Mathematical
Inequalities
& Applications
Volume 6, Number 1 (2003), 23–36

FURTHER INEQUALITIES FOR THE EXPECTATION AND VARIANCE OF A RANDOM VARIABLE DEFINED ON A FINITE INTERVAL

N. S. BARNETT, P. CERONE, S. S. DRAGOMIR

(*communicated by N. Elezović*)

Abstract. Some new elementary inequalities for the expectation and the variance of a continuous random variable defined on a finite interval are given.

1. Introduction

Let X be a continuous random variable having the probability density function $f:[a,b]\to(0,\infty)$ and cumulative distribution function $F:[a,b]\to[0,1]$.

In a recent paper [11], the authors pointed out a number of inequalities for the expectation, $E(X)$ and the variance, $\sigma^2(X)$ from which we cite the following:

$$0\leqslant\sigma^2(X)\leqslant[b-E(X)][E(X)-a]\leqslant\frac{1}{4}(b-a)^2; \tag{1}$$

$$\begin{aligned}0&\leqslant[b-E(X)][E(X)-a]-\sigma^2(X)\\&\leqslant\begin{cases}\frac{(b-a)^3}{6}\|f\|_\infty\\[B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|f\|_p,\\\text{provided } f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\end{cases}\end{aligned} \tag{2}$$

where $B(\cdot,\cdot)$ is Euler's Beta function. That is, we recall

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\ \ \alpha,\beta>0.$$

We note that the proof of (2) was based on the use of the elementary identity (see (2.5) in [11]):

$$[E(X)-a][b-E(X)]-\sigma^2(X)=\int_a^b(b-t)(t-a)f(t)\,dt. \tag{3}$$

Mathematics subject classification (2000): 26D15, 60E15.
Key words and phrases: Expectation, variance, analytic inequalities.

Paper MIA-06-03

For a more general result incorporating (3) see [4].

Moreover, if $m \leqslant f \leqslant M$ a.e. on $[a,b]$, then

$$\frac{m(b-a)^3}{6} \leqslant [b-E(X)][E(X)-a]-\sigma^2(X) \leqslant \frac{M(b-a)^3}{6} \tag{4}$$

and

$$\left| [b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^3}{6} \right| \leqslant \frac{\sqrt{5}(b-a)^3(M-m)}{60}. \tag{5}$$

In this current paper, we point out some additional results.

2. The Results

LEMMA 1. *Let X be a continuous random variable having the cumulative distribution function $F:[a,b] \to [0,1]$. Then,*

$$\begin{aligned} \sigma^2(X) &= (b-E(X))(E(X)-a) \\ &\quad + \frac{1}{b-a}\int_a^b \int_a^b (t-\tau)(F(\tau)-F(t))\, d\tau dt. \end{aligned} \tag{6}$$

Proof. Using integration by parts, we have

$$\begin{aligned} \sigma^2(X) &= \int_a^b (t-E(X))^2 \, dF(t) \\ &= (b-E(X))^2 - 2\int_a^b (t-E(X))F(t)\,dt. \end{aligned} \tag{7}$$

Further, using Korkine's identity,

$$\begin{aligned} \frac{1}{b-a}\int_a^b h(t)g(t)\,dt &= \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \\ &\quad + \frac{1}{2(b-a)^2}\int_a^b \int_a^b (h(t)-h(\tau))(g(t)-g(\tau))\,d\tau dt, \end{aligned}$$

we have

$$\begin{aligned} \int_a^b (t-E(X))F(t)\,dt &= \frac{1}{b-a}\int_a^b (t-E(X))\,dt \int_a^b F(t)\,dt \\ &\quad + \frac{1}{2(b-a)}\int_a^b \int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt. \end{aligned} \tag{8}$$

Since,

$$\int_a^b (t-E(X))\,dt = (b-a)\left(\frac{b+a}{2}-E(X)\right)$$

and

$$\int_a^b F(t)\,dt = b - E(X),$$

then, by (7) and (8),

$$\begin{aligned}
\sigma^2(X) &= (b-E(X))^2 - 2\left[\frac{b+a-2E(X)}{2}\cdot(b-E(X))\right.\\
&\quad\left.+\frac{1}{2(b-a)}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt\right]\\
&= (b-E(X))^2 - (b+a-2E(X))(b-E(X))\\
&\quad -\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt\\
&= (b-E(X))(E(X)-a) - \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt
\end{aligned}$$

and the lemma is proved.

