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HEINZ–KATO–FURUTA–TYPE INEQUALITIES WITH BOUNDS AND EQUALITY CONDITIONS

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Abstract. In this article we present several new genaralized Heinz-Kato-Furuta-type inequalities. The main tools used here are the Cauchy-Schwarz inequality, basic properties of the polar decomposition of an operator, and, of course, the Furuta inequality. Such considerations enable us to obtain inequalities very easily. The bounds and equality conditions for genaralized Heinz-Kato-Furuta-type inequalities are given.

1. Notations and introduction

Throughout this paper it is to be understood that the capital letters mean bounded linear operators on a Hilbert space H over the field $\mathbb{C}$ of complex numbers. $T = U \mid T \mid$ is the polar decomposition of T with U the partial isometry; U^*U and UU^* are the initial and the final projections, respectively; and $\mid T \mid$ is the positive square root of the positive operator T^*T such that $N(U) = N(\mid T \mid)$, where $N(A)$ denotes the kernel of A. T is a positive operator (written $T \geqslant O$) in case $(Tx, x) \geqslant 0$ for all $x \in$H. If S and T are Hermitian, we write $T \geqslant S$ in case $T - S \geqslant O$.

The so called Heinz-Kato-Furuta inequality is as follows: Let $A, B \geqslant O$ such that $T^*T \leqslant A^2$ and $TT^* \leqslant B^2$. Then

$$\mid (T \mid T \mid^{\alpha+\beta-1} x, y) \mid \leqslant \parallel \mid T \mid^{\alpha} x \parallel \parallel \mid T^* \mid^{\beta} y \parallel \leqslant \parallel A^{\alpha} x \parallel \parallel B^{\beta} y \parallel$$

for all $x, y \in$H, $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geqslant 1$.

The first inequality is obtained by the Cauchy-Schwarz inequality, and the second one is due to the Löwner-Heinz formula, which is the inequality $A^{\alpha} \geqslant B^{\alpha}$ for $\alpha \in [0, 1]$, if $A \geqslant B \geqslant O$; but the inequality does not hold in general for $\alpha > 1$. Recently, the Heinz-Kato-Furuta inequality has been extensively considered and extended by so many authors (cf. References). In this article we present some further generalizations, and show that the easiest way to obtain such type of inequalities is by the Cauchy-Schwarz inequality, some basic properties of the polar decomposition of an operator, and the Furuta inequality. It should be pointed out that such approach is different

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from the ones in references. Equality conditions for inequalities are given, and we also consider the bounds of inequalities. Consequentally, inequalities related to the Löwner-Heinz formula are presented as a corollary.

2. Basic lemma

The next result is the main tool in our discussion of Heinz-Kato-Furuta-type inequalities, equality conditions, and bounds of inequalities.

LEMMA. *Let $x,y,z \in H$. Then the following are equivalent.*

(i) $|(x,y)| \leqslant \|x\| \|y\|$ *(Cauchy-Schwarz inequality).*
Equality holds if and only if $x = \delta y$, $\delta \in \mathbb{C}$. *Moreover, the bound is*

$$\frac{\|x\|^2\|y\|^2 - |(x,y)|^2}{\|x\|^2} \leqslant \frac{\|ry - x\|^2}{r^2}$$

if $x \neq 0$ *and any real number* $r \neq 0$.

(ii) If $(y,z) = 0$, *then*

$$\|z\|^2 |(x,y)|^2 \leqslant \|y\|^2 \left[\|z\|^2\|x\|^2 - |(x,z)|^2\right];$$

or,

$$\|z\|^2 |(x,y)| \leqslant \|y\| \left\| \|z\|^2 x - (x,z)z \right\|.$$

Equality holds if and only if $y = \delta \left[x - \dfrac{(x,z)z}{\|z\|^2}\right]$, $\delta \in \mathbb{C}$ *and* $z \neq 0$.
Moreover, the bound is

$$\frac{|(x,z)|^2}{\|z\|^2} \leqslant \frac{\|x\|^2\|y\|^2 - |(x,y)|^2}{\|y\|^2} \leqslant \frac{\|rx - y\|^2}{r^2}$$

for any real number $r \neq 0$, *and* $z \neq 0 \neq y$.

