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OPERATOR MONOTONE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. We propose a notion of operator monotonicity for functions of several variables, which extends the well known notion of operator monotonicity for functions of only one variable. The notion is chosen such that a fundamental relationship between operator convexity and operator monotonicity for functions of one variable is extended also to functions of several variables.

1. Introduction and main result

The notion of operator convexity for functions of several variables has been extensively studied in the literature. The first step is to define the functional calculus for functions of several variables. This can be done in the following way:

Let $I_1, \ldots, I_k$ be real intervals and let $f : I_1 \times \cdots \times I_k \to \mathbf{R}$ be a Borel measurable and essentially bounded function. Let $x = (x_1, \ldots, x_k)$ be a k-tuple of bounded self-adjoint operators on Hilbert spaces $H_1, \ldots, H_k$ such that the spectrum of x_i is contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If

$$x_i = \int_{I_i} \lambda_i \, E_i(d\lambda_i) \qquad i = 1, \ldots, k$$

is the spectral decomposition of x_i, we define

$$f(x) = \int_{I_1 \times \cdots \times I_k} f(\lambda_1, \ldots, \lambda_k) \, E_1(d\lambda_1) \otimes \cdots \otimes E_k(d\lambda_k) \tag{1}$$

as a bounded self-adjoint operator on $H_1 \otimes \cdots \otimes H_k$, cf. [4, 1, 9]. If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction extends the definition of Korányi [9] for functions of two variables and have the property that

$$f(x_1, \ldots, x_k) = f_1(x_1) \otimes \cdots \otimes f_k(x_k),$$

whenever f can be separated as a product $f(t_1, \ldots, t_k) = f_1(t_1) \cdots f_k(t_k)$ of k functions each depending on only one variable.

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REMARK 1.1. One might consider the functional calculus only for commuting operators $x_1,\dots,x_k$ on a single Hilbert space H and define

$$f_{\rm com}(x_1,\dots,x_k)=\int f(\lambda_1,\dots,\lambda_k)\,dE(\lambda_1,\dots,\lambda_k)$$

as an operator on H, where E is the product measure of the commuting spectral measures associated with each of the operators. This approach was suggested by Pedersen and Lieb in [12, 10]. Our definition in equation (1) can then be written as

$$f(x_1,\dots,x_k)=f_{\rm com}(x_1\otimes 1\otimes\cdots\otimes 1,\dots,1\otimes\cdots\otimes 1\otimes x_k)$$

for arbitrary non-commuting operators $x_1,\dots,x_k$ on H. If however the operators $x_1,\dots,x_k$ do commute, then there is a self-adjoint projection P on $H\otimes\cdots\otimes H$ with range isomorphic to H such that $f_{\rm com}(x_1,\dots,x_k)=Pf(x_1,\dots,x_k)P$. The two approaches are thus essentially equivalent.

Once the functional calculus is defined, we say that a function $f:I_1\times\cdots\times I_k\to\mathbf{R}$ is operator convex, if f is continuous and the operator inequality

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y)\qquad\forall\lambda\in[0,1]$$

holds for all k-tuples of self-adjoint operators $x=(x_1,\dots,x_k)$ and $y=(y_1,\dots,y_k)$ in the domain of f acting on any Hilbert spaces $H_1,\dots,H_k$. The definition is meaningful since also the k-tuple $\lambda x+(1-\lambda)y$ is in the domain of f. We say that f is matrix convex of order $(n_1,\dots,n_k)$, if the operator inequality holds for operators on Hilbert spaces of finite dimensions $(n_1,\dots,n_k)$.

The aim of this paper is to define the notion of an operator monotone function also for functions of several variables. The definition should, when restricted to functions of only one variable, be a simple reformulation of the ordinary condition for such functions. We also want the following theorem to be valid.

THEOREM 1.2. *Let $f:[0,\alpha_1[\times\cdots\times[0,\alpha_k[\to\mathbf{R}$ be a continuous real function. The following statements are equivalent:*

(i) f is operator convex, and $f(r_1,\dots,r_k)\leqslant 0$ if $r_i=0$ for some $i=1,\dots,k$.
(ii) The function $g:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ defined by setting

$$g(r_1,\dots,r_k)=r_1^{-1}\cdots r_k^{-1}f(r_1,\dots,r_k)$$

is operator monotone.

The theorem above is known to be valid for functions of one variable [7, 2.4 Theorem], and the extension to functions of several variables seems to be very natural. Our notion of operator monotonicity for functions of several variables is ultimately given in Definition 2.14, but it depends on intermediary notions and results given in Definition 2.1, Definition 2.2, Definition 2.3, and Corollary 2.13.

Before proceeding with this programme, we shall briefly discuss other possible definitions of operator monotonicity for functions of several variables, which we ultimately have rejected.

PROPOSITION 1.3. *Let f be a non-negative continuous function of k variables defined in the first quadrant $[0,\infty[\times\cdots\times[0,\infty[$. If f is matrix concave of order $(n_1,\ldots,n_k)$, then*

$$0\leqslant x_i\leqslant y_i \quad i=1,\ldots,k \qquad \Rightarrow \qquad f(x)\leqslant f(y)$$

for arbitrary k-tuples of positive semi-definite matrices $x=(x_1,\ldots,x_k)$ and $y=(y_1,\ldots,y_k)$ of order $(n_1,\ldots,n_k)$.

Proof. Let the appropriate k-tuples of matrices be chosen and take $\lambda\in[0,1[$. We set $z_i=\lambda(1-\lambda)^{-1}(y_i-x_i)$ and notice that

$$\lambda y_i=\lambda x_i+(1-\lambda)z_i \qquad \text{and} \qquad z_i\geqslant 0$$

for $i=1,\ldots,k$. Since f is matrix concave and non-negative we obtain

$$f(\lambda y)\geqslant\lambda f(x)+(1-\lambda)f(z)\geqslant\lambda f(x)$$

where $z=(z_1,\ldots,z_k)$. The result now follows by letting λ tend to one. □

The converse is not true. The function of two variables $f(r_1,r_2)=r_1r_2$ is indeed matrix increasing of any order in the sense that

$$f(x_1,x_2)=x_1\otimes x_2\leqslant y_1\otimes y_2=f(y_1,y_2)$$

for $0\leqslant x_1\leqslant y_1$ and $0\leqslant x_2\leqslant y_2$, but it is not even concave as a real function. However, the situation is quite different for functions of only one variable. Mathias [11] showed that a function, defined on the positive real half-line and matrix monotone of order n, is matrix concave of order $[n/2]$. It follows from [3, 7], although not stated explicitely, that a function, defined on the real positive half-line and matrix monotone of order $4n$, is matrix concave of order n. We may reproduce Mathias' result by proving that a function $f:[0,\infty[\to\mathbf{R}$, matrix monotone of order $2n$, is matrix concave of order n, and the following very simple argument will do. Let x_1,x_2 be positive definite matrices of order n and notice [3] that to a given $\varepsilon>0$ the inequality