REMARK 1. Since the mapping F is monotonic nondecreasing on $[a,b]$, then

$$(t-\tau)(F(\tau)-F(t)) \leqslant 0 \text{ for all } t,\tau\in[a,b]; \tag{9}$$

which implies that

$$\sigma^2(X) \leqslant [b-E(X)][E(X)-a], \tag{10}$$

an inequality that was proved in [11] and [12] using two different methods.

The inequality (10) can be improved as follows.

THEOREM 1. *With the assumptions in Lemma 1,*

$$\begin{aligned}
&(b-E(X))(E(X)-a)-\sigma^2(X)\\
&\qquad \geqslant 2\left|\int_a^b |t|\,F(t)\,dt - \frac{1}{b-a}(b-E(X))\int_a^b |t|\,dt\right| \geqslant 0.
\end{aligned} \tag{11}$$

Proof. In [13], S. S. Dragomir proved the following refinement of Chebychev's inequality

$$T(h,g) \geqslant \max\{|T(h,|g|)|,|T(|h|,g)|,|T(|h|,|g|)|\} \geqslant 0, \tag{12}$$

provided (h,g) are synchronous on $[a,b]$, that is,

$$(h(t)-h(\tau))(g(t)-g(\tau)) \geqslant 0 \text{ for all } t,\tau\in[a,b]$$

and

$$T(h,g) := \frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt.$$

If we define $h(t) = t$, $t \in [a,b]$, then from (6)

$$\begin{aligned} T(h,F) &= \frac{1}{b-a}\int_a^b \int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt \\ &= \frac{1}{b-a}\left[(b-E(X))(E(X)-a)-\sigma^2(X)\right]. \end{aligned}$$

Now, from (12),

$$\begin{aligned} T(|h|,F) &= \frac{1}{b-a}\int_a^b \int_a^b |t|\,F(t)\,dt - \frac{1}{(b-a)^2}\int_a^b |t|\,dt \int_a^b F(t)\,dt, \\ T(h,|F|) &= T(h,F), \\ T(|h|,|F|) &= T(|h|,F). \end{aligned}$$

Using the result (12), we get (11).

REMARK 2. If $a \leqslant b \leqslant 0$ or $0 \leqslant a \leqslant b$, then the first inequality in (11) becomes an identity and is of no special interest.
If $a < 0 < b$, however, then,

$$\begin{aligned} \int_a^b |t|\,F(t)\,dt &= -\int_a^0 tF(t)\,dt + \int_0^b tF(t)\,dt; \\ \frac{1}{b-a}\int_a^b |t|\,dt &= \frac{1}{b-a}\left[\frac{a^2+b^2}{2}\right] \end{aligned}$$

and by (11), we get

$$\begin{aligned} &(b-E(X))(E(X)-a)-\sigma^2(X) \\ &\qquad \geqslant 2\left|\int_0^b tF(t)\,dt - \int_a^0 tF(t)\,dt - \frac{a^2+b^2}{2(b-a)}(b-E(X))\right| \geqslant 0. \quad (13) \end{aligned}$$

Assume that $f(x)$, $f : [a,b] \to (0,\infty)$ is the p.d.f. of X, then the following theorem holds.

THEOREM 2. *With the assumptions in Lemma 1,*

$$\begin{aligned} 0 &\leqslant (b-E(X))(E(X)-a)-\sigma^2(X) \\ &\leqslant \begin{cases} \frac{(b-a)^3}{6}\|f\|_\infty & \text{if} \quad f \in L_\infty[a,b]; \\ \\ \frac{2q^2(b-a)^{2+\frac{1}{q}}\|f\|_p}{(2q+1)(3q+1)} & \text{if } f \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \end{aligned} \quad (14)$$

where $\|\cdot\|_p$ $(p \geqslant 1)$ *are the usual Lebesgue norms.*

Proof. Using (6),

$$\begin{aligned} 0 &\leqslant (b-E(X))(E(X)-a)-\sigma^2(X) \\ &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right)dtd\tau. \end{aligned} \tag{15}$$

By the modulus property, we have

$$\begin{aligned} 0 &\leqslant (b-E(X))(E(X)-a)-\sigma^2(X) \\ &= \frac{1}{b-a}\left|\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right)dtd\tau\right| \\ &\leqslant \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t f(u)\,du\right|dtd\tau =: M. \end{aligned} \tag{16}$$