(iii) Let $\{e_i\}_1^n$ *be a set of unit vectors and let the set* $\{u_i\}_1^n \subseteq H$ *be defined as follows:* $u_0 = x$, *and* $u_i = u_{i-1} - (u_{i-1},e_i)e_i$, $i = 1,2,\ldots,n$. *If* $(y,e_i) = 0$ *for all* i, *then*

$$|(x,y)|^2 + \|y\|^2 \sum_i |(u_{i-1},e_i)|^2 \leqslant \|x\|^2\|y\|^2.$$

Equality holds if and only if $u_n = \delta y$, $\delta \in \mathbb{C}$. *Moreover, the bound is*

$$\frac{\|y\|^2 \left[\|x\|^2 - \sum_i |(u_{i-1},e_i)|^2\right] - |(x,y)|^2}{\|x\|^2 - \sum_i |(u_{i-1},e_i)|^2} \leqslant \frac{\left\| ry - \left[x - \sum_i (u_{i-1},e_i)e_i\right] \right\|^2}{r^2}$$

for any real number $r \neq 0$, *and* $\|x\|^2 \neq \sum_i |(u_{i-1},e_i)|^2$.

(iv) If $(y,z_i)=0,\ i=1,2,\dots n$, *for given nonzero vectors* $\{z_i\}_1^n$, *then*

$$|(x,y)|^2+\sum_i \frac{|(x,z_i)|^2\|y\|^2}{\sum_j |(z_i,z_j)|}\leqslant \|x\|^2\|y\|^2 .$$

Proof. Define a function of any real number $r\neq 0$ by

$$f(r)=\|x\|^2\|ry-x\|^2-r^2[\|y\|^2\|x\|^2-|(y,x)|^2].$$

Then

$$f(r)=r^2|(y,x)|^2-2r\,\mathrm{Re}\,(y,x)\,\|x\|^2+\|x\|^4\geqslant 0$$

since Re $\alpha\leqslant|\alpha|$, and we have the required bound in (i).

(i) implies (ii). Just simplify the Cauchy-Schwarz inequality

$$|(\|z\|^2x-(x,z)z,\|z\|^2y)|^2\leqslant\|\|z\|^2x-(x,z)z\|^2\|z\|^4\|y\|^2,$$

and notice that $\|\|z\|^2x-(x,z)z\|^2=\|z\|^2\,[\|z\|^2\|x\|^2-|(x,z)|^2]$.

Equality holds if and only if $\|z\|^2x-(x,z)z$ and $\|z\|^2y$ are proportional, i.e., $y=\delta x+\gamma z,\ \delta,\gamma\in\mathbb{C}$. Since $0=(y,z)=(\delta x+\gamma z,z)=\delta(x,z)+\gamma\|z\|^2$, $\gamma=\dfrac{-\delta(x,z)}{\|z\|^2}$ and we have equality condition. Moreover, the bound, after rearranging the inequality, follows from (i).

(ii) implies (i). In (ii) we may choose a vector z such that $(x,z)=(y,z)=0$.

(i) implies (iii). By definition of the set $\{u_i\}_1^n$, a straightforward verification shows that

$$u_n=x-\sum_i(u_{i-1},e_i)e_i;\qquad (u_n,y)=(x,y);$$

and

$$\|u_n\|^2=\|x\|^2-\sum_i|(u_{i-1},e_i)|^2 .$$

Now, all we have to do is substituting above relations into the Cauchy-Schwarz inequality $|(u_n,y)|^2\leqslant\|u_n\|^2\|y\|^2$. Equality condition and the bound follow from (i).

(iii) implies (i). Let $n=1$ and choose a unit vector e_1 such that $(x,e_1)=(y,e_1)=0$.