$$V^*\begin{pmatrix} x_1 & 0\\ 0 & x_2\end{pmatrix}V=\frac{1}{2}\begin{pmatrix} x_1+x_2 & x_2-x_1\\ x_2-x_1 & x_1+x_2\end{pmatrix}\leqslant\begin{pmatrix} 2^{-1}(x_1+x_2)+\varepsilon & 0\\ 0 & \lambda\end{pmatrix}$$

is valid for a sufficiently large $\lambda>0$, where

$$V=\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1\\ 1 & 1\end{pmatrix}$$

is a unitary block matrix of order $2n\times 2n$. We then obtain

$$\frac{1}{2}\begin{pmatrix} f(x_1)+f(x_2) & f(x_2)-f(x_1)\\ f(x_2)-f(x_1) & f(x_1)+f(x_2)\end{pmatrix}=V^*\begin{pmatrix} f(x_1) & 0\\ 0 & f(x_2)\end{pmatrix}V$$

$$=f\left(V^*\begin{pmatrix} x_1 & 0\\ 0 & x_2\end{pmatrix}V\right)\leqslant\begin{pmatrix} f(2^{-1}(x_1+x_2)+\varepsilon) & 0\\ 0 & f(\lambda)\end{pmatrix}$$

and consequently

$$\frac{f(x_1)+f(x_2)}{2} \leqslant f\left(\frac{x_1+x_2}{2}+\varepsilon\right)$$

from which the statement follows by letting ε tend to zero, since matrix monotone functions of order greater or equal to two are continuous (even continuously differentiable).

We shall finally mention that Korányi and others have considered a notion of operator monotonicity for functions of two variables defined on I^2 where $I =]-1,1[$. The notion is closely connected to the theory of analytic functions of several variables, and in particular to a generalization of the Riesz-Herglotz formula [9, 13]. According to this theory the function

$$g(r_1,r_2) = \frac{r_1 r_2}{(1+r_1)(1+r_2)} \qquad r_1,r_2 \in]0,1[$$

would be called operator monotone, but this is not consistent with Theorem 1.2 as the continuous function

$$f(r_1,r_2) = \frac{r_1^2\, r_2^2}{(1+r_1)(1+r_2)} \qquad r_1,r_2 \in [0,1[$$

is not operator convex. Korányi's notion of operator monotononicity leads to no significant distinction between functions of one and two variables as does the theory of operator convex functions.

2. Decompositions and monotonicity

DEFINITION 2.1. Let x be a positive invertible operator acting on a Hilbert space H. We say that an l-tuple $(y_1,\dots,y_l)$ of positive invertible operators on H is a decomposition of x (of length l) if

$$y_1+\cdots+y_l = x. \tag{2}$$

The l-tuple $a=(a_1,\dots,a_l)$ defined by setting $a_i = x^{-1/2}y_i^{1/2}$ for $i=1,\dots,l$ is called the associated unitary row.

We recall [1] that an l-tuple $a=(a_1,\dots,a_l)$ of operators on a Hilbert space H is said to be a unitary row, if there exists a unitary operator U on the direct sum of l copies of H such that $(a_1,\dots,a_l)$ is the first row in the $l\times l$ block matrix representation of U. The equation

$$a_1a_1^* + \cdots + a_la_l^* = 1 \qquad \text{(the identity on } H) \tag{3}$$

is a necessary, but in general not sufficient condition for $a=(a_1,\dots,a_l)$ to be a unitary row.

The row $a=(a_1,\dots,a_l)$ associated with the decomposition of x in the definition above satisfy condition (3) since

$$a_1a_1^* + \cdots + a_la_l^* = x^{-1/2}y_1x^{-1/2} + \cdots + x^{-1/2}y_lx^{-1/2} = 1.$$

Araki and the author proved that an l-tuple $a=(a_1,\dots,a_l)$ satisfying condition (3) is a unitary row, if $\dim\ker a_i=\dim\ker a_i^*$ for at least one $i=1,\dots,l$. The condition is trivially satisfied in this case since the operators $a_1,\dots,a_l$ are invertible. The l-tuple $a=(a_1,\dots,a_l)$ in Definition 2.1 is therefore indeed a unitary row. We notice that $y_i=a_i^*xa_i$ for $i=1,\dots,l$.

DEFINITION 2.2. An index is a pair (l,j) of integers, where $l\geqslant 2$ and $0\leqslant j\leqslant l-1$.

DEFINITION 2.3. Let $f:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ be a real function. The constants $\alpha_1,\dots,\alpha_k$ may be plus infinity.

(*i*) We say that f is operator monotone of index (l,j), if f is continuous and

$$\operatorname{diag}\Big(f(y_{t_1 1},\dots,y_{t_k k})\Big)_{|t|=j\,(\mathrm{mod}\,l)}\leqslant f(x_1,\dots,x_k)L_{l^{k-1}} \qquad (*)$$

for every k-tuple $x=(x_1,\dots,x_k)$ in the domain of f acting on any Hilbert spaces $H_1,\dots,H_k$ and all decompositions

$$y_{1i}+\cdots+y_{li}=x_i \qquad i=1,\dots,k$$

where $L_{l^{k-1}}$ is the $l^{k-1}\times l^{k-1}$ block matrix with the unit operator on the tensor product $H_1\otimes\cdots\otimes H_k$ in each entry. The index t is a multi-index of the form $t=(t_1,\dots,t_k)$, where $t_i=1,\dots,l$ for $i=1,\dots,k$ and weight $|t|=t_1+\cdots+t_k$.

(*ii*) We say that f is matrix monotone of index (l,j) and order $(n_1,\dots,n_k)$, if the same inequalities $(*)$ are satisfied for operators acting only on Hilbert spaces $H_1,\dots,H_k$ of finite dimensions $(n_1,\dots,n_k)$.

It is not difficult to establish that a continuous function is operator monotone of index (l,j), if and only if it is matrix monotone of index (l,j) and all orders $(n_1,\dots,n_k)$. The proof follows a suggestion by Löwner as reported by Bendat and Sherman [2, Lemma 2.2] and can easily be adapted to the present situation. Furthermore, consider k-tuples $(m_1,\dots,m_k)$ and $(n_1,\dots,n_k)$ such that $m_i\leqslant n_i$ for $i=1,\dots,k$. If a function is matrix convex of order $(n_1,\dots,n_k)$ then it is also matrix convex of order $(m_1,\dots,m_k)$. Likewise, if a function is matrix monotone of index (l,j) and order $(n_1,\dots,n_k)$, then it is also matrix monotone of index (l,j) and order $(m_1,\dots,m_k)$.