If $f\in L_\infty[a,b]$, then we can write,

$$\left|\int_\tau^t f(u)\,du\right| \leqslant |t-\tau|\,\|f\|_\infty$$

for all $t,\tau\in[a,b]$, and so

$$\begin{aligned} M &\leqslant \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|\,\|f\|_\infty\,dtd\tau \\ &= \frac{\|f\|_\infty}{b-a}\int_a^b\int_a^b (t-\tau)^2\,dtd\tau = \frac{\|f\|_\infty (b-a)^3}{6}. \end{aligned}$$

For the second part, we apply Hölder's integral inequality to write:

$$\begin{aligned} \left|\int_\tau^t f(u)\,du\right| &\leqslant \left|\int_\tau^t du\right|^{\frac{1}{q}}\left|\int_\tau^t f^p(u)\,du\right|^{\frac{1}{p}} \leqslant |t-\tau|^{\frac{1}{q}}\left(\int_a^b f^p(u)\,du\right)^{\frac{1}{p}} \\ &= |t-\tau|^{\frac{1}{q}}\,\|f\|_p, \end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

In addition,

$$\begin{aligned} M &\leqslant \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|^{\frac{1}{q}}\,\|f\|_p\,dtd\tau \\ &= \frac{\|f\|_p}{b-a}\int_a^b\left[\int_a^t (t-\tau)^{1+\frac{1}{q}}\,d\tau + \int_t^b (\tau-t)^{1+\frac{1}{q}}\,d\tau\right]dt \\ &= \frac{2\,\|f\|_p\,(b-a)^{2+\frac{1}{q}}}{\left(2+\frac{1}{q}\right)\left(3+\frac{1}{q}\right)} \end{aligned}$$

and the second inequality in (14) is proved.

REMARK 3. Inequality (2) and Theorem 2 provide the same bound for the quantity $[b-E(X)][E(X)-a]-\sigma^2(X)$ which is $\frac{(b-a)^3}{6}\|f\|_\infty$ if $f\in L_\infty[a,b]$. A computer simulation using Maple 6 shows that the second bound in (2) is better than the corresponding bound in Theorem 2, but we don't have an analytic proof of this.

Using the Cauchy-Buniakowsky-Schwartz inequality, we have the following inequality.

THEOREM 3. *If X and F are as in Lemma 1, then,*

$$\begin{aligned} 0 &\leqslant (b-E(X))(E(X)-a)-\sigma^2(X) \\ &\leqslant \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\|F\|_2^2-(b-E(X))^2\right]^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Proof. Using the Cauchy-Buniakowsky-Schwartz integral inequality for double integrals,

$$\begin{aligned} &\left|\int_a^b\int_a^b (t-\tau)(F(\tau)-F(t))\,dtd\tau\right| \\ &\leqslant \left(\int_a^b\int_a^b (t-\tau)^2\,dtd\tau\right)^{\frac{1}{2}}\left(\int_a^b\int_a^b (F(t)-F(\tau))^2\,dtd\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{18}$$

However,

$$\begin{aligned} \int_a^b\int_a^b (t-\tau)^2\,dtd\tau &= \frac{(b-a)^4}{6}, \\ \int_a^b\int_a^b (F(\tau)-F(t))^2\,dtd\tau &= 2\left[(b-a)\int_a^b F^2(t)\,dt-\left(\int_a^b F(t)\,dt\right)^2\right] \\ &= 2\left[(b-a)\|F\|_2^2-(b-E(X))^2\right] \end{aligned}$$

and, by (18),

$$\left|\int_a^b\int_a^b (t-\tau)(F(\tau)-F(t))\,d\tau dt\right| \leqslant \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\|F\|_2^2-(b-E(X))^2\right]^{\frac{1}{2}}$$

and the inequality in (17) is proved.

If it is assumed that the mapping f is convex on $[a,b]$, then the following result can be obtained.

