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ON A NEW INEQUALITY SIMILAR TO HARDY–HILBERT'S INEQUALITY

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Abstract. In this paper, a new inequality similar to Hardy-Hilbert's inequality with a best constant factor is given. As applications, we consider its equivalent form and their associated integral inequalities.

1. Introduction

If $a_n, b_n \geqslant 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the famous Hardy-Hilbert's inequality (see Hardy et al. [1]) is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is important in analisys and its applications (see Mitrinovic et al. [2]). Recently, Yang, Gao and Debnath [3,4,5] gave (1.1) some strengthened versions. By introducing a parameter, Yang [6] gave an extension of (1.1) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda \sin(\pi/p)} \left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\lambda \sin(\pi/p)}$ ($0 < \lambda \leqslant \min\{p,q\}$) is the best possible. For $p = q = 2$ in (1.1), Yang [7] gave a new extensions as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^\lambda} < \frac{1}{(AB)^{\lambda/2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2\right\}^{\frac{1}{2}}, \tag{1.3}$$

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where the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)/(AB)^{\frac{\lambda}{2}}$ $(0<\lambda\leqslant 2,\ A,B>0,\ B(u,v)$ is the β function) is the best possible. By introducing a nonnegative and homogeneous function of degree $-t$ $(t>0)$, Kuang and Debnath [8] gave (1.1) some new generalizations.

The major objective of this paper is to build a new inequality with a best constant factor similar to (1.1), which is related the double series form as

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln m+\ln n}=\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn}.$$

For this, we must estimate the weight coefficient of the form

$$\omega_r(n)=\sum_{m=2}^{\infty}\frac{1}{m\ln mn}\left(\frac{\ln n}{\ln m}\right)^{\frac{1}{r}}\quad (r=p,\ q>1,\ n\in\mathbf{N}\setminus\{1\}). \tag{1.4}$$

2. Lemma and main results

LEMMA 2.1. *For* $0<\varepsilon<q-1$ *(* $q>1$ *), we have*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln mn}\cdot\frac{1}{m(\ln m)^{(1+\varepsilon)/p}}\cdot\frac{1}{n(\ln n)^{(1+\varepsilon)/q}}>\frac{1}{\varepsilon}\left[\frac{\pi}{\sin(\pi/p)}+o(1)\right]\ (\varepsilon\to 0^+). \tag{2.1}$$

Proof. Obviously, we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln mn}\cdot\frac{1}{m(\ln m)^{(1+\varepsilon)/p}}\cdot\frac{1}{n(\ln n)^{(1+\varepsilon)/q}}\\&\qquad>\int_e^{\infty}\int_e^{\infty}\frac{1}{\ln xy}\cdot\frac{1}{x(\ln x)^{(1+\varepsilon)/p}}\cdot\frac{1}{y(\ln y)^{(1+\varepsilon)/q}}dydx.\end{aligned} \tag{2.2}$$

Setting $u=\dfrac{\ln y}{\ln x}$ in the following, for $x\geqslant e$ and $0<\varepsilon<q-1$, we obtain

$$\begin{aligned}\int_e^{\infty}\frac{1}{\ln xy}\cdot\frac{1}{y(\ln y)^{(1+\varepsilon)/q}}dy&=\frac{1}{(\ln x)^{(1+\varepsilon)/q}}\int_{1/\ln x}^{\infty}\frac{1}{(1+u)}\cdot\frac{1}{u^{(1+\varepsilon)/q}}du\\&=\frac{1}{(\ln x)^{(1+\varepsilon)/q}}\left[\int_0^{\infty}\frac{1}{(1+u)}\cdot\frac{1}{u^{(1+\varepsilon)/q}}du-\int_0^{1/\ln x}\frac{1}{(1+u)}\cdot\frac{1}{u^{(1+\varepsilon)/q}}du\right]\\&>\frac{1}{(\ln x)^{(1+\varepsilon)/q}}\left[\int_0^{\infty}\frac{1}{(1+u)}\cdot\frac{1}{u^{(1+\varepsilon)/q}}du-\int_0^{1/\ln x}\frac{1}{u^{(1+\varepsilon)/q}}du\right]\\&=\frac{1}{(\ln x)^{(1+\varepsilon)/q}}\left[\frac{\pi}{\sin(\pi/p)}+o(1)\right]-\frac{q}{(q-1-\varepsilon)}\cdot\frac{1}{\ln x}.\end{aligned}$$