PROPOSITION 2.4. *A continuous real function $f:]0,\alpha[\to\mathbf{R}$ is operator monotone of any given index (l,j), if and only if it is operator monotone. Likewise is f matrix monotone of any given index (l,j) and order n, if and only if it is matrix monotone of order n.*

Proof. If we set $k=1$, the inequality $(*)$ reads

$$\begin{aligned} f(y_{l1})&\leqslant f(x_1) &&\text{for } j=0\\ f(y_{j1})&\leqslant f(x_1) &&\text{for } j=1,\dots,l-1\end{aligned}$$

where $y_{11}+\cdots+y_{l1}=x_1$ is a decomposition of x_1. These inequalities are trivially satisfied if f is operator monotone. If on the other hand one of the above inequalities are satisfied for a given index (l,j) and all decompositions of any x_1 in the domain of f, then f is operator monotone. The same reasoning applies to matrix monotone functions. $\square$

PROPOSITION 2.5. *Let $f:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ be a continuous function and consider for $i=1,\dots,k$ the function of one variable*

$$g_i(r_i)=f(r_1,\dots,r_k)$$

obtained from f by keeping all variables fixed except the ith variable. If f is operator monotone of some index (l,j), then g_i is operator monotone. Likewise, if f is matrix monotone of some index (l,j) and order $(n_1,\dots,n_k)$, then g_i is matrix monotone of order n_i.

Proof. Let f be operator monotone (or matrix monotone) of some index (l,j) and assume $i=1$. We choose operators $y\leqslant x$ in the domain of g_1. For some sufficiently small $\varepsilon>0$ we set

$$y_{m1}=\begin{cases} y & m=j+1\\ \varepsilon+(x-y)/(l-1) & m\neq j+1\end{cases}\qquad\text{and}\qquad y_{m2}=\begin{cases} r_2 & m=l-1\\ \varepsilon & m\neq l-1,\end{cases}$$

and for $p=3,\dots,k$

$$y_{mp}=\begin{cases} r_p & m=l\\ \varepsilon & m\neq l.\end{cases}$$

We thus have the decompositions $y_{11}+\cdots+y_{l1}=x+(l-1)\varepsilon$ and

$$y_{1p}+\cdots+y_{lp}=r_p+(l-1)\varepsilon\qquad p=2,\dots,l.$$

By only considering the index $t=(j+1,l-1,l,\dots,l)$ with length $|t|=j\,(\mathrm{mod}\,l)$ in $(*)$, we obtain the inequality

$$f(y,r_2,\dots,r_p)\leqslant f(x+(l-1)\varepsilon,r_2+(l-1)\varepsilon,\dots,r_l+(l-1)\varepsilon)$$

from which the inequality $g_1(y)\leqslant g_1(x)$ is derived by letting ε tend to zero. $\square$

To further investigate the content of Definition 2.3 we set $k=2$ and $l=2$. The inequality $(*)$ then reads

$$\begin{pmatrix} f(y_{11},y_{12}) & 0\\ 0 & f(y_{21},y_{22})\end{pmatrix}\leqslant\begin{pmatrix} f(x_1,x_2) & f(x_1,x_2)\\ f(x_1,x_2) & f(x_1,x_2)\end{pmatrix}\qquad j=0$$

and

$$\begin{pmatrix} f(y_{11},y_{22}) & 0\\ 0 & f(y_{21},y_{12})\end{pmatrix}\leqslant\begin{pmatrix} f(x_1,x_2) & f(x_1,x_2)\\ f(x_1,x_2) & f(x_1,x_2)\end{pmatrix}\qquad j=1$$

for decompositions $y_{11}+y_{21}=x_1$ and $y_{12}+y_{22}=x_2$. This is so because the solutions to the equation $|t|=t_1+t_2=j\,(\mathrm{mod}\,2)$ are the multi-indices $(1,1),(2,2)$ for $j=0$ and $(1,2),(2,1)$ for $j=1$. If we set $k=2$ and $l=3$ the inequality $(*)$ reads

$$\begin{pmatrix} f(y_{11},y_{22}) & 0 & 0\\ 0 & f(y_{21},y_{12}) & 0\\ 0 & 0 & f(y_{31},y_{32})\end{pmatrix}\leqslant f(x_1,x_2)\begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\qquad j=0$$

for decompositions $y_{11}+y_{21}+y_{31}=x_1$ and $y_{12}+y_{22}+y_{32}=x_2$. This is so because the solutions to the equation $|t|=t_1+t_2=0\,(\mathrm{mod}\,3)$ are the multi-indices $(1,2),(2,1),(3,3)$. Finally, if we set $k=3$ and $l=2$ the inequality $(*)$ reads

$$\begin{pmatrix} f(y_{11},y_{12},y_{23}) & 0 & 0 & 0 \\ 0 & f(y_{11},y_{22},y_{13}) & 0 & 0 \\ 0 & 0 & f(y_{21},y_{12},y_{13}) & 0 \\ 0 & 0 & 0 & f(y_{21},y_{22},y_{23}) \end{pmatrix}$$

$$\leqslant f(x_1,x_2,x_3)L_4 \qquad j=0$$

for decompositions $y_{11}+y_{21}=x_1$, $y_{12}+y_{22}=x_2$ and $y_{13}+y_{23}=x_3$. This is so because the solutions to the equation $|t|=t_1+t_2+t_3=0\,(\mathrm{mod}\,2)$ are the multi-indices $(1,1,2),(1,2,1),(2,1,1)$ and $(2,2,2)$.

THEOREM 2.6. *Let $f:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ be a continuous, real function. If the function $g:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ defined by*

$$g(r_1,\ldots,r_k)=r_1^{-1}\cdots r_k^{-1}f(r_1,\ldots,r_k)$$

is matrix monotone of some index (l,j) and order $(l,\ldots,l)$, then f is convex.

Proof. We consider the simple root $\beta=e^{2\pi i/l}$ of the polynomial X^l-1 and set

$$u=\mathrm{diag}\,(\beta^p)_{p=1}^l$$

which is a unitary matrix acting on $\mathbf{C}^l$. We introduce projections

$$P_j=(u^*)^jPu^j \qquad j=1,\ldots,l$$

where P defined by

$$P=\frac{1}{l}\begin{pmatrix} 1\cdots 1 \\ \vdots\ddots\vdots \\ 1\cdots 1 \end{pmatrix}$$

is a one-dimensional projection acting on $\mathbf{C}^l$. We notice that

$$P_j=\frac{1}{l}\left(\beta^{(q-p)j}\right)_{p,q=1}^l \qquad j=1,\ldots,l$$

and consequently

$$\sum_{j=1}^l P_j=\frac{1}{l}\left(\sum_{j=1}^l\beta^{(q-p)j}\right)_{p,q=1}^l=E_l$$

where E_l is the $l\times l$ identity matrix. The projections $P_1,\ldots,P_l$ are thus mutually orthogonal. Let $x_{1i},\ldots,x_{li}$ be real numbers in $]0,\alpha_i[$ and set

$$x_i=\mathrm{diag}\left(x_{ji}\right)_{j=1}^l \qquad i=1,\ldots,k.$$

The k-tuple $((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)$ is for a sufficiently small $\varepsilon>0$ in the domain of g. We define

$$y_{ji}=x_i^{1/2}(P_j+\varepsilon)x_i^{1/2}\qquad j=1,\dots l;\ i=1,\dots,k$$

and calculate

$$y_{1i}+\cdots+y_{li}=x_i^{1/2}\left(\sum_{j=1}^{l}(P_j+\varepsilon)\right)x_i^{1/2}=(1+l\varepsilon)x_i\qquad i=1,\dots,k.$$

