \infty$$

which proves the lemma.

THEOREM 4.2. *Let $p,\ q \in \mathcal{B}$ and let*

$$\ell^{\{p_n\}} \hookrightarrow \ell^{\{q_n\}}.$$

Then $p - q \in \mathcal{P}$.

Proof. Assume $p - q \notin \mathcal{P}$. Set $\varepsilon_n = \frac{p_n - q_n}{p_n}$ for $n \in \mathbb{Z}$. According to Lemma 3.3, $\{\varepsilon_n\} \notin \mathcal{P}$. Moreover, $\varepsilon_n < 1$ for $n \in \mathbb{Z}$. Lemma 3.5 gives the existence of $b \in \mathcal{M}$, $0 \leqslant b$, such that

$$\sum_{n\in\mathbb{Z}} b_n \leqslant 1 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \infty.$$

Set $a_n = b_n^{1/p_n}$, $n \in \mathbb{Z}$. Then

$$\sum_{n\in\mathbb{Z}} a_n^{p_n} = \sum_{n\in\mathbb{Z}} b_n \leqslant 1$$

and

$$\sum_{n\in\mathbb{Z}} a_n^{q_n} = \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \infty$$

which yields with Lemma 2.5 $\ell^{\{p_n\}} \not\hookrightarrow \ell^{\{q_n\}}$, and the proof is complete.

As an easy consequence of Remark 3.2 we have the following theorem.

THEOREM 4.3. *Let $p,\ q \in \mathcal{B}$. Then the norms in spaces ℓ^{p_n} and ℓ^{q_n} are equivalent if and only if $|p - q| \in \mathcal{P}$.*

5. Shift operators

In this section we show that the uniform boundedness of shift operators is equivalent to the existence of a real $r \geqslant 1$ such that the norms in the spaces $\ell^{\{p_n\}}$ and ℓ^r are equivalent.

Let $p \in \mathcal{B}$ be fixed in this section.

DEFINITION 5.1. For each $k \in \mathbb{Z}$ define a shift operator S_k from $\mathcal{M}$ into itself by

$$(S_k a)_n = a_{n-k},\ a \in \mathcal{M},\ n \in \mathbb{Z}.$$

Set

$$D = \sup\{\|S_k\|_{\{p_n \to p_n\}}; k \in \mathbb{Z}\}.$$

LEMMA 5.2. *Let $r \in [1,\infty)$ be such that the norms in the spaces $\ell^{\{p_n\}}$ and ℓ^r are equivalent. Then $D < \infty$.*

Proof. Let c satisfy $c^{-1}\|a\|_{\{p_n\}} \leqslant \|a\|_r \leqslant c\|a\|_{\{p_n\}}$ for all $a \in \mathcal{M}$. Let $k \in \mathbb{Z}$ be arbitrary. Since $\|S_k\|_{\{r\to r\}} = 1$, we immediately obtain

$$\|S_k\|_{\{p_n\to p_n\}} \leqslant \|\mathrm{Id}\|_{\{p_n\to r\}}\ \|S_k\|_{\{r\to r\}}\ \|\mathrm{Id}\|_{\{r\to p_n\}} \leqslant c^2.$$

Thus, $D \leqslant c^2$, which finishes the proof.

Next, we will prove the converse implication.

LEMMA 5.3. *Assume that*

$$\lim_{n\to\infty} |p_{n+1} - p_n| \neq 0.$$

Then either S_1 or S_{-1} is unbounded on $\ell^{\{p_n\}}$.

Proof. According to the assumptions, there exists $\alpha > 0$ such that $|p_{n+1} - p_n| \geqslant \alpha$ for infinitely many positive integers $n_1 < n_2 < \dots$. Set

$$\mathbb{P} = \{n \in \mathbb{N}; p_n - p_{n+1} \geqslant \alpha\} \text{ and } \mathbb{Z}_- = \{n \in \mathbb{N}; p_n - p_{n+1} \leqslant -\alpha\}.$$

Then either $\mathbb{P}$ or $\mathbb{Z}_-$ is infinite.