Hence we find

$$\begin{aligned}
&\int_e^\infty \int_e^\infty \frac{1}{\ln xy} \cdot \frac{1}{x(\ln x)^{(1+\varepsilon)/p}} \cdot \frac{1}{y(\ln y)^{(1+\varepsilon)/q}} dydx \\
&\quad > \int_e^\infty \frac{1}{x(\ln x)^{(1+\varepsilon)}} \left[\frac{\pi}{\sin(\pi/p)} + o(1)\right] dx - \frac{q}{(q-1-\varepsilon)} \int_e^\infty \frac{1}{x(\ln x)^{1+(1+\varepsilon)/p}} dx \\
&\quad = \left[\frac{\pi}{\sin(\pi/p)} + o(1)\right] \frac{1}{\varepsilon} - \frac{qp}{(q-1-\varepsilon)(1+\varepsilon)} \ (\varepsilon \to 0^+). \qquad (2.3)
\end{aligned}$$

By (2.2) and (2.3), it follows that (2.1) is valid. The lemma is proved.

THEOREM 2.1. *If* $a_n, b_n \geqslant 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \sum_{n=2}^\infty n^{p-1} a_n^p < \infty$, $0 < \sum_{n=2}^\infty n^{q-1} b_n^q < \infty$, *then we have*

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=2}^\infty n^{p-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^\infty n^{q-1} b_n^q\right\}^{\frac{1}{q}}, \qquad (2.4)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ *is the best possible. In particular, for* $p = q = 2$, *we have*

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{\sum_{n=2}^\infty n a_n^2 \sum_{n=2}^\infty n b_n^2\right\}^{\frac{1}{2}}. \qquad (2.5)$$

Proof. By Hölder's inequality and (1.4), we have

$$\begin{aligned}
&\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln mn} \\
&= \sum_{n=2}^\infty \sum_{m=2}^\infty \left[\frac{a_m}{(\ln mn)^{1/p}} \left(\frac{\ln m}{\ln n}\right)^{\frac{1}{pq}} \left(\frac{m^{1/q}}{n^{1/p}}\right)\right] \left[\frac{b_n}{(\ln mn)^{1/q}} \left(\frac{\ln n}{\ln m}\right)^{\frac{1}{pq}} \left(\frac{n^{1/p}}{m^{1/q}}\right)\right] \\
&\leqslant \left\{\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{1}{\ln mn} \left(\frac{\ln m}{\ln n}\right)^{\frac{1}{q}} \left(\frac{m^{p-1}}{n}\right) a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{1}{\ln mn} \left(\frac{\ln n}{\ln m}\right)^{\frac{1}{p}} \left(\frac{n^{q-1}}{m}\right) b_n^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=2}^\infty \omega_q(m) m^{p-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^\infty \omega_p(n) n^{q-1} b_n^q\right\}^{\frac{1}{q}}. \qquad (2.6)
\end{aligned}$$

For $r = p, q$ and $n \geqslant 2$ in (1.4), setting $u = \frac{\ln x}{\ln n}$ in the following integral, we find

$$\omega_r(n) < \int_1^\infty \frac{1}{x \ln nx} \left(\frac{\ln n}{\ln x}\right)^{\frac{1}{r}} dx = \int_0^\infty \frac{1}{(1+u)u^{1/r}} du = \frac{\pi}{\sin \pi(1 - \frac{1}{r})}. \qquad (2.7)$$

In view of $\sin(\pi/p) = \sin(\pi/q)$, by (2.6) and (2.7), we have (2.4).

For $0 < \varepsilon < q-1$, setting $\tilde{a}_n$ and $\tilde{b}_n$ as

$$\tilde{a}_m = \frac{1}{m(\ln m)^{(1+\varepsilon)/p}}, \ \tilde{b}_n = \frac{1}{n(\ln n)^{(1+\varepsilon)/q}}, \text{ for } m, n \in \mathbf{N} \setminus \{1\},$$

then we have

$$\left\{\sum_{n=2}^{\infty} n^{p-1}\tilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}\tilde{b}_n^q\right\}^{\frac{1}{q}} = \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1+\varepsilon}}$$
$$= \frac{1}{2(\ln 2)^{1+\varepsilon}} + \frac{1}{3(\ln 3)^{1+\varepsilon}} + \sum_{n=4}^{\infty}\frac{1}{n(\ln n)^{1+\varepsilon}}$$
$$< \frac{1}{2(\ln 2)^{1+\varepsilon}} + \frac{1}{3(\ln 3)^{1+\varepsilon}} + \int_e^{\infty}\frac{1}{x(\ln x)^{1+\varepsilon}}dx$$
$$= \frac{1}{2(\ln 2)^{1+\varepsilon}} + \frac{1}{3(\ln 3)^{1+\varepsilon}} + \frac{1}{\varepsilon} = \frac{1}{\varepsilon}(1+o(1)) \ \ (\varepsilon \to 0^+). \tag{2.8}$$