Since g is matrix monotone of index (l,j) and order $(l,\dots,l)$ it follows that

$$\operatorname{diag}\Bigl(g(y_{t_1 1},\dots,y_{t_k k})\Bigr)_{|t|=j\,(\operatorname{mod} l)}\leqslant g((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)L_{l^{k-1}}$$

or inserting $g(r_1,\dots,r_k)=r_1^{-1}\cdots r_k^{-1}f(r_1,\dots,r_k)$ that

$$\begin{aligned}&\operatorname{diag}\Bigl((y_{s_1 1}^{-1/2}\otimes\cdots\otimes y_{s_k k}^{-1/2})f(y_{s_1 1},\dots,y_{s_k k})(y_{s_1 1}^{-1/2}\otimes\cdots\otimes y_{s_k k}^{-1/2})\Bigr)_{|s|=j\,(\operatorname{mod} l)}\\&\leqslant c_\varepsilon^{-k}\Bigl((x_1^{-1/2}\otimes\cdots\otimes x_k^{-1/2})f(c_\varepsilon x_1,\dots,c_\varepsilon x_k)(x_1^{-1/2}\otimes\cdots\otimes x_k^{-1/2})\Bigr)_{|t|=|s|=j\,(\operatorname{mod} l)}\end{aligned}$$

where $c_\varepsilon=1+l\varepsilon$. Multiplying to the left and to the right with the self-adjoint matrix

$$\operatorname{diag}\Bigl(y_{s_1 1}^{1/2}\otimes\cdots\otimes y_{s_k k}^{1/2}\Bigr)_{|s|=j\,(\operatorname{mod} l)}$$

we obtain

$$\begin{aligned}&\operatorname{diag}\Bigl(f(y_{s_1 1},\dots,y_{s_k k})\Bigr)_{|s|=j\,(\operatorname{mod} l)}\\&\leqslant c_\varepsilon^{-k}\Bigl((y_{t_1 1}^{1/2}x_1^{-1/2}\otimes\cdots\otimes y_{t_k k}^{1/2}x_k^{-1/2})f(c_\varepsilon x_1,\dots,c_\varepsilon x_k)\times\\&\qquad\qquad(x_1^{-1/2}y_{s_1 1}^{1/2}\otimes\cdots\otimes x_k^{-1/2}y_{s_k k}^{1/2})\Bigr)_{|t|=|s|=j\,(\operatorname{mod} l)}.\end{aligned}$$

We introduce for $s_i=1,\dots,l$ and $i=1,\dots,k$ the $l\times l$ matrix

$$Q_{s_i i}=\frac{1}{x_{1i}+\cdots+x_{li}}\Bigl(x_{pi}^{1/2}x_{qi}^{1/2}\beta^{(q-p)s_i}\Bigr)_{p,q=1}^{l}.$$

It is an easy calculation to show that $Q_{s_i i}$ is a projection and that

$$x_i^{1/2}P_{s_i}x_i^{1/2}=\frac{x_{1i}+\cdots+x_{li}}{l}Q_{s_i i}\qquad s_i=1,\dots,l;\ i=1,\dots,k.$$

Multiplying the above inequality from the left and the right with the projection

$$\operatorname{diag}\Bigl(Q_{s_1 1}\otimes\cdots\otimes Q_{s_k k}\Bigr)_{|s|=j\,(\operatorname{mod} l)}$$

and letting ε tend to zero we thus obtain

$$
\begin{aligned}
& f\left(\frac{x_{11}+\cdots+x_{l1}}{l},\ldots,\frac{x_{1k}+\cdots+x_{lk}}{l}\right)\operatorname{diag}\Big(Q_{s_1 1}\otimes\cdots\otimes Q_{s_k k}\Big)_{|s|=j\,(\mathrm{mod}\,l)}\\
&\leqslant\left(\frac{x_{11}+\cdots+x_{l1}}{l}\right)^{-1}\cdots\left(\frac{x_{1k}+\cdots+x_{lk}}{l}\right)^{-1}\times \\
&\Big((x_1^{1/2}P_{t_1}\otimes\cdots\otimes x_k^{1/2}P_{t_k})f(x_1,\ldots,x_k)(P_{s_1}x_1^{1/2}\otimes\cdots\otimes P_{s_k}x_k^{1/2})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}
\end{aligned}
\tag{4}
$$

where we used that

$$
Q_{t_i i}\,y_{t_i i}^{1/2}x_i^{-1/2}\to\left(\frac{x_{1i}+\cdots+x_{li}}{l}\right)^{1/2}Q_{t_i i}\,x_i^{-1/2}=\left(\frac{x_{1i}+\cdots+x_{li}}{l}\right)^{-1/2}x_i^{1/2}P_{t_i}
$$

as ε tends to zero. We notice that (4) is an $l^{k-1}\times l^{k-1}$ block matrix inequality of $l^k\times l^k$ matrices. Let us in order to examine the inequality calculate the entry

$$
\begin{aligned}
&\Big[x_1^{1/2}P_{t_1}\otimes\cdots\otimes x_k^{1/2}P_{t_k}\Big]_{pq}=\Big[x_1^{1/2}P_{t_1}\Big]_{p_1q_1}\cdots\Big[x_k^{1/2}P_{t_k}\Big]_{p_kq_k}\\
&= x_{p_11}^{1/2}\,l^{-1}\beta^{(q_1-p_1)t_1}\cdots x_{p_kk}^{1/2}\,l^{-1}\beta^{(q_k-p_k)t_k}=l^{-k}\beta^{(q-p)\cdot t}\,x_{p_11}^{1/2}\cdots x_{p_kk}^{1/2}
\end{aligned}
$$

for $p=(p_1,\ldots,p_k)$ and $q=(q_1,\ldots,q_k)$ with $p_1,\ldots,p_k,q_1,\ldots,q_k=1,\ldots,l$. We proceed to calculate the entry

$$
\begin{aligned}
&\Big[(x_1^{1/2}P_{t_1}\otimes\cdots\otimes x_k^{1/2}P_{t_k})f(x_1,\ldots,x_k)(P_{s_1}x_1^{1/2}\otimes\cdots\otimes P_{s_k}x_k^{1/2})\Big]_{pq}\\
&=\sum_{u_1,\ldots,u_k=1}^{l} l^{-k}\beta^{(u-p)\cdot t}\,x_{p_11}^{1/2}\cdots x_{p_kk}^{1/2}\Big[f(x_1,\ldots,x_k)(P_{s_1}x_1^{1/2}\otimes\cdots\otimes P_{s_k}x_k^{1/2})\Big]_{uq}\\
&=\sum_{u_1,\ldots,u_k=1}^{l} l^{-k}\beta^{(u-p)\cdot t}\,x_{p_11}^{1/2}\cdots x_{p_kk}^{1/2}f(x_{u_11},\ldots,x_{u_kk})l^{-k}\beta^{(q-u)\cdot s}x_{q_11}^{1/2}\cdots x_{q_kk}^{1/2}\\
&=l^{-2k}\beta^{q\cdot s-p\cdot t}x_{p_11}^{1/2}\cdots x_{p_kk}^{1/2}\,x_{q_11}^{1/2}\cdots x_{q_kk}^{1/2}\sum_{u_1,\ldots,u_k=1}^{l}\beta^{(t-s)\cdot u}f(x_{u_11},\ldots,x_{u_kk})
\end{aligned}
$$