Assume first that $\mathbb{P}$ is infinite. Choose $\gamma \in \mathbb{R}$ such that

$$\gamma\left(1 - \frac{\alpha}{p^*}\right) \leqslant 1 < \gamma. \tag{8}$$

Let $\pi : \mathbb{P} \to \mathbb{N}$ be one-to-one mapping and let $a \in \mathcal{M}$ be given by

$$a_n = \begin{cases} (\pi(n))^{-\frac{\gamma}{p_n}}, & n \in \mathbb{P}, \\ 0, & n \notin \mathbb{P}. \end{cases}$$

By (8) we have

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{n\in\mathbb{P}} (\pi(n))^{-\gamma} = \sum_{k=1}^{\infty} k^{-\gamma} < \infty$$

and

$$\sum_{n\in\mathbb{Z}}((S_1a)_n)^{p_n} = \sum_{n\in\mathbb{Z}}(a_{n-1})^{p_n} = \sum_{n\in\mathbb{Z}}(a_n)^{p_{n+1}} = \sum_{n\in\mathbb{P}}(\pi(n))^{-\gamma p_{n+1}/p_n}$$
$$\geqslant \sum_{n\in\mathbb{P}}((\pi(n))^{-\gamma(p_n-\alpha)/p_n}) \geqslant \sum_{n\in\mathbb{P}}(\pi(n))^{-\gamma(1-\alpha/p^*)} \geqslant \sum_{k=1}^{\infty} k^{-\gamma(1-\alpha/p^*)} = \infty.$$

Thus, S_1 is unbounded.

If $\mathbb{Z}_-$ is infinite then analogously S_{-1} is unbounded, which proves the lemma.

As an easy consequence we obtain the following lemma.

LEMMA 5.4. *Let* $D < \infty$. *Then*

$$\lim_{n\to\infty}|p_{n+1}-p_n| = \lim_{n\to-\infty}|p_{n+1}-p_n| = 0.$$

LEMMA 5.5. *Let* $\lim_{n\to\infty}|p_{n+1}-p_n| = 0$. *Denote* $\underline{p} = \liminf_{n\to\infty} p_n$, $\overline{p} = \limsup_{n\to\infty} p_n$. *Let* $\underline{p} < \overline{p}$. *Then for any* $c > 1$ *there exists* $m \in \mathbb{Z}$ *such that* $\|S_m\|_{\{p_n\to p_n\}} \geqslant c$.

Proof. Let $c > 1$. Assume $\underline{p} < \overline{p}$. Let $\delta = \frac{1}{3}(\overline{p}-\underline{p})$ and $\{b_n\}_{n\in\mathbb{N}}$, $b_n > 0$, be a sequence satisfying

$$\sum_{n=1}^{\infty}(b_n)^{\overline{p}-\delta} \leqslant 1 \text{ and } \sum_{n=1}^{\infty}(b_n)^{\underline{p}+\delta} = \infty. \tag{9}$$

Then there exists $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{N}(b_n)^{\underline{p}+\delta} \geqslant c^{p^*}. \tag{10}$$

According to the assumption $\lim_{n\to\infty}|p_{n+1}-p_n| = 0$, there are $n_1, n_2 \in \mathbb{N}$, $n_2 > n_1+N$, such that for any $1 \leqslant s \leqslant N$ it is $p_{(n_1+s)} > \overline{p}-\delta$ and $p_{(n_2+s)} < \underline{p}+\delta$. Let $a \in \mathcal{M}$ be given by

$$a_n = \begin{cases} b_{n-n_1} & \text{if } n \in \{n_1+s; 1\leqslant s\leqslant N\}, \\ 0 & \text{if } n \notin \{n_1+s; 1\leqslant s\leqslant N\}. \end{cases}$$

Set $m = n_2 - n_1$. By (9), we have $b_n \leqslant 1$ and consequently,

$$\sum_{n\in\mathbb{Z}}(a_n)^{p_n} = \sum_{s=1}^{N}(b_s)^{p_{(n_1+s)}} \leqslant \sum_{s=1}^{N}(b_s)^{\overline{p}-\delta} \leqslant 1.$$

Using (10), we obtain

$$\sum_{n\in\mathbb{Z}}((S_ma)_n)^{p_n} = \sum_{n\in\mathbb{Z}}(a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}}(a_n)^{p_{n+m}} =$$
$$\sum_{s=1}^{N}(b_s)^{p_{(n_1+s+m)}} = \sum_{s=1}^{N}(b_s)^{p_{(n_2+s)}} \geqslant \sum_{s=1}^{N}(b_s)^{\underline{p}+\delta} \geqslant c^{p^*}.$$

By Lemma 2.8 we have $\|S_m\|_{\{p_n \to p_n\}} \geqslant c$, which proves the lemma.

