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ON NEW PROOFS OF WILKER'S INEQUALITIES INVOLVING TRIGONOMETRIC FUNCTIONS

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(*communicated by J. Pečarić*)

Abstract. In the note, some new proofs for inequalities involving trigonometric functions are given.

1. Introduction

In [10], J. B. Wilker proposed that

(a) If $0 < x < \frac{\pi}{2}$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

(b) There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{2}$$

for $0 < x < \frac{\pi}{2}$.

In [9], the inequality (1) was proved, and the following inequalities were also obtained

$$2 + \frac{8}{45}x^3 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x. \tag{3}$$

The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ are best possible, that is, they can not be replaced by smaller or larger numbers respectively.

The inequalities in (1) and (3) are called Wilker's inequalities in [4].

In this note, we will give new proofs for the inequalities in (1) and (3).

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2. A new proof of inequality (1)

The inequality (1) can be rewritten as

$$\sin^2 x\cos x + x\sin x > 2x^2\cos x. \tag{4}$$

Let

$$g(x) = \sin^2 x\cos x + x\sin x - 2x^2\cos x, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{5}$$

$$h(x) = 2\sin x\cos^2 x - 3x\cos x + (1+x^2)\sin x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{6}$$

Direct calculation yields

$$\begin{aligned} g'(x) &= 2\sin x\cos^2 x - \sin^3 x + \sin x + x\cos x - 4x\cos x + 2x^2\sin x \\ &= (x^2 - \sin^2 x)\sin x + 2\sin x\cos^2 x - 3x\cos x + (1+x^2)\sin x \\ &= (x^2 - \sin^2 x)\sin x + h(x), \\ h'(x) &= 2\cos^3 x - 4\sin^2 x\cos x - 3\cos x + 3x\sin x + 2x\sin x + (1+x^2)\cos x \\ &= (x^2 - \sin^2 x)\cos x + 5(x - \sin x\cos x)\sin x. \end{aligned}$$

Since $x > \sin x$ for $x > 0$, we have $h'(x) > 0$, $h(x)$ is increasing. From $h(0) = 0$, we obtain $h(x) > 0$, and then $g'(x) = (x^2 - \sin^2 x)\sin x + h(x) > 0$, the function $g(x)$ is increasing. From $g(0) = 0$ and $g\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, we get $0 < g(x) < \frac{\pi}{2}$ for $x \in \left(0, \frac{\pi}{2}\right)$.

The proof of inequality (1) is complete.

3. A new proof of inequalities in (3)

Define

$$\psi(x) = \frac{\sin 2x}{2x^5} + \frac{1}{x^4} - \frac{2\cot x}{x^3} \tag{7}$$

for $0 < x < \frac{\pi}{2}$. Easy computation yields

$$\psi'(x) = -\frac{5\sin 2x}{2x^6} + \frac{\cos 2x}{x^5} - \frac{4}{x^5} + \frac{6\cos x}{x^4\sin x} + \frac{2}{x^3\sin^2 x}. \tag{8}$$

It is well-known [1, p. 226–227] that

$$\sin 2x = 2x - \frac{4}{3}x^3 + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+5}}{(2n+5)!}x^{2n+5}, \tag{9}$$

$$\cot x = \frac{1}{x} - \frac{1}{3}x - \sum_{n=0}^{\infty}\frac{2^{2n+4}B_{n+2}}{(2n+4)!}x^{2n+3}, \tag{10}$$

where B_n denotes the n-th Bernoulli number, which is defined in [1, p. 228] by

$$\frac{t}{e^t - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}B_k}{(2k)!}t^{2k}, \quad |t| < 2\pi. \tag{11}$$

Therefore, by direct computation, we have

$$\psi(x)=\sum_{n=0}^{\infty}\frac{2^{2n+4}}{(2n+5)!}\left\{2(2n+5)B_{n+2}+(-1)^n\right\}x^{2n}. \tag{12}$$

From the identity in [1, p. 231]

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{\pi^{2n}\cdot 2^{2n-1}}{(2n)!}B_n, \tag{13}$$

by mathematical induction, for $n>2$, we have

$$2(2n+5)B_{n+2}=\frac{4\cdot(2n+5)!}{(2\pi)^{2n+4}}\sum_{k=1}^{\infty}\frac{1}{k^{2n+4}}>\frac{4\cdot(2n+5)!}{(2\pi)^{2n+4}}>1, \tag{14}$$

then $\phi''(x)\geqslant 0$, where $\phi(x)=\psi\left(\sqrt{x}\right)$, and $\phi'(x)$ is increasing on $(0,\frac{\pi^2}{4})$. Since $\phi'\left(\left(\frac{\pi}{2}\right)^2\right)=\frac{1}{\pi}\psi'\left(\frac{\pi}{2}\right)=\left(\frac{2}{\pi}\right)^4\cdot\left(1-\frac{10}{\pi^2}\right)<0$, hence $\phi'(x)<0$, and then $\phi(x)$ is decreasing, that is $\psi(x)$ is decreasing on $(0,\frac{\pi}{2})$, then we have

$$\frac{8}{45}=\psi(0)>\psi(x)>\psi\left(\frac{\pi}{2}\right)=\frac{16}{\pi^4},\quad x\in\left(0,\frac{\pi}{2}\right). \tag{15}$$

Inequalities in (15) are equivalent to those in (3). The proof of inequalities in (3) is complete.

REMARK 1. For details about Bernoulli numbers, also please refer to [3, 6, 8].

REMARK 2. Using Tchebysheff's integral inequality, many inequalities involving the function $\frac{\sin x}{x}$ are constructed in [7].

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