where we used that $f(x_1,\ldots,x_k)$ is a diagonal matrix with $f(x_{u_11},\ldots,x_{u_kk})$ as the uth diagonal entry, and finally calculate the diagonal entry

$$
\big[Q_{q_11}\otimes\cdots\otimes Q_{q_kk}\big]_{qq}=[Q_{q_11}]_{q_1q_1}\cdots[Q_{q_kk}]_{q_kq_k}=x_{q_11}\cdots x_{q_kk}\prod_{i=1}^{k}(x_{1i}+\cdots+x_{li})^{-1}.
$$

We obtain from (4) an inequality between $l^{k-1}\times l^{k-1}$ matrices by retaining the (t,s)-entry in each (t,s)-block on both sides of the inequality and discarding all other entries. We then insert the entries calculated above in the inequality obtained in this way and get

$$f\left(\frac{x_{11}+\cdots+x_{l1}}{l},\ldots,\frac{x_{1k}+\cdots+x_{lk}}{l}\right)\prod_{i=1}^{k}(x_{1i}+\cdots+x_{li})^{-1}\times$$
$$\operatorname{diag}\Big(x_{s_1 1}\cdots x_{s_k k}\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$
$$\leqslant l^{-k}\prod_{i=1}^{k}(x_{1i}+\cdots+x_{li})^{-1}\times$$
$$\left(x_{t_1 1}^{1/2}\cdots x_{t_k k}^{1/2}x_{s_1 1}^{1/2}\cdots x_{s_k k}^{1/2}\sum_{u_1,\ldots,u_k=1}^{l}\beta^{s\cdot s-t\cdot t+(t-s)\cdot u}f\left(x_{u_1 1},\ldots,x_{u_k k}\right)\right)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.$$

Multiplying from the left and from the right with the self-adjoint matrix

$$\operatorname{diag}\left(x_{s_1 1}^{-1/2}\cdots x_{s_k k}^{-1/2}\prod_{i=1}^{k}(x_{1i}+\cdots+x_{li})^{1/2}\right)_{|s|=j\,(\mathrm{mod}\,l)}$$

we obtain

$$\begin{aligned}&f\left(\frac{x_{11}+\cdots+x_{l1}}{l},\ldots,\frac{x_{1k}+\cdots+x_{lk}}{l}\right)E_{l^{k-1}}\\&\leqslant l^{-k}\sum_{u_1,\ldots,u_k=1}^{l}f\left(x_{u_1 1},\ldots,x_{u_k k}\right)\left(\beta^{s\cdot s-t\cdot t+(t-s)\cdot u}\right)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.\end{aligned}\tag{5}$$

We define for each $u=(u_1,\ldots,u_k)$ with $u_1,\ldots,u_k=1,\ldots,l$ an $l^{k-1}\times l^{k-1}$ matrix Π_u by setting

$$\Pi_u=l^{-(k-1)}\left(\beta^{s\cdot s-t\cdot t+(t-s)\cdot u}\right)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.\tag{6}$$

It is an easy calculation to show that the matrices Π_u are self-adjoint projections, and the inequality (5) can in terms of these projections be written as

$$\begin{aligned}&f\left(\frac{x_{11}+\cdots+x_{l1}}{l},\ldots,\frac{x_{1k}+\cdots+x_{lk}}{l}\right)E_{l^{k-1}}\\&\leqslant\frac{1}{l}\sum_{u_1,\ldots,u_k=1}^{l}f\left(x_{u_1 1},\ldots,x_{u_k k}\right)\Pi_u.\end{aligned}\tag{7}$$

Because of

$$\sum_{u_1,\ldots,u_k=1}^{l}\beta^{(s-t)\cdot u}=l^k\delta_{ts}$$

it follows that

$$\sum_{u_1,\ldots,u_k=1}^{l}\Pi_u=lE_{l^{k-1}}.\tag{8}$$

Since each index (t,s) in each Π_u satisfy $|t| = |s| = j\,(\mathrm{mod}\,l)$, confer equation (6), it follows that $\Pi_u = \Pi_v$ for each v on the form

$$v = (v_1,\dots,v_k) = (u_1 + i\,(\mathrm{mod}\,l),\dots,u_k + i\,(\mathrm{mod}\,l)) \qquad i = 0,1,\dots,l-1. \quad (9)$$

We also notice that for each u there are exactly l different indices in (9). It follows that each projection is counted l times in the sum (8). Two projections are consequently either orthogonal, or identical with their indices connected as in (9). Setting $u = (1,\dots,1)$ and multiplying (7) with Π_u we obtain

$$\begin{aligned} &f\left(\frac{x_{11}+\cdots+x_{l1}}{l},\dots,\frac{x_{1k}+\cdots+x_{lk}}{l}\right)\Pi_u \\ &\leqslant \frac{1}{l}\Big(f(x_{11},\dots,x_{1k})+\cdots+f(x_{l1},\dots,x_{lk})\Big)\Pi_u. \end{aligned}$$

Therefore f is convex. □

A matrix monotone function may tend to minus infinity as the argument of the function approaches a point located on an axis, but it cannot go too fast.

COROLLARY 2.7. *Let $g :]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to \mathbf{R}$ be a continuous function, which is matrix monotone of some index (l,j) and order $(l,\dots,l)$. To each subset of the domain of g of the form $]0,\beta_1[\times\cdots\times]0,\beta_k[$ where $\beta_1,\dots,\beta_k < \infty$, there is a constant $C \geqslant 0$ such that*

$$g(r_1,\dots,r_k) \geqslant -\frac{C}{r_1\cdots r_k} \qquad (r_1,\dots,r_k) \in]0,\beta_1[\times\cdots\times]0,\beta_k[.$$

Proof. The function $f :]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to \mathbf{R}$ given by

$$f(r_1,\dots,r_k) = r_1\cdots r_k\, g(r_1,\dots,r_k)$$

is convex by the preceeding theorem, and it is therefore bounded from below on bounded subsets of the domain. □

To proceed, we need the following slight generalization of [1, Theorem 1.2].