As an easy consequence we obtain the following lemma.

LEMMA 5.6. *Let $D < \infty$. Then there exist limits $\lim_{n\to\infty} p_n$ and $\lim_{n\to-\infty} p_n$.*

LEMMA 5.7. *Let $D < \infty$. Then $\lim_{n\to\infty} p_n = \lim_{n\to-\infty} p_n$.*

Proof. Set

$$p_l = \lim_{n\to-\infty} p_n,\ p_r = \lim_{n\to\infty} p_n. \tag{11}$$

Let $p_l \neq p_r$. Without loss of the generality we can assume $p_l > p_r$. Let $c > 1$ be an arbitrary real number. Set $\delta = \frac{1}{3}(p_l - p_r)$. Let $0 < b_k$ satisfy

$$\sum_{n=1}^{\infty} (b_n)^{p_l - \delta} \leqslant 1 \text{ and } \sum_{n=1}^{\infty} (b_n)^{p_r + \delta} = \infty.$$

According to (11) and $p_l > p_r$ there is $N_1 \in \mathbb{N}$ such that $p_n \geqslant p_l - \delta$ for $n \leqslant -N_1$ and $p_n \leqslant p_r + \delta$ for $n \geqslant N_1$. Take $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and

$$\sum_{n=N_1}^{N_2} (b_n)^{p_r+\delta} \geqslant c^{p^*}.$$

Let $a \in \mathcal{M}$ be given by

$$a_n = \begin{cases} b_{-n} & \text{if } -N_2 \leqslant n \leqslant -N_1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $m = N_1 + N_2$. Since $0 \leqslant a_n \leqslant 1$ for $n \in \mathbb{Z}$ we obtain

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{n=-N_2}^{-N_1} (b_{-n})^{p_n} = \sum_{n=N_1}^{N_2} (b_n)^{p_{-n}} \leqslant \sum_{n=N_1}^{N_2} (b_n)^{p_l-\delta} \leqslant 1$$

and

$$\sum_{n\in\mathbb{Z}} ((S_m a)_n)^{p_n} = \sum_{n\in\mathbb{Z}} (a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}} (a_n)^{p_{n+m}} =$$
$$\sum_{n=-N_2}^{-N_1} (b_{-n})^{p_{n+m}} = \sum_{n=N_1}^{N_2} (b_n)^{p_{m-n}} \geqslant \sum_{n=N_1}^{N_2} (b_n)^{p_r+\delta} \geqslant c^{p^*}.$$

Thus, by Lemma 2.8, $\|S_m\|_{\{p_n \to p_n\}} \geqslant c$.

In what follows we use the following convention. Given a real number r we keep the same symbol for the constant mapping $r : \mathbb{Z} \to \mathbb{R}$ given by $r_k = r$ for all $k \in \mathbb{Z}$.

LEMMA 5.8. *Let $r = \lim_{n\to-\infty} p_n = \lim_{n\to\infty} p_n$ and let $\{p_n - r\} \notin \mathcal{P}$. Then for any $c > 1$ there is $m \in \mathbb{Z}$ such that $\|S_m\|_{\{p_n \to p_n\}} \geqslant c$.*

Proof. Since $\{p_n - r\} \notin \mathcal{P}$ we have by Lemma 3.3 that $\{1 - \frac{r}{p_n}\} \notin \mathcal{P}$. Set $\delta_n = 1 - \frac{r}{p_n}$, $n \in \mathbb{Z}$. By Lemma 3.5 there is a $b \in \mathcal{M}$, $0 \leqslant b_n$, such that

$$\sum_{k\in\mathbb{Z}} b_n \leqslant 1 \text{ and } \sum_{n\in\mathbb{Z}} b_n^{1-\delta_n} = \infty. \tag{12}$$

Given $N \in \mathbb{N}$ denote

$$\mathbb{Z}(N) = \{n \in \mathbb{Z}; -N \leqslant n \leqslant N\}. \tag{13}$$

Let $c > 1$ be an arbitrary real number. Fix $N \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{Z}(N)} b_n^{1-\delta_n} \geqslant 2c^{p^*}.$$

By (12) we have $0 \leqslant b_n \leqslant 1$ and due to the fact that the set $\mathbb{Z}(N)$ is finite we can choose $\varepsilon > 0$ with

$$\sum_{n\in\mathbb{Z}(N)} b_n^{1-\delta_n+\varepsilon/p_n} = \sum_{n\in\mathbb{Z}(N)} b_n^{(r+\varepsilon)/p_n} \geqslant c^{p^*}. \tag{14}$$