If the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (2.4) is not the best possible, then there exists a positive number $K < \frac{\pi}{\sin(\pi/p)}$, such that (2.4) is valid if we change $\frac{\pi}{\sin(\pi/p)}$ to K. In particular, we have

$$\varepsilon\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{\ln mn} < \varepsilon K\left\{\sum_{n=2}^{\infty} n^{p-1}\tilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}\tilde{b}_n^q\right\}^{\frac{1}{q}}.$$

By (2.1) and (2.8), it follows that

$$\frac{\pi}{\sin(\pi/p)} + o(1) < K(1+o(1)) \ (\varepsilon \to 0^+),$$

and we have $\frac{\pi}{\sin(\pi/p)} \leqslant K$. This contradicts the fact that $K < \frac{\pi}{\sin(\pi/p)}$. Hence the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (2.4) is the best possible. The theorem is proved.

REMARK 1. Inequality (2.4) is more similar to the following Muholland's inequality (see [9]):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{mn\ln mn} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty} n^{-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{-1}b_n^q\right\}^{\frac{1}{q}}. \tag{2.9}$$

THEOREM 2.2. *If* $a_n \geqslant 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \sum_{n=2}^{\infty} n^{p-1}a_n^p < \infty$, *then we have*

$$\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\ln mn}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=2}^{\infty} n^{p-1}a_n^p, \tag{2.10}$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ *is the best possible. Inequalities (2.10) and (2.4) are equivalent. In particular, for* $p = q = 2$, *we have*

$$\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\ln mn}\right)^2 < \pi^2\sum_{n=2}^{\infty} na_n^2. \tag{2.11}$$

Proof. Since $\sum_{n=2}^{\infty} n^{p-1}a_n^p > 0$, there exists $k_0 \geqslant 2$, such that for any $k > k_0$, $\sum_{n=2}^{k} n^{p-1}a_n^p > 0$, and $b_n(k) = \frac{1}{n}\Big(\sum_{m=2}^{k} \frac{a_m}{\ln mn}\Big)^{p-1} > 0$. Then we have

$$0 < \sum_{n=2}^{k} n^{q-1}b_n^q(k) = \sum_{n=2}^{k} \frac{1}{n}\Big(\sum_{m=2}^{k} \frac{a_m}{\ln mn}\Big)^p = \sum_{n=2}^{k}\sum_{m=2}^{k} \frac{a_m b_n(k)}{\ln mn}. \tag{2.12}$$

If we set $\tilde{a}_n = a_n$ and $\tilde{b}_n = b_n(k)$, for $n = 2,3,\ldots,k$; and $\tilde{a}_n = \tilde{b}_n = 0$, for $n > k$, by using (2.4), we may have

$$\begin{aligned}\sum_{n=2}^{k}\sum_{m=2}^{k} \frac{a_m b_n(k)}{\ln mn} &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{\ln mn}\\ &< \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=2}^{\infty} n^{p-1}\tilde{a}_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=2}^{\infty} n^{q-1}\tilde{b}_n^q\Big\}^{\frac{1}{q}}\\ &= \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=2}^{k} n^{p-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=2}^{k} n^{q-1}b_n^q(k)\Big\}^{\frac{1}{q}}.\end{aligned}$$

Hence by (2.12), we have

$$0 < \sum_{n=2}^{k} n^{q-1}b_n^q(k) = \sum_{n=2}^{k} \frac{1}{n}\Big(\sum_{m=2}^{k} \frac{a_m}{\ln mn}\Big)^p < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^p \sum_{n=2}^{k} n^{p-1}a_n^p.$$

It follows that $0 < \sum_{n=2}^{\infty} n^{q-1}b_n^q(\infty) < \infty$, since $\sum_{n=2}^{\infty} n^{p-1}a_n^p < \infty$. Hence by (2.4), we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n}\Big(\sum_{m=2}^{\infty} \frac{a_m}{\ln mn}\Big)^p &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n(\infty)}{\ln mn}\\ &< \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=2}^{\infty} n^{p-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=2}^{\infty} n^{q-1}b_n^q(\infty)\Big\}^{\frac{1}{q}}\\ &= \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=2}^{\infty} n^{p-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=2}^{\infty} \frac{1}{n}\Big(\sum_{m=2}^{\infty} \frac{a_m}{\ln mn}\Big)^p\Big\}^{\frac{1}{q}}.\end{aligned}$$

By simplification, we have (2.10).