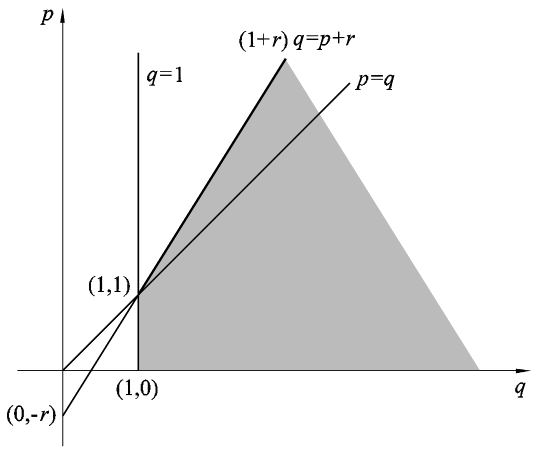
That (i) implies (iv) was proved in [2, Lemma 1]. Conversely, let $n=1$ in particular and choose z_1 such that $(y,z_1)=(x,z_1)=0$. Then (iv) implies (i).

3. Generalized Heinz-Kato-Furuta-type inequalities

Let us recall the following well-known basic relations which is essential as we shall use them in computations and simplifications. Let $T=U|T|$ be the polar decomposition as in section 1. Then $U^*U=I$, the identity operator; and for $m>0$, $|T^*|^m=U|T|^mU^*$ holds in general [5, p. 752].

Inevitably we have to mention the Furuta inequality and its domain as follows.

THEOREM F (FURUTA INEQUALITY). *If* $A \geqslant B \geqslant O$, *then for each* $r \geqslant 0$,
(i) $(B^{r/2}A^pB^{r/2})^{1/q} \geqslant (B^{r/2}B^pB^{r/2})^{1/q}$ *and*
(ii) $(A^{r/2}A^pA^{r/2})^{1/q} \geqslant (A^{r/2}B^pA^{r/2})^{1/q}$
hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



It should be pointed out that conditions $r,p \geqslant 0$, $q \geqslant 1$, *and* $(1+r)q \geqslant p+r$ *of the Furuta inequality are the best possible for the inequality to be valid [15]. The inequality is an excellent generalization of the Löwner-Heinz formula, and is found to be useful in the theory of general operator inequalities.*

Now we are ready to present generalized Heinz-Kato-Furuta inequalities. Due to Lemma, Theorem 1 below includes equality conditions and Theorem 2 contains bounds.

THEOREM 1. *Let* $A,B \geqslant O$ *with* $T^*T \leqslant A^2$ *and* $TT^* \leqslant B^2$. *Then for each* $r,s \geqslant 0$, $p,q \geqslant 1$, $\alpha,\beta \in [0,1]$ *with* $(1+2r)\alpha+(1+2s)\beta \geqslant 1$, *we have*

(i) $|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 \leqslant \||T|^{(1+2r)\alpha}x\|^2\||T^*|^{(1+2s)\beta}y\|^2$
$\leqslant ((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x)(|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y)$ *for* $x,y \in H$.
The first inequality becomes equality if and only if

$$|T|^{(1+2r)\alpha}x = \delta |T|^{(1+2s)\beta}U^*y, \qquad \delta \in \mathbb{C}.$$

(ii) $|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 + \dfrac{\||T^*|^{(1+2s)\beta}y\|^2|(|T|^{2(1+2r)\alpha}x,z)|^2}{\||T|^{(1+2r)\alpha}z\|^2}$
$\leqslant \||T|^{(1+2r)\alpha}x\|^2\||T^*|^{(1+2s)\beta}y\|^2$

$\leqslant ((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x)(|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y)$
for $x,y,z \in H$ *such that* $|T|^{(1+2r)\alpha}z \neq 0$ *and* $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z,y) = 0$.