Taking this ε we can find $n_1 \in \mathbb{Z}$ such that $p_n < r+\varepsilon$ for all $n \geqslant n_1$. Set $m = n_1+N$. Then $p_{n+m} < r + \varepsilon$ for all $n \in \mathbb{Z}(N)$ and, by (14),

$$\sum_{n\in\mathbb{Z}(N)} b_n^{p(n+m)/p_n} \geqslant \sum_{n\in\mathbb{Z}(N)} b_n^{(r+\varepsilon)/p_n} \geqslant c^{p^*}. \tag{15}$$

Let $a \in \mathcal{M}$ be given by

$$a_n = \begin{cases} (b_n)^{1/p_n} & \text{if } n \in \mathbb{Z}(N) \\ 0 & \text{if } n \notin \mathbb{Z}(N). \end{cases}$$

Then by (12) and (15) we obtain

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{n\in\mathbb{Z}(N)} b_n \leqslant 1$$

and

$$\sum_{n\in\mathbb{Z}} ((S_m a)_n)^{p_n} = \sum_{n\in\mathbb{Z}} (a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}} (a_n)^{p_{n+m}} = \sum_{n\in\mathbb{Z}(N)} (b_n)^{p(n+m)/p_n} \geqslant c^{p^*}.$$

Thus, due to Lemma 2.8, we have $\|S_m\|_{\{p_n\to p_n\}} \geqslant c$, which proves the lemma.

LEMMA 5.9. *Let* $r = \lim_{n\to-\infty} p_n = \lim_{n\to\infty} p_n$ *and* $\{r - p_n\} \notin \mathbb{P}$. *Then for any* $c > 1$ *there is* $m \in \mathbb{Z}$ *such that* $\|S_m\|_{\{p_n\to p_n\}} \geqslant c$.

Proof. The proof is analogous to that of Lemma 5.8 and therefore we will proceed faster. Given $c > 1$ set $\delta_n = 1 - \frac{p_n}{r}$. Since $\{\delta_n\} \notin \mathcal{P}$, there is $b_n \in \mathcal{M}$ and $N \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{Z}} b_n \leqslant 1 \text{ and } \sum_{n\in\mathbb{Z}(N)} b_n^{1-\delta_n} \geqslant 2c^{p^*} \tag{16}$$

where $\mathbb{Z}(N)$ is given by (13). Take $\varepsilon > 0$ such that

$$\sum_{n\in\mathbb{Z}(N)} b_n^{(1-\delta_n)r/(r-\varepsilon)} = \sum_{n\in\mathbb{Z}(N)} b_n^{p_n/(r-\varepsilon)} \geqslant c^{p^*}. \tag{17}$$

Find $n_1 \in \mathbb{Z}$ satisfying $p_n \geqslant r-\varepsilon$ if $n \geqslant n_1$. Set $m = n_1 + N$. Define $a \in \mathcal{M}$ by

$$a_n = \begin{cases} (b_{n-m})^{1/p_n} & \text{if } n-m \in \mathbb{Z}(N) \\ 0 & \text{if } n-m \notin \mathbb{Z}(N). \end{cases}$$

Then by (16) and (17) we obtain

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} \leqslant \sum_{n\in\mathbb{Z}} b_n \leqslant 1$$

and

$$\sum_{n\in\mathbb{Z}} ((S_{-m}a)_n)^{p_n} = \sum_{n\in\mathbb{Z}} (a_{n+m})^{p_n} = \sum_{n\in\mathbb{Z}(N)} (b_n)^{p_n/p_{(n+m)}} \geqslant c^{p^*}.$$

Thus, due to Lemma 2.8, we have $\|S_{-m}\|_{\{p_n\to p_n\}} \geqslant c$, which proves the lemma.

LEMMA 5.10. *Let $D < \infty$. Then there exists $r \in [1,\infty)$ such that the norms in $\ell^{\{p_n\}}$ and in ℓ^r are equivalent.*

This lemma with Lemma 5.2 immediately give the following theorem.

THEOREM 5.11. *The following statements are equivalent:*

(i) $D < \infty$;

(ii) *there is $r \in [1,\infty)$ such that the norms in $\ell^{\{p_n\}}$ and in ℓ^r are equivalent.*

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