On the other hand, suppose that (2.10) is valid, then by Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n}{\ln mn} &= \sum_{n=2}^{\infty}\Big(\frac{1}{n^{1/p}}\sum_{m=2}^{\infty} \frac{a_m}{\ln mn}\Big)(n^{\frac{1}{p}} b_n)\\ &\leqslant \Big\{\sum_{n=2}^{\infty} \frac{1}{n}\Big(\sum_{m=2}^{\infty} \frac{a_m}{\ln mn}\Big)^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=2}^{\infty} n^{q-1}b_n^q\Big\}^{\frac{1}{q}}.\end{aligned} \tag{2.13}$$

By using (2.10) in (2.13), we have (2.4).

We have showed that inequality (2.10) and (2.4) are equivalent. If the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ in (2.10) is not the best possible, then by (2.13) we may show that the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (2.4) is not the best possible. This is a contradiction.

The theorem is proved.

3. The associated integral inequalities

Define the weight function $\varpi_r(x)$ as

$$\varpi_r(x) = \int_1^\infty \frac{1}{y\ln xy}\left(\frac{\ln x}{\ln y}\right)^{\frac{1}{r}} dy \ (r = p,\ q > 1,\ x \geqslant 1). \tag{3.1}$$

Setting $u = \frac{\ln y}{\ln x}$ in (3.1), we have

$$\varpi_r(x) = \int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{r}} du = \frac{\pi}{\sin\pi(1-\frac{1}{r})} \ (r = p,\ q > 1,\ x \geqslant 1). \tag{3.2}$$

THEOREM 3.1. *If $f,g \geqslant 0$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $0 < \int_1^\infty x^{p-1}f^p(x)dx < \infty$, $0 < \int_1^\infty x^{q-1}g^q(x)dx < \infty$, then we have*

$$\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{\ln xy}dxdy < \frac{\pi}{\sin(\pi/p)}\left\{\int_1^\infty x^{p-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_1^\infty x^{q-1}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{3.3}$$

$$\int_1^\infty \frac{1}{y}\left(\int_1^\infty \frac{f(x)}{\ln xy}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_1^\infty x^{p-1}f^p(x)dx, \tag{3.4}$$

where both the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (3.3) and the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ in (3.4) are the best possible. Inequalities (3.3) and (3.4) are equivalent.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{\ln xy}dxdy \\
&= \int_1^\infty\int_1^\infty \left[\frac{f(x)}{(\ln xy)^{1/p}}\left(\frac{\ln x}{\ln y}\right)^{\frac{1}{pq}}\frac{x^{1/q}}{y^{1/p}}\right]\left[\frac{g(y)}{(\ln xy)^{1/q}}\left(\frac{\ln y}{\ln x}\right)^{\frac{1}{pq}}\frac{y^{1/p}}{x^{1/q}}\right]dxdy \\
&< \left\{\int_1^\infty\int_1^\infty \frac{f^p(x)}{\ln xy}\left(\frac{\ln x}{\ln y}\right)^{\frac{1}{q}}\frac{x^{p-1}}{y}dydx\right\}^{\frac{1}{p}}\left\{\int_1^\infty\int_1^\infty \frac{g^q(y)}{\ln xy}\left(\frac{\ln y}{\ln x}\right)^{\frac{1}{p}}\frac{y^{q-1}}{x}dxdy\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3.5}$$

If (3.5) takes the form of equality, then there exists numbers a and b, such that (cf. [9,p.29])

$$a\frac{f^p(x)}{\ln xy}\left(\frac{\ln x}{\ln y}\right)^{\frac{1}{q}}\frac{x^{p-1}}{y} = b\frac{g^q(y)}{\ln xy}\left(\frac{\ln y}{\ln x}\right)^{\frac{1}{p}}\frac{y^{q-1}}{x} \ \text{a.e. in } (1,\infty)\times(1,\infty).$$

Then we have $ax^{p-1}f^p(x)\ln x = by^{q-1}g^q(y)\ln y$ a.e. in $(1,\infty)\times(1,\infty)$. Hence we have

$$ax^{p-1}f^p(x)\ln x = by^{q-1}g^q(y)\ln y = constant \text{ a.e. in } (1,\infty)\times(1,\infty),$$

which contradicts the facts that $0<\int_1^\infty x^{p-1}f^p(x)dx<\infty$ and $0<\int_1^\infty x^{q-1}g^q(x)dx <\infty$. It follows that (3.5) takes the form of strict inequality, and by (3.1) we have

$$\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{\ln xy}dxdy < \Big\{\int_1^\infty \varpi_q(x)x^{p-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_1^\infty \varpi_p(y)y^{q-1}g^q(y)dy\Big\}^{\frac{1}{q}}.$$

Hence by(3.2), we have (3.3).