The first inequality becomes equality if and only if

$$| T^* |^{(1+2s)\beta} y = \delta T \mid T \mid^{(1+2r)\alpha-1} [x - \frac{(| T |^{2(1+2r)\alpha} x, z)z}{\|| T |^{(1+2r)\alpha} z \|^2}], \qquad \delta \in C.$$

(iii) $|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 + \frac{\| |T|^{(1+2r)\alpha}x \|^2 \, |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z, y)|^2}{\| |T|^{(1+2r)\alpha}z \|^2}$

$$\leqslant \|| T |^{(1+2r)\alpha} x \|^2 \|| T^* |^{(1+2s)\beta} y \|^2$$

$$\leqslant ((| T |^{2r} A^{2p} \mid T |^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)(| T^* |^{2s} B^{2q} \mid T^* |^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$$

for $x, y, z \in H$ *such that* $| T |^{(1+2r)\alpha} z \neq 0$ *and* $(| T |^{2(1+2r)\alpha} x, z) = 0$.

The first inequality becomes equality if and only if

$$| T |^{(1+2r)\alpha} x = \delta \left[| T |^{(1+2s)\beta} U^* y - \frac{(y, T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} z) \mid T \mid^{(1+2r)\alpha} z}{\|| T |^{(1+2r)\alpha} z \|^2} \right],$$

$\delta \in C$.

(iv) Let the set $\{u_i\}_1^n$ *be the same as in (iii) of Lemma, then*

$$| (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2 + \sum_i \frac{| (u_{i-1}, U \mid T \mid^{(1+2r)\alpha} z_i) |^2 \|| T^* |^{(1+2s)\beta} y \|^2}{\|| T |^{(1+2r)\alpha} z_i \|^2}$$

$$\leqslant \|| T |^{(1+2r)\alpha} x \|^2 \|| T^* |^{(1+2s)\beta} y \|^2$$

$$\leqslant ((| T |^{2r} A^{2p} \mid T |^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)(| T^* |^{2s} B^{2q} \mid T^* |^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$$

for $x, y, z_i \in H$ *such that* $(T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} z_i, y) = 0$ *and* $| T |^{(1+2r)\alpha} z_i \neq 0$, $i = 1, 2, \ldots, n$.

The first inequality becomes equality if and only if $u_n = \delta \mid T^* |^{(1+2s)\beta} y$, $\delta \in C$.

(v) $|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2(1+2r)\alpha}x, z_i)|^2 \, \| |T^*|^{(1+2s)\beta}y \|^2}{\sum_j |(|T|^{2(1+2r)\alpha}z_i, z_j)|}$

$$\leqslant \|| T |^{(1+2r)\alpha} x \|^2 \|| T^* |^{(1+2s)\beta} y \|^2$$

$$\leqslant ((| T |^{2r} A^{2p} \mid T |^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)(| T^* |^{2s} B^{2q} \mid T^* |^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$$

for $x, y, z_i \in H$ *such that* $\sum_j | (| T |^{2(1+2r)\alpha} z_i, z_j) | \neq 0$ *and* $(T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} z_i, y) = 0$, $i, j = 1, 2, \ldots, n$.

Proof. It is easy to verify that the second inequality in each five inequalities above follows by the Furuta inequality (i) (as in Theorem F).

Now, replace x by $U \mid T \mid^{(1+2r)\alpha} x$, y by $| T^* |^{(1+2s)\beta} y$, and z by $U \mid T \mid^{(1+2r)\alpha} z$ in every inequality in Lemma , and use the basic properties of the polar decomposition for T mentioned in above to simplify relations. In fact, the following relations hold: $\| x \| = \|| T |^{(1+2r)\alpha} x \|$; $\| y \| = \|| T^* |^{(1+2s)\beta} y \|$; $\| z \| = \|| T |^{(1+2r)\alpha} z \|$; $(x, y) = (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y)$; $(z, y) = (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} z, y)$; and $(x, z) = (| T |^{2(1+2r)\alpha} x, z)$. It follows that (i) and (ii) are obtained from (i) and (ii) in Lemma, respectively; (iii) is from (ii) in Lemma with x and y interchanged; while (iv) is from (iii) in Lemma by letting $e_i = z_i/\parallel z_i \parallel$, $i = 1, 2, \ldots, n$; and finally (v) follows by (iv) in Lemma.