THEOREM 4. *Assume that the p.d.f., $f:[a,b]\to(0,\infty)$ is convex. Then we have the inequality*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2 f\left(\frac{t+\tau}{2}\right)d\tau dt \\ &\leqslant [b-E(X)][E(X)-a]-\sigma^2(X) \\ &\leqslant \frac{(b-a)^2}{3}+\sigma^2(X)-(b-E(X))(E(X)-a). \end{aligned} \tag{19}$$

Proof. Using the Hermite-Hadamard inequality,

$$f\left(\frac{t+\tau}{2}\right) \leqslant \frac{\int_t^\tau f(u)\,du}{\tau - t} \leqslant \frac{f(t)+f(\tau)}{2} \tag{20}$$

for all $t, \tau \in [a,b]$, $t \neq \tau$, we have

$$(t-\tau)^2 f\left(\frac{t+\tau}{2}\right) \leqslant (t-\tau)(F(t)-F(\tau)) \leqslant \frac{f(t)+f(\tau)}{2}(t-\tau)^2 \tag{21}$$

for all $t, \tau \in [a,b]$.

Integrating (21) on $[a,b]^2$ and using the representation (6), gives:

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2 f\left(\frac{t+\tau}{2}\right) dt d\tau \\
&\qquad \leqslant \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\, dt d\tau \\
&\qquad = [b-E(X)][E(X)-a]-\sigma^2(X) \\
&\qquad \leqslant \frac{1}{b-a}\int_a^b\int_a^b \frac{f(t)+f(\tau)}{2}(t-\tau)^2\, dt d\tau.
\end{aligned} \tag{22}$$

Now

$$\begin{aligned}
&\int_a^b\int_a^b (t-\tau)^2\left[\frac{f(t)+f(\tau)}{2}\right] dt d\tau \\
&\quad = \int_a^b\int_a^b (t-\tau)^2 f(t)\, d\tau dt = \int_a^b \left[\int_a^b (t-\tau)^2 d\tau\right] f(t)\, dt \\
&\quad = \int_a^b \left[\frac{(b-t)^3+(t-a)^3}{3}\right] f(t)\, dt \\
&\quad = \frac{(b-a)}{3}\int_a^b \left[(b-t)^2-(b-t)(t-a)+(t-a)^2\right] f(t)\, dt \\
&\quad = \frac{b-a}{3}\int_a^b \left[(b-a)^2-3(b-t)(t-a)\right] f(t)\, dt \\
&\quad = \frac{(b-a)^3}{3}-(b-a)\int_a^b (b-t)(t-a) f(t)\, dt \\
&\quad = \frac{(b-a)^3}{3}+(b-a)\left[\sigma^2(X)-(b-E(X))(E(X)-a)\right]
\end{aligned} \tag{23}$$

on using the identity (3) (see [11]).

Hence, from (23) and the above,

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2\left[\frac{f(t)+f(\tau)}{2}\right] dt d\tau \\
&\qquad = \frac{(b-a)^2}{3}+\left[\sigma^2(X)-(b-E(X))(E(X)-a)\right],
\end{aligned}$$

and the second part of (19) is proved.

REMARK 4. The second inequality in (19) is equivalent to:

$$[b-E(X)][E(X)-a] \leqslant \sigma^2(X)+\frac{1}{6}(b-a)^2. \tag{24}$$

REMARK 5. For $b-a<\frac{1}{\sqrt{3}}$, then the result of Theorem 5 is better than that of Theorem 4. For $b-a>\frac{1}{\sqrt{3}}$, the opposite applies. It must be remembered that Theorem 4 relies on f being convex whereas Theorem 3 does not.

The following representation for the absolutely continuous p.d.f., $f:[a,b]\to\mathbb{R}$ holds.

LEMMA 2. *Let X be a random variable having the p.d.f., $f:[a,b]\to\mathbb{R}$ absolutely continuous on $[a,b]$. Then we have*

$$\begin{aligned}\sigma^2(X) &= (b-E(X))(E(X)-a)-\frac{(b-a)^2}{6}\\ &+ \frac{1}{2(b-a)}\int_a^b\int_a^b(t-\tau)\left(\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau. \end{aligned}\tag{25}$$

Proof. We use the following identity which holds for the absolutely continuous mapping $g:[a,b]\to\mathbb{R}$

$$\int_a^b g(u)\,du=\frac{g(a)+g(b)}{2}(b-a)-\int_a^b\left(u-\frac{a+b}{2}\right)g'(u)\,du, \tag{26}$$

can be easily proven by using the integration by parts formula.

