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A NOTE ON A THEOREM OF BORWEIN, BORWEIN, FEE AND GIRGENSOHN

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(*communicated by J. Borwein*)

Abstract. A simple application of a theorem of [1] shows that the product of the volume of the unit ball of ℓ_p^n by the volume of the unit ball of its dual space is increasing for $1 \leqslant p \leqslant 2$.

For $\alpha > 1$ and $p > 0$ let

$$V_\alpha(p) = 2^\alpha \frac{\Gamma\left(1+\frac{1}{p}\right)^\alpha}{\Gamma\left(1+\frac{\alpha}{p}\right)}.$$

If $\alpha = n$ is a positive integer and $p \geqslant 1$ then $V_n(p)$ is $\mathrm{vol}_n(B(\ell_p^n))$ – the volume of the unit ball of the n-dimensional normed space ℓ_p^n. In [1] it is proved (Theorem 2.1) that

THEOREM 1. ([1]) *For* $\alpha > 1$, $p, q > 0$, $p \neq q$ *and* $\lambda \in (0,1)$

$$V_\alpha(p)^\lambda V_\alpha(q)^{1-\lambda} < V_\alpha\left(\frac{1}{\frac{\lambda}{p}+\frac{1-\lambda}{q}}\right).$$

The case $q = p' = p/(p-1)$ and $\lambda = 1/2$ of Theorem 1 recovers, in the special case $X = \ell_p^n$, the well known Blaschke-Santalò inequality

$$\mathrm{vol}_n(B(X))\mathrm{vol}_n(B(X^*)) \leqslant \mathrm{vol}_n(B(\ell_2^n))^2 \tag{1}$$

where $B(X)$ is the unit ball of the n-dimensional normed space X and X^* is the dual space of X (X is represented as $\mathbb{R}^n$ equipped with a norm and duality is via the standard scalar product).

The truth of the, so called, *inverse Santalò inequality*:

$$\mathrm{vol}_n(B(X))\mathrm{vol}_n(B(X^*)) \geqslant \mathrm{vol}_n(B(\ell_1^n))\mathrm{vol}_n(B(\ell_\infty^n)) \tag{2}$$

is still an open problem, however, (2) is known to be true in certain special cases, for example, if $B(X)$ or $B(X^*)$ is a zonoid [2]. Since $B(\ell_q^n)$ is a zonoid if $q \geqslant 2$, (2) is true for $X = \ell_p^n$.

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A simple observation shows that, in fact, Theorem 1 supplies a unified proof of Blaschke-Santalò and inverse Santalò inequalities for $X = \ell_p^n$, $1 \leqslant p \leqslant \infty$. More is true:

THEOREM 2. *For* $\alpha > 1$ *and* $1 \leqslant p < q \leqslant 2$, *with* $p' = p/(p-1)$, $q' = q/(q-1)$, *we have*

$$V_\alpha(p)V_\alpha(p') < V_\alpha(q)V_\alpha(q'). \tag{3}$$

Proof. We may assume $1 < p < q < 2$. Let $\lambda, \mu \in (0,1)$. By Theorem 1 we have

$$V_\alpha(p)^\lambda V_\alpha(p')^{1-\lambda} < V_\alpha\left(\frac{p}{2\lambda + p(1-\lambda) - 1}\right), \tag{4}$$

$$V_\alpha(p)^{1-\mu} V_\alpha(p')^{\mu} < V_\alpha\left(\frac{p'}{2\mu + p'(1-\mu) - 1}\right). \tag{5}$$

We look for λ and μ that satisfy

$$q = \frac{p}{2\lambda + p(1-\lambda) - 1}, \quad q' = \frac{p'}{2\mu + p'(1-\mu) - 1}$$

and get $\lambda = ((p/q) + 1 - p)/(2-p)$. Note that, since $(p/2) < (p/q) < 1$, we have $1/2 < \lambda < 1$. We get as well $\mu = ((p'/q') + 1 - p')/(2-p')$ and it is easily checked that $\mu = \lambda$. Hence, multiplying the inequalities (4) and (5) together, we get (3). □

REFERENCES

[1] D. BORWEIN, J. BORWEIN, G. FEE AND R. GIRGENSOHN, *Refined convexity and special cases of the Blaschke-Santalò inequality*, Mathematical Inequalities & Applications **4** (2001), 631–638.

[2] S. REISNER, *Zonoids with minimal volume-product*, Math. Zeitschrift **192** (1986), 339–346.

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