THEOREM 2.8. *Let f be a real, continuous function of k variables defined on the domain $I_1\times\cdots\times I_k$ where $I_1,\dots,I_k$ are intervals containing zero and let (l,j) be any index. The following statements are equivalent:*

(*i*) *f is operator convex and $f(r_1,\dots,r_k) \leqslant 0$ if $r_i = 0$ for some $i = 1,\dots,k$.*

(*ii*) *The operator inequality*

$$\begin{aligned} &\mathrm{diag}\Big(f(a^*_{s_1 1}x_1a_{s_1 1},\dots,a^*_{s_k k}x_ka_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)} \\ &\leqslant \Big((a^*_{t_1 1}\otimes\cdots\otimes a^*_{t_k k})f(x_1,\dots,x_k)(a_{s_1 1}\otimes\cdots\otimes a_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)} \end{aligned}$$

is valid for all unitary rows $a_i = (a_{1i},\dots,a_{li})$ of length l acting on any Hilbert space H_i for $i = 1,\dots,k$ and all k-tuples $(x_1,\dots,x_k)$ of self-adjoint operators in the domain of f acting on $H_1,\dots,H_k$.

(*iii*) *The operator inequality*

$$\operatorname{diag}\Big(f\,(p_{s_1 1}x_1 p_{s_1 1},\dots,p_{s_k k}x_k p_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$

$$\leqslant \Big((p_{t_1 1}\otimes\cdots\otimes p_{t_k k})f\,(x_1,\dots,x_k)(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

is valid for all partitions of unity $p_{1i}+\cdots+p_{li}=1$ *on any Hilbert space* H_i *by orthogonal projections for each* $i=1,\dots,k$ *and all* k*-tuples* $(x_1,\dots,x_k)$ *of self-adjoint operators in the domain of* f *acting on* $H_1,\dots,H_k$.

The indices s,t *in* (*ii*) *and* (*iii*) *are multi-indices of the form* $s=(s_1,\dots,s_k)$, *where* $s_i=1,\dots,l$ *for* $i=1,\dots,k$ *with weight* $|s|=s_1+\cdots+s_k$.

In the reference [1] the sufficiency of (*ii*) and (*iii*) in order to obtain (*i*) were only established for indices of the form $(l,0)$. However, rewriting of the original proof shows, mutatis mutandis, that the inequalities are indeed sufficient for the operator convexity of f for any index. The theorem above is stated for more general domains of the function f than in the original reference, cf. the discussion in the survey article [5]. It has the following version for functions of matrices [5].

THEOREM 2.9. *Let* f *be a real, continuous function of* k *variables defined on the domain* $I_1\times\cdots\times I_k$ *where* $I_1,\dots,I_k$ *are intervals containing zero and let* (l,j) *be any index. Let* $(n_1,\dots,n_k)$ *be a* k*-tuple of natural numbers and consider the statements:*

(*i*) f *is matrix convex of order* $(ln_1,\dots,ln_k)$ *and* $f\,(r_1,\dots,r_k)\leqslant 0$ *if* $r_i=0$ *for some* $i=1,\dots,k$.

(*ii*) *The matrix inequality*

$$\operatorname{diag}\Big(f\,(a_{s_1 1}^* x_1 a_{s_1 1},\dots,a_{s_k k}^* x_k a_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$

$$\leqslant \Big((a_{t_1 1}^*\otimes\cdots\otimes a_{t_k k}^*)f\,(x_1,\dots,x_k)(a_{s_1 1}\otimes\cdots\otimes a_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

is valid for all unitary rows $a_i=(a_{1i},\dots,a_{li})$ *of length* l *acting on a Hilbert space* H_i *of dimension* n_i *for* $i=1,\dots,k$ *and all* k*-tuples* $(x_1,\dots,x_k)$ *of self-adjoint operators in the domain of* f *acting on* $H_1,\dots,H_k$.

(*iii*) *The matrix inequality*

$$\operatorname{diag}\Big(f\,(p_{s_1 1}x_1 p_{s_1 1},\dots,p_{s_k k}x_k p_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$

$$\leqslant \Big((p_{t_1 1}\otimes\cdots\otimes p_{t_k k})f\,(x_1,\dots,x_k)(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

is valid for all partitions of unity $p_{1i}+\cdots+p_{li}=1$ *on a Hilbert space* H_i *of dimension* n_i *by orthogonal projections for each* $i=1,\dots,k$ *and all* k*-tuples* $(x_1,\dots,x_k)$ *of self-adjoint operators in the domain of* f *acting on* $H_1,\dots,H_k$.

(iv) *The matrix inequality*

$$\operatorname{diag}\Big(f(p_{s_1 1}x_1p_{s_1 1},\dots,p_{s_k k}x_kp_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$
$$\leqslant \Big((p_{t_1 1}\otimes\cdots\otimes p_{t_k k})f(x_1,\dots,x_k)(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

is valid for all partitions of unity $p_{1i}+\cdots+p_{li}=1$ *on a Hilbert space* H_i *of dimension* ln_i *by orthogonal projections for each* $i=1,\dots,k$ *and all* k*-tuples* $(x_1,\dots,x_k)$ *of self-adjoint operators in the domain of* f *acting on* $H_1,\dots,H_k$.

(v) f *is matrix convex of order* $(n_1,\dots,n_k)$ *and* $f(r_1,\dots,r_k)\leqslant 0$ *if* $r_i=0$ *for some* $i=1,\dots,k$.

The implications $(i)\Rightarrow(ii)\Rightarrow(iii)$ *and* $(iv)\Rightarrow(v)$ *are then valid.*

The indices s,t in (ii), (iii) and (iv) are multi-indices of the form $s=(s_1,\dots,s_k)$, where $s_i=1,\dots,l$ for $i=1,\dots,k$ with weight $|s|=s_1+\cdots+s_k$. Since the Hilbert spaces in (ii) are finite dimensional, it follows that any row $a_i=(a_{1i},\dots,a_{li})$ satisfying condition (3) is unitary.

THEOREM 2.10. *Let* $f:[0,\alpha_1[\times\cdots\times[0,\alpha_k[\to\mathbf{R}$ *be a continuous real function such that* $f(r_1,\dots,r_k)\leqslant 0$ *if* $r_i=0$ *for some* $i=1,\dots,k$. *The constants* $\alpha_1,\dots,\alpha_k$ *may be plus infinity. If* f *is matrix convex of order* $(ln_1,\dots,ln_k)$ *for some integer* $l\geqslant 2$ *and some* k*-tuple of natural numbers* $(n_1,\dots,n_k)$, *then the function*

$$g(r_1,\dots,r_k)=r_1^{-1}\cdots r_k^{-1}f(r_1,\dots,r_k)\qquad (r_1,\dots,r_k)\in]0,\alpha_1[\times\cdots\times]0,\alpha_k[$$

is matrix monotone of index (l,j) *and order* $(n_1,\dots,n_k)$ *for* $j=0,1,\dots,l-1$.