We know that

$$\begin{aligned}&(E(X)-a)(b-E(X))-\sigma^2(X)\\ &=\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)\int_\tau^t f(u)\,du\,dtd\tau\\ &=\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)\left[\frac{f(t)+f(\tau)}{2}(t-\tau)-\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right]dtd\tau\\ &=\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)^2\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau\\ &\quad-\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)\left(\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau. \end{aligned}\tag{27}$$

However, observe that (see the proof of Theorem 4)

$$\begin{aligned}L &:= \frac{1}{b-a}\int_a^b\int_a^b(t-\tau)^2\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau\\ &= \sigma^2(X)+\frac{1}{3}\left[(E(X)-b)^2-(E(X)-a)(b-E(X))+(E(X)-a)^2\right]. \end{aligned}$$

Using (27), we have

$$\begin{aligned}(E(X)-a)(b-E(X))-\sigma^2(X) &= \sigma^2(X)+\frac{1}{3}\left[(E(X)-b)^2-(E(X)-a)(b-E(X))+(E(X)-a)^2\right]\\ &-\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)\left(\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau,\end{aligned}$$

which is clearly equivalent to (25).

Using Lemma 2, we are able to obtain the following bounds.

THEOREM 5. *Assume that f is as in Lemma 2, then we have the inequality*

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \leqslant \begin{cases}\frac{\|f'\|_\infty}{80}(b-a)^4 & \text{if } f'\in L_\infty[a,b];\\ \frac{q^2\|f'\|_p}{2(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}(b-a)^{3+\frac{1}{q}} & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f'\|_1}{24}(b-a)^3. & \end{cases} \tag{28}$$

Proof. Using the equality (25), we may write

$$\begin{aligned}&\left|\sigma^2(X)-(b-E(X))(E(X)-a)+\frac{(b-a)^2}{6}\right|\\ &\quad\leqslant\frac{1}{2(b-a)}\int_a^b\int_a^b|t-\tau|\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right|dtd\tau:=N.\end{aligned}\tag{29}$$

Now, it may be easily shown that,

$$\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right|\leqslant\|f'\|_\infty\frac{(t-\tau)^2}{4}$$

for all $t,\tau\in[a,b]$.

Also, by Hölder's integral inequality, we may write

$$\begin{aligned}\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| &\leqslant \left|\int_\tau^t|f'(u)|^p\,du\right|^{\frac{1}{p}}\left|\int_\tau^t\left|u-\frac{t+\tau}{2}\right|^q du\right|^{\frac{1}{q}}\\ &\leqslant \|f'\|_p\frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\end{aligned}$$

for all $t,\tau\in[a,b]$, and further,

$$\begin{aligned}\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| &\leqslant \sup_{u\in[t,\tau]}\left|u-\frac{t+\tau}{2}\right|\left|\int_\tau^t|f'(u)|\,du\right| \\ &\leqslant \frac{|t-\tau|}{2}\|f'\|_1 .\end{aligned}$$

Consequently,

$$\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right|\leqslant\begin{cases}\|f'\|_\infty\frac{(t-\tau)^2}{4} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_p\frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} & \text{if } f'\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f'\|_1\frac{|t-\tau|}{2} & \text{if } f'\in L_1[a,b].\end{cases} \tag{30}$$

Using (30), we may write, from (29), for f' belonging to the obvious Lebesgue space $L_p[a,b]$, $p\geqslant 1$,

$$N\leqslant\begin{cases}\frac{\|f'\|_\infty}{8(b-a)}\int_a^b\int_a^b|t-\tau|^3\,dtd\tau, \\ \frac{\|f'\|_p}{4(q+1)^{\frac{1}{q}}(b-a)}\int_a^b\int_a^b|t-\tau|^{2+\frac{1}{q}}\,dtd\tau, \\ \frac{\|f'\|_1}{4(b-a)}\int_a^b\int_a^b(t-\tau)^2\,dtd\tau.\end{cases} \tag{31}$$

Now, since some straight forward algebra shows that

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^3\,dtd\tau &= \int_a^b\left[\int_a^t(t-\tau)^3\,d\tau+\int_t^b(\tau-t)^3\,d\tau\right]dt=\frac{(b-a)^5}{10}, \\ \int_a^b\int_a^b|t-\tau|^{2+\frac{1}{q}}\,dtd\tau &= \int_a^b\left[\int_a^t(t-\tau)^{2+\frac{1}{q}}\,d\tau+\int_t^b(\tau-t)^{2+\frac{1}{q}}\,d\tau\right]dt \\ &= \frac{2q^2(b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)}\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_a^b(t-\tau)^2\,dtd\tau &= \int_a^b\left[\int_a^t(t-\tau)^2\,d\tau+\int_t^b(\tau-t)^2\,d\tau\right]dt \\ &= \frac{(b-a)^4}{6},\end{aligned}$$

we obtain the desired inequality (28) from using (31) and (29).