THEOREM 2. *Let $A, B \geqslant O$ with $T^*T \leqslant A^2$ and $TT^* \leqslant B^2$. Then for each $r, s \geqslant 0$, $p, q \geqslant 1$, $\alpha, \beta \in [0,1]$ with $(1+2r)\alpha + (1+2s)\beta \geqslant 1$, we have*

(i) In inequality (i) of Theorem 1 the bound of the first inequality is

$$\frac{\||T|^{(1+2r)\alpha}x\|^2\||T^*|^{(1+2s)\beta}y\|^2 - |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2}{\||T|^{(1+2r)\alpha}x\|^2} \leqslant \frac{1}{r^2}\|r|T^*|^{(1+2s)\beta}y - U|T|^{(1+2r)\alpha}x\|^2$$

if $|T|^{(1+2r)\alpha}x \neq 0$ and any real number $r \neq 0$.

(ii) In inequality (ii) of Theorem 1 the bound of the first inequality is

$$\frac{|(|T|^{2(1+2r)\alpha}x, z)|^2}{\||T|^{(1+2r)\alpha}z\|^2} \leqslant \frac{\||T|^{(1+2r)\alpha}x\|^2\||T^*|^{(1+2s)\beta}y\|^2 - |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2}{\||T^*|^{(1+2s)\beta}y\|^2} \leqslant \frac{1}{r^2}\|rU|T|^{(1+2r)\alpha}x - |T^*|^{(1+2s)\beta}y\|^2$$

if $|T|^{(1+2r)\alpha}z \neq 0 \neq |T^*|^{(1+2s)\beta}y$ and any real number $r \neq 0$.

(iii) In inequality (iii) of Theorem 1 the bound of the first inequality is

$$\frac{|(y, T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z)|^2}{\||T|^{(1+2r)\alpha}z\|^2} \leqslant \frac{\||T|^{(1+2r)\alpha}x\|^2\||T^*|^{(1+2s)\beta}y\|^2 - |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2}{\||T|^{(1+2r)\alpha}x\|^2} \leqslant \frac{1}{r^2}\|r|T^*|^{(1+2s)\beta}y - U|T|^{(1+2r)\alpha}x\|^2$$

if $|T|^{(1+2r)\alpha}x \neq 0 \neq |T|^{(1+2r)\alpha}z$ and any real number $r \neq 0$.

Proof. With the same replacements for x, y and z as in Theorem 1, this is consequences of bounds in Lemma. More precisely, (i) and (ii) are obtained from (i) and (ii) in Lemma, respectively; and (iii) is from (ii) in Lemma with x and y interchanged.

It should be pointed out that every inequality in Theorem 1 is equivalent to the Furuta inequality (i) (as in Theorem F), and this is essentially well-known. Correspondingly, the next result is characterizations of the Löwner-Heinz formula in terms of Heinz-Kato-Furuta-type inequalities which are consequences of Theorem 1 and 2.

COROLLARY. *Let* $A,B \geqslant O$ *such that* $T^*T \leqslant A^2$ *and* $TT^* \leqslant B^2$. *Then for* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \geqslant 1$, *the following are equivalent.*

(i) The Löwner-Heinz formula: $A^\alpha \geqslant B^\alpha$ *if* $A \geqslant B \geqslant O$;

(ii) The Heinz-Kato-Furuta inequality, i.e.,

$$| (T \mid T \mid^{\alpha+\beta-1} x,y) | \leqslant \| \mid T \mid^\alpha x \| \| \mid T^* \mid^\beta y \| \leqslant \| A^\alpha x \| \| B^\beta y \|$$

for $x,y \in H$.

The first inequality becomes equality if and only if $\mid T \mid^\alpha x = \delta \mid T \mid^\beta U^* y$, $\delta \in \mathbb{C}$.

Moreover, the bound of the first inequality is

$$\frac{\| \mid T \mid^\alpha x \|^2 \| \mid T^* \mid^\beta y \|^2 - | (T \mid T \mid^{\alpha+\beta-1} x,y) |^2}{\| \mid T \mid^\alpha x \|^2} \leqslant \frac{\| r \mid T^* \mid^\beta y - U \mid T \mid^\alpha x \|^2}{r^2}$$

if $\mid T \mid^\alpha x \neq 0$ *and any real number* $r \neq 0$.