Proof. Let $(x_1,\dots,x_k)$ be any k-tuple of positive invertible operators in the domain of f acting on Hilbert spaces $H_1,\dots,H_k$ of dimensions $n_1,\dots,n_k$ and let

$$y_{1i}+\cdots+y_{li}=x_i$$

be any decomposition of x_i of length l for each $i=1,\dots,k$. We set

$$a_{s_i i}=x_i^{-1/2}y_{s_i i}^{1/2}\qquad s_i=1,\dots,l;\ i=1,\dots,k$$

and observe that

$$y_{s_i i}=a_{s_i i}^*x_ia_{s_i i}\qquad s_i=1,\dots,l;\ i=1,\dots,k.$$

If f is matrix convex of order $(ln_1,\dots,ln_k)$ we may apply Jensen's matrix inequality for functions of several variables, cf. Theorem 2.9 $(i)\Rightarrow(ii)$. For each $j=0,1,\dots,l-1$ we have

$$\operatorname{diag}\Big(f(y_{s_1 1},\dots,y_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$
$$=\operatorname{diag}\Big(f(a_{s_1 1}^*x_1a_{s_1 1},\dots,a_{s_k k}^*x_ka_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$

$$\begin{aligned}
&\leqslant \Big((a^*_{t_1 1}\otimes\cdots\otimes a^*_{t_k k})f(x_1,\dots,x_k)(a_{s_1 1}\otimes\cdots\otimes a_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}\\
&= \Big((y^{1/2}_{t_1 1}x_1^{-1/2}\otimes\cdots\otimes y^{1/2}_{t_k k}x_k^{-1/2})f(x_1,\dots,x_k)\times\\
&\qquad (x_1^{-1/2}y^{1/2}_{s_1 1}\otimes\cdots\otimes x_k^{-1/2}y^{1/2}_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}\\
&= \Big((y^{1/2}_{t_1 1}\otimes\cdots\otimes y^{1/2}_{t_k k})g(x_1,\dots,x_k)(y^{1/2}_{s_1 1}\otimes\cdots\otimes y^{1/2}_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}
\end{aligned}$$

and multiplying to the left and to the right with the self-adjoint operator

$$C=\mathrm{diag}\Big(y^{-1/2}_{s_1 1}\otimes\cdots\otimes y^{-1/2}_{s_k k}\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$

in the above inequality, we obtain

$$\mathrm{diag}\Big((y^{-1}_{s_1 1}\otimes\cdots\otimes y^{-1}_{s_k k})f(y_{s_1 1},\dots,y_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}\leqslant\Big(g(x_1,\dots,x_k)\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

or equivalently

$$\mathrm{diag}\Big(g(y_{s_1 1},\dots,y_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}\leqslant g(x_1,\dots,x_k)L_{l^{k-1}}$$

showing that g is matrix monotone of index (l,j) and order $(n_1,\dots,n_k)$. □

THEOREM 2.11. *Let $f:[0,\alpha_1[\times\cdots\times[0,\alpha_k[\to\mathbf{R}$ be a continuous, real function and suppose the function*

$$g(r_1,\dots,r_k)=r_1^{-1}\cdots r_k^{-1}f(r_1,\dots,r_k)\qquad (r_1,\dots,r_k)\in]0,\alpha_1[\times\cdots\times]0,\alpha_k[$$

is matrix monotone of some index (l,j) and order $(ln_1,\dots,ln_k)$. Then the following statements are valid:

(i) *$f(r_1,\dots,r_k)\leqslant 0$ if $r_i=0$ for some $i=1,\dots,k$.*
(ii) *f is matrix convex of order $(n_1,\dots,n_k)$.*

Proof. Since g is an increasing function in each coordinate, cf. Proposition 2.5, the first statement follows.

Let $(x_1,\dots,x_k)$ be a k-tuple of positive invertible operators in the domain of f acting on Hilbert spaces $H_1,\dots,H_k$ of dimensions $ln_1,\dots,ln_k$ and let

$$p_{1i}+\cdots+p_{li}=1\qquad i=1,\dots,k$$

be resolutions of the identity on H_i of length l. We choose a positive ε such that $(1+l\varepsilon)x$ is in the domain of f and set

$$y_{s_i i}=x_i^{1/2}(p_{s_i i}+\varepsilon)x_i^{1/2}\qquad s_i=1,\dots,l;\ i=1,\dots,k.$$

We consider the decompositions

$$y_{1i}+\cdots+y_{li}=(1+l\varepsilon)x_i\qquad i=1,\dots,k$$

and use the assumption to obtain

$$\operatorname{diag}\Bigl(g(y_{s_1 1},\dots,y_{s_k k})\Bigr)_{|s|=j\,(\mathrm{mod}\,l)} \leqslant g((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)L_{l^{k-1}}.$$

We introduce the diagonal block matrix

$$C=\operatorname{diag}\Bigl(x_1^{1/2}(p_{s_1 1}+\varepsilon)x_1 p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}(p_{s_k k}+\varepsilon)x_k p_{s_k k}\Bigr)_{|s|=j\,(\mathrm{mod}\,l)}$$

and multiply to the left with C^* and to the right with C in the above inequality to obtain

$$\begin{aligned}&\operatorname{diag}\Bigl((p_{s_1 1}x_1(p_{s_1 1}+\varepsilon)x_1^{1/2}\otimes\cdots\otimes p_{s_k k}x_k(p_{s_k k}+\varepsilon)x_k^{1/2})g(y_{s_1 1},\dots,y_{s_k k})\times\\&\qquad(x_1^{1/2}(p_{s_1 1}+\varepsilon)x_1p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}(p_{s_k k}+\varepsilon)x_kp_{s_k k})\Bigr)_{|s|=j\,(\mathrm{mod}\,l)}\\&\leqslant\Bigl((p_{t_1 1}x_1(p_{t_1 1}+\varepsilon)x_1^{1/2}\otimes\cdots\otimes p_{t_k k}x_k(p_{t_k k}+\varepsilon)x_k^{1/2})g((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)\\&\qquad\times(x_1^{1/2}(p_{s_1 1}+\varepsilon)x_1p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}(p_{s_k k}+\varepsilon)x_kp_{s_k k})\Bigr)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.\end{aligned}$$

Inserting

$$\begin{aligned}g(y_{s_1 1},\dots,y_{s_k k})&=(x_1^{-1/2}(p_{s_1 1}+\varepsilon)^{-1}x_1^{-1/2}\otimes\cdots\otimes x_k^{-1/2}(p_{s_k k}+\varepsilon)^{-1}x_k^{-1/2})\times\\&\qquad f(x_1^{1/2}(p_{s_1 1}+\varepsilon)x_1^{1/2},\dots,x_k^{1/2}(p_{s_k k}+\varepsilon)x_k^{1/2})\end{aligned}$$

and

$$\begin{aligned}&g((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)=\\&(1+l\varepsilon)^{-k}(x_1^{-1/2}\otimes\cdots\otimes x_k^{-1/2})f((1+l\varepsilon)x_1,\dots,(1+l\varepsilon)x_k)(x_1^{-1/2}\otimes\cdots\otimes x_k^{-1/2})\end{aligned}$$

in the inequality, and then letting ε tend to zero we obtain

$$\begin{aligned}&\operatorname{diag}\Bigl((p_{s_1 1}x_1^{1/2}\otimes\cdots\otimes p_{s_k k}x_k^{1/2})f(x_1^{1/2}p_{s_1 1}x_1^{1/2},\dots,x_k^{1/2}p_{s_k k}x_k^{1/2})\times\\&\qquad(x_1^{1/2}p_{s_1 1}x_1p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}p_{s_k k}x_kp_{s_k k})\Bigr)_{|s|=j\,(\mathrm{mod}\,l)}\\&\leqslant\Bigl((p_{t_1 1}x_1p_{t_1 1}\otimes\cdots\otimes p_{t_k k}x_kp_{t_k k})f(x_1,\dots,x_k)\times\\&\qquad(p_{s_1 1}x_1p_{s_1 1}\otimes\cdots\otimes p_{s_k k}x_kp_{s_k k})\Bigr)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.\end{aligned}$$