The following representation for the mappings whose derivatives are absolutely continuous on $[a,b]$ also holds.

LEMMA 3. *Let X be a random variable having the p.d.f. $f:[a,b]\to\mathbb{R}$ and with the property that $f':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. We have*

$$\sigma^2(X) = (b-E(X))(E(X)-a) - \frac{(b-a)^2}{6} + \frac{1}{4(b-a)}\int_a^b\int_a^b (t-\tau)\int_\tau^t (t-u)(u-\tau)f''(u)\,du\,dt\,d\tau. \tag{32}$$

Proof. We use the following identity which holds for the mappings g whose derivatives are absolutely continuous:

$$\int_a^b g(u)\,du = \frac{g(a)+g(b)}{2}(b-a) - \frac{1}{2}\int_a^b (b-u)(u-a)g''(u)\,du \tag{33}$$

and can easily be proven by using the integration by parts formula twice.

We know that

$$(b-E(X))(E(X)-a) - \sigma^2(X) = \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\int_t^\tau f(u)\,du\,dt\,d\tau$$

and then, using the representation (33) writen for f instead of g, and proceeding as in the proof of Lemma 2, we end up with the identity (32).

Using the representation of Lemma 3, we are able to obtain the following bounds.

THEOREM 6. *Assume that f is as in Lemma 3. Then we have the inequality*

$$\left|[b-E(X)][E(X)-a] - \sigma^2(X) - \frac{(b-a)^2}{6}\right| \leqslant \begin{cases} \frac{\|f''\|_\infty}{360}(b-a)^5 & \text{if } f''\in L_\infty[a,b] \\ \frac{\|f''\|_q}{2(4p+1)(5p+1)}[B(p+1,p+1)]^{\frac{1}{p}}(b-a)^{4+\frac{1}{p}} & \text{if } f''\in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{\|f''\|_1}{160}(b-a)^4, \end{cases} \tag{34}$$

where the p-norms are taken on the interval $[a,b]$.

Proof. Using the equality (32), we may write

$$\left|\sigma^2(X) - [b-E(X)][E(X)-a] - \frac{(b-a)^2}{6}\right| \leqslant \frac{1}{4(b-a)}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| dt\,d\tau := K.$$

First of all, let us observe that

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leqslant \|f''\|_{\infty}\left|\int_{\tau}^{t}(t-u)(u-\tau)\,du\right| \leqslant \frac{\|f''\|_{\infty}}{6}|t-\tau|^{3},$$

for all $t,\tau\in[a,b]$.

Further, by Holder's integral inequality, we obtain

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leqslant \|f''\|_{q}\left|\int_{\tau}^{t}|t-u|^{p}|u-\tau|^{p}\,du\right|^{\frac{1}{p}} = \|f''\|_{q}|t-\tau|^{2+\frac{1}{p}}[B(p+1,p+1)]^{\frac{1}{p}}$$

for all $t,\tau\in[a,b]$, where B is the Beta function of Euler and $\frac{1}{p}+\frac{1}{q}=1$; $p>1$.

Also, we have

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leqslant \|f''\|_{1}\max_{u\in[\tau,t]}|(t-u)(u-\tau)| = \frac{|t-\tau|^{2}}{4}\|f''\|_{1}$$

for all $t,\tau\in[a,b]$.

Consequently, we may state the inequality

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leqslant \begin{cases} \frac{\|f''\|_{\infty}}{6}|t-\tau|^{3} & \text{if } f''\in L_{\infty}[a,b]; \\ \|f''\|_{q}[B(p+1,p+1)]^{\frac{1}{p}}|t-\tau|^{2+\frac{1}{p}} & \text{if } f''\in L_{q}[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{|t-\tau|^{2}}{4}\|f''\|_{1}, \end{cases} \tag{35}$$

for all $t,\tau\in[a,b]$.