(iii) $| (T \mid T \mid^{\alpha+\beta-1} x,y) |^2 + \dfrac{\| \mid T^* \mid^\beta y \|^2 | (\mid T \mid^{2\alpha} x,z) |^2}{\| \mid T \mid^\alpha z \|^2}$

$$\leqslant \| \mid T \mid^\alpha x \|^2 \| \mid T^* \mid^\beta y \|^2 \leqslant \| A^\alpha x \|^2 \| B^\beta y \|^2$$

for $x,y,z \in H$ *such that* $\mid T \mid^\alpha z \neq 0$ *and* $(T \mid T \mid^{\alpha+\beta-1} z,y) = 0$.

The first inequality becomes equality if and only if

$$\mid T^* \mid^\beta y = \delta T \mid T \mid^{\alpha-1} [x - \frac{(\mid T \mid^{2\alpha} x,z) z}{\| \mid T \mid^\alpha z \|^2}], \qquad \delta \in C.$$

Moreover, the bound of the first inequality is

$$\frac{| (\mid T \mid^{2\alpha} x,z) |^2}{\| \mid T \mid^\alpha z \|^2} \leqslant \frac{\| \mid T \mid^\alpha x \|^2 \| \mid T^* \mid^\beta y \|^2 - | (T \mid T \mid^{\alpha+\beta-1} x,y) |^2}{\| \mid T^* \mid^\beta y \|^2}$$
$$\leqslant \frac{1}{r^2} \| rU \mid T \mid^\alpha x - \mid T^* \mid^\beta y \|^2$$

if $\mid T \mid^\alpha z \neq 0 \neq \mid T^* \mid^\beta y$ *and any real number* $r \neq 0$.

(iv) $| (T \mid T \mid^{\alpha+\beta-1} x,y) |^2 + \dfrac{\| \mid T \mid^\alpha x \|^2 | (T \mid T \mid^{\alpha+\beta-1} z,y) |^2}{\| \mid T \mid^\alpha z \|^2}$

$$\leqslant \| \mid T \mid^\alpha x \|^2 \| \mid T^* \mid^\beta y \|^2 \leqslant \| A^\alpha x \|^2 \| B^\beta y \|^2$$

for $x,y,z \in H$ *such that* $\mid T \mid^\alpha z \neq 0$ *and* $(\mid T \mid^{2\alpha} x,z) = 0$.

The first inequality becomes equality if and only if

$$\mid T \mid^\alpha x = \delta [\mid T \mid^\beta U^* y - \frac{(y, T \mid T \mid^{\alpha+\beta-1} z) \mid T \mid^\alpha z}{\| \mid T \mid^\alpha z \|^2}], \qquad \delta \in C.$$

Moreover, the bound of the first inequality is

$$\frac{| (y, T \mid T \mid^{\alpha+\beta-1} z) |^2}{\| \mid T \mid^\alpha z \|^2} \leqslant \frac{\| \mid T \mid^\alpha x \|^2 \| \mid T^* \mid^\beta y \|^2 - | (T \mid T \mid^{\alpha+\beta-1} x,y) |^2}{\| \mid T \mid^\alpha x \|^2}$$
$$\leqslant \frac{1}{r^2} \| r \mid T^* \mid^\beta y - U \mid T \mid^\alpha x \|^2$$

if $|T|^{\alpha} z \neq 0 \neq |T|^{\alpha} x$ *and any real number* $r \neq 0$.

(v) Let the set $\{u_i\}_1^n$ *be the same as in (iii) of Lemma, then*

$$|(T|T|^{\alpha+\beta-1}x,y)|^2 + \sum_i \frac{|(u_{i-1}, U|T|^{\alpha} z_i)|^2 \| |T^*|^{\beta} y\|^2}{\| |T|^{\alpha} z_i\|^2} \leqslant \| |T|^{\alpha} x\|^2 \| |T^*|^{\beta} y\|^2 \leqslant \| A^{\alpha} x\|^2 \| B^{\beta} y\|^2$$

for $x, y, z_i \in H$ *such that* $(T|T|^{\alpha+\beta-1} z_i, y) = 0$ *and* $|T|^{\alpha} z_i \neq 0$, $i = 1, 2, \ldots, n$. *The first inequality becomes equality if and only if* $u_n = \delta |T^*|^{\beta} y$, $\delta \in C$.