The identity

$$\begin{aligned}&f(x_1^{1/2}p_{s_1 1}x_1^{1/2},\dots,x_k^{1/2}p_{s_k k}x_k^{1/2})(x_1^{1/2}p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}p_{s_k k})=\\&\qquad(x_1^{1/2}p_{s_1 1}\otimes\cdots\otimes x_k^{1/2}p_{s_k k})f(p_{s_1 1}x_1p_{s_1 1},\dots,p_{s_k k}x_kp_{s_k k})\end{aligned}$$

follows by first considering polynomials and then applying Weierstrass' approximation theorem. Inserting the identity in the inequality above we obtain

$$\begin{aligned}&\operatorname{diag}\Bigl((p_{s_1 1}x_1p_{s_1 1}\otimes\cdots\otimes p_{s_k k}x_kp_{s_k k})f(p_{s_1 1}x_1p_{s_1 1},\dots,p_{s_k k}x_kp_{s_k k})\times\\&\qquad(p_{s_1 1}x_1p_{s_1 1}\otimes\cdots\otimes p_{s_k k}x_kp_{s_k k})\Bigr)_{|s|=j\,(\mathrm{mod}\,l)}\\&\leqslant\Bigl((p_{t_1 1}x_1p_{t_1 1}\otimes\cdots\otimes p_{t_k k}x_kp_{t_k k})f(x_1,\dots,x_k)\times\\&\qquad(p_{s_1 1}x_1p_{s_1 1}\otimes\cdots\otimes p_{s_k k}x_kp_{s_k k})\Bigr)_{|t|=|s|=j\,(\mathrm{mod}\,l)}\end{aligned}$$

and hence

$$\operatorname{diag}\Big((p_{s_1 1}\otimes\cdots\otimes p_{s_k k})f\,(p_{s_1 1}x_1p_{s_1 1},\ldots,p_{s_k k}x_kp_{s_k k})(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$
$$\leqslant\Big((p_{t_1 1}\otimes\cdots\otimes p_{t_k k})f\,(x_1,\ldots,x_k)(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}.$$

Because of (i) we obtain

$$f\,(p_{s_1 1}x_1p_{s_1 1},\ldots,p_{s_k k}x_kp_{s_k k})$$
$$\leqslant(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})f\,(p_{s_1 1}x_1p_{s_1 1},\ldots,p_{s_k k}x_kp_{s_k k})(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})$$

and consequently

$$\operatorname{diag}\Big(f\,(p_{s_1 1}x_1p_{s_1 1},\ldots,p_{s_k k}x_kp_{s_k k})\Big)_{|s|=j\,(\mathrm{mod}\,l)}$$
$$\leqslant\Big((p_{t_1 1}\otimes\cdots\otimes p_{t_k k})f\,(x_1,\ldots,x_k)(p_{s_1 1}\otimes\cdots\otimes p_{s_k k})\Big)_{|t|=|s|=j\,(\mathrm{mod}\,l)}$$

which is Jensen's matrix inequality. We thus deduce, cf. Theorem 2.9 $(iv)\Rightarrow(v)$, that f is matrix convex of order $(n_1,\ldots,n_k)$. □

One may think that the preceeding theorem, which ensures matrix convexity of f, could replace Theorem 2.6 which with similar conditions only imparts ordinary convexity on f. However, it is essential in the proof of the preceeding theorem that f is defined also on the axes, while this is not required in Theorem 2.6. This problem can easily be overcome for functions of only one variable by making a small translation of the matrix monotone function g. This remedy is not available for functions of several variables, since the translation of the decomposition of an operator no longer is a decomposition of the translated operator, cf. equation (2).

COROLLARY 2.12. *Let* $g:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ *be a continuous real function. If* g *is matrix monotone of some index* (l,j) *and order* $(lmn_1,\ldots,lmn_k)$ *for a natural number* m *and a* k*-tuple of natural numbers* $(n_1,\ldots,n_k)$, *then it is matrix monotone of index* (m,h) *and order* $(n_1,\ldots,n_k)$ *for* $h=0,1,\ldots,m-1$.

Proof. The real and continuous function f defined by

$$f\,(r_1,\ldots,r_k)=r_1\cdots r_k\,g(r_1,\ldots,r_k)\qquad 0<r_i<\alpha_i\quad\text{for}\quad i=1,\ldots,k$$

is convex by Theorem 2.6. It therefore extends to a continuous function

$$\tilde f:[0,\alpha_1[\times\cdots\times[0,\alpha_k[\to\mathbf{R},$$

and it follows that $\tilde f(r_1,\ldots,r_k)\leqslant 0$ if $r_i=0$ for some $i=1,\ldots,k$. The function $\tilde f$ is matrix convex of order $(mn_1,\ldots,mn_k)$ by Theorem 2.11. The function g is thus matrix monotone of index (m,h) and order $(n_1,\ldots,n_k)$ for $h=0,1,\ldots,m-1$ by Theorem 2.10. □

COROLLARY 2.13. *Let* $g:]0,\alpha_1[\times\cdots\times]0,\alpha_k[\to\mathbf{R}$ *be a continuous function. If* g *is operator monotone of some index, then it is operator monotone of all indices.*

DEFINITION 2.14. We say that a continuous function $g :]0, \alpha_1[\times \cdots \times]0, \alpha_k[\to \mathbf{R}$ is operator monotone, if it is operator monotone of some and hence operator monotone of all indices.

Proof (of Theorem 1.2): The statement follows by combining Theorem 2.10, Theorem 2.11 and Definition 2.14. □

The simplest example of operator convex functions satisfying the boundary conditions in Theorem 1.2 are the negative constants. The function

$$g(r_1, \ldots, r_k) = -r_1^{-1} \cdots r_k^{-1}$$

defined in the first (open) quadrant is thus operator monotone, cf. also Corollary 2.7. The set of operator monotone functions defined on a given domain is a weakly closed convex cone, but the constant function $g(r_1, \ldots, r_k) = 1$ is not operator monotone for $k \geqslant 2$. This must indeed be so since the function $(r_1, r_2) \to r_1 r_2$ is not convex.

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