Using (35) and the definition of K above, we may write

$$K \leqslant \begin{cases} \frac{\|f''\|_{\infty}}{24(b-a)}\int_{a}^{b}\int_{a}^{b}(t-\tau)^{4}\,dt\,d\tau & \text{if } f''\in L_{\infty}[a,b]; \\ \frac{\|f''\|_{q}}{4(b-a)}[B(p+1,p+1)]^{\frac{1}{p}}\int_{a}^{b}\int_{a}^{b}|t-\tau|^{3+\frac{1}{p}}\,dt\,d\tau & \text{if } f''\in L_{q}[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{\|f''\|_{1}}{16(b-a)}\int_{a}^{b}\int_{a}^{b}|t-\tau|^{3}\,dt\,d\tau. \end{cases} \tag{36}$$

Now, since some straight forward algebra shows that

$$\int_a^b \int_a^b (t-\tau)^4 \, dt \, d\tau = \frac{(b-a)^6}{15}.$$

$$\begin{aligned}
\int_a^b \int_a^b |t-\tau|^{3+\frac{1}{p}} \, dt \, d\tau &= \int_a^b \left[\int_a^t (t-\tau)^{3+\frac{1}{p}} \, d\tau + \int_t^b (\tau - t)^{3+\frac{1}{p}} \, d\tau \right] dt \\
&= \int_a^b \left[\frac{(t-a)^{4+\frac{1}{p}} + (b-t)^{4+\frac{1}{p}}}{4+\frac{1}{p}} \right] dt \\
&= \frac{2p^2 (b-a)^{5+\frac{1}{p}}}{(4p+1)(5p+1)}
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b \int_a^b |t-\tau|^3 \, dt \, d\tau &= \int_a^b \left[\int_a^t (t-\tau)^3 \, d\tau + \int_t^b (\tau - t)^3 \, d\tau \right] dt \\
&= \int_a^b \left[\frac{(t-a)^4 + (b-t)^4}{4} \right] dt = \frac{(b-a)^5}{10},
\end{aligned}$$

then by (36), we deduce the desired inequality (34).

REFERENCES

[1] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.

[2] M. MATIĆ, J. E. PEČARIĆ AND N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Ineq. & Appl.*, **2**(3) (1999), 343–361.

[3] P. CERONE AND S. S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, *RGMIA Res. Rep. Coll.* , **2**(4) (1999), Article 8. `http://rgmia.vu.edu.au/v2n4.html`

[4] P. CERONE AND S. S. DRAGOMIR, On some inequalities for the expectation and variance, *Korean J. Comput. & Appl. Math.,* **8**(2) (2001), 357–380.

[5] N. S. BARNETT AND S. S. DRAGOMIR, An inequality of Ostrowski's type for cumulative distribution functions, *Kyungpook Math. J.,* **39**(2) (1999), 303–311. *RGMIA Research Rep. Coll.*, 1,**1**(1998), Article 1., `http://rgmia.vu.edu.au/v1n1.html`

[6] S. S. DRAGOMIR, New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications, *Math. Ineq. & Appl.*, **2**(2) (1999), 183–194.

[7] S. S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74–82.

[8] S. S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43–47.

[9] S. S. DRAGOMIR, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 145–156.

[10] S. S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397–415.

[11] N. S. BARNETT AND S. S. DRAGOMIR, Some elementary inequalities for the expectation and variance of a random variable whose PDF is defined on a finite interval, *accepted, Theory Ineq. & Appl.,* Ed. S. S. Dragomir, Y. J. Cho and Y. J. Kim, Nova Science Publishers (in press). *RGMIA Re. Rep. Coll.*, **2**(7) (1999), Article 12., `http://rgmia.vu.edu.au/v2n7.html`

[12] N. S. BARNETT, P. CERONE, S. S. DRAGOMIR AND J. ROUMELIOTIS, Some inequalities for the dispersion of a random variable whose PDF is defined on a finite interval, *J. Ineq. Pure & Appl. Math.,* **2***(1) (2001), Article 1. RGMIA Re. Rep. Coll.*, **2**(7) (1999), Article 6., http://rgmia.vu.edu.au/v2n7.html

[13] S. S. DRAGOMIR, Some improvement of Čebyšev's inequality for isotonic functionals, *Atti. Se. Mat. Fasc. Univ. Modena*, **41** (1993), 473–481.

(Received January 18, 2000)

N. S. Barnett
School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
Victoria 8001, Australia
e-mail: neil@matilda.vu.edu.au
URL: http://sci.vu.edu.au/staff/neilb.html

P. Cerone
e-mail: pc@matilda.vu.edu.au
URL: http://sci.vu.edu.au/staff/peterc.html

S. S. Dragomir
e-mail: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html