(vi) $$|(T|T|^{\alpha+\beta-1}x,y)|^2 + \sum_i \frac{|(|T|^{2\alpha} x, z_i)|^2 \| |T^*|^{\beta} y\|^2}{\sum_j |(|T|^{2\alpha} z_i, z_j)|} \leqslant \| |T|^{\alpha} x\|^2 \| |T^*|^{\beta} y\|^2 \leqslant \| A^{\alpha} x\|^2 \| B^{\beta} y\|^2$$

for $x, y, z_i \in H$ *such that* $\sum_j |(|T|^{2\alpha} z_i, z_j)| \neq 0$ *and* $(T|T|^{\alpha+\beta-1} z_i, y) = 0$, $i, j = 1, 2, \ldots, n$.

Proof. All we have to do is let $r = s = 0$ in every inequality in Theorem 1 and 2, and let $r = 0$ and $\frac{p}{q} = \alpha$ (as $1 \geqslant \frac{p}{q} \geqslant 0$) in (i) of Theorem F. Of course, the proof can be directly done by Lemma and Theorem F.

4. Remarks

(1) In the proof of Theorem 1 and 2 we adapted replacements of x by $U|T|^{(1+2r)\alpha} x$, y by $|T^*|^{(1+2s)\beta} y$, and z by $U|T|^{(1+2r)\alpha} z$ in Lemma. We may use different replacements to get more different generalized Heinz-Kato-Furuta-type inequalities as long as the inequality is of the form

$$|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 + X \leqslant \| |T|^{(1+2r)\alpha} x\|^2 \| |T^*|^{(1+2s)\beta} y\|^2$$

with a suitable term X. For example, in (iii) of Lemma let $e_i = z_i / \| z_i \|$, $i = 1, 2, \ldots, n$, and replace x by $U|T|^{(1+2r)\alpha} x$, y by $|T^*|^{(1+2s)\beta} y$, z by $|T|^{(1+2r)\alpha} z$, and u_{i-1} by $|T| u_{i-1}$. Then

$$|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 + \sum_i \frac{|(|T|^{2(1+2r)\alpha} u_{i-1}, z_i)|^2 \| |T^*|^{(1+2s)\beta} y\|^2}{\| |T|^{(1+2r)\alpha} z_i\|^2} \leqslant \| |T|^{(1+2r)\alpha} x\|^2 \| |T^*|^{(1+2s)\beta} y\|^2 .$$

for $x, y, z_i \in$ H such that $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1} z_i, y) = 0$ and $|T|^{(1+2r)\alpha} z_i \neq 0$, $i = 1, 2, \ldots, n$. The first inequality becomes equality if and only if $u_n = \delta |T^*|^{(1+2s)\beta} y$, $\delta \in C$.

In particular, if $r = s = 0$, then the inequality is presicely [1, Theorem 3.4] with a long and complicated proof there. Even if $r = s = 0$ and $\alpha + \beta = 1$, the proof is not simple as was shown in [10, Theorem 4].

(2) Equivalence of statements (i) in Theorem 1 and the Furuta inequality (i) (as in Theorem F) was proved in [8, Theorem 1 and Section 2].

(3) The first inequality in (iii) of Corollary was proved in [3, Theorem 1] by using the positivity of the Gram matrix.

(4) A special case of the first inequality in (iii) of Corollary appeared in [10, Theorem 1] with a constructive long proof.

(5) (v) in Theorem 1 appeared in [2, Theorem 8] which relies on [2, Theorem 4] with a complicated proof. Equality condition of the inequality (v) in Theorem 1 may be considered if we scrutinize equality condition of (iv) in Lemma via [2, Lemma 1].

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