For $0<\varepsilon<q-1$, setting $\tilde{f}$, $\tilde{g}$ as

$$\tilde{f}(x)=\tilde{g}(x)=0, \text{ for } x\in[1,e);$$
$$\tilde{f}(x)=\frac{1}{x(\ln x)^{(1+\varepsilon)/p}},\ \tilde{g}(x)=\frac{1}{x(\ln x)^{(1+\varepsilon)/q}}, \text{ for } x\in[e,\infty),$$

then we have

$$\Big\{\int_1^\infty x^{p-1}\tilde{f}(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_1^\infty x^{q-1}\tilde{g}(x)dx\Big\}^{\frac{1}{q}} = \int_e^\infty \frac{1}{x(\ln x)^{(1+\varepsilon)}}dx = \frac{1}{e}. \tag{3.6}$$

If the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (3.3) is not the best possible, then there exists a positive number $k<\frac{\pi}{\sin(\pi/p)}$, such that (3.3) is valid if we change $\frac{\pi}{\sin(\pi/p)}$ to k. In particular, by (2.3), we have

$$\Big[\frac{\pi}{\sin(\pi/p)}+o(1)\Big]\frac{1}{\varepsilon}-\frac{qp}{(q-1-\varepsilon)(1+\varepsilon)} < \int_1^\infty\int_1^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{\ln xy}dxdy < \frac{k}{\varepsilon}\ (\varepsilon\to 0^+).$$

Hence we find $\frac{\pi}{\sin(\pi/p)}\leqslant k$. This contradiction follows that the constant factor $\frac{\pi}{\sin(\pi/p)}$ in (3.3) is the best possible.

Since $\int_1^\infty x^{p-1}f^p(x)dx>0$, then there exists $T_0\geqslant 1$, such that for any $T>T_0$, $\int_1^T x^{p-1}f^p(x)dx>0$, and $g(y,T)=\frac{1}{y}\Big(\int_1^T\frac{f(x)}{\ln xy}dx\Big)^{p-1}>0$ ($y\in(1,\infty)$). By (3.3), we have

$$0<\int_1^T y^{q-1}g^q(y,T)dy=\int_1^T\frac{1}{y}\Big(\int_1^T\frac{f(x)}{\ln xy}\Big)^p dx=\int_1^T\int_1^T\frac{f(x)g(y,T)}{\ln xy}dxdy$$
$$<\frac{\pi}{\sin(\pi/p)}\Big\{\int_1^T x^{p-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_1^T y^{q-1}g^q(y,Y)dy\Big\}^{\frac{1}{q}}. \tag{3.7}$$

Then we find

$$\int_1^T\frac{1}{y}\Big(\int_1^T\frac{f(x)}{\ln xy}\Big)^p dx=\int_1^T y^{q-1}g^q(y,T)dy<\Big[\frac{\pi}{\sin(\pi/p)}\Big]^p\int_1^T x^{p-1}f^p(x)dx. \tag{3.8}$$

It follows that $0 < \int_1^\infty y^{q-1} g^q(y,\infty)dy < \infty$. For $T \to \infty$, still by (3.3), neither (3.7) nor (3.8) takes the form of equality, and we have (3.4).

On the other hand, if (3.4) is valid, then by Hölder's inequality, we have

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{\ln xy} dxdy = \int_1^\infty \left[\frac{1}{y^{1/p}} \int_1^\infty \frac{f(x)}{\ln xy} dx\right] [y^{\frac{1}{p}} g(y)] dy$$
$$\leqslant \left\{\int_1^\infty \frac{1}{y}\left(\int_1^\infty \frac{f(x)}{\ln xy}\right)^p dx\right\}^{\frac{1}{p}} \left\{\int_1^\infty y^{q-1} g^q(y) dy\right\}^{\frac{1}{q}}. \tag{3.9}$$

Hence by (3.4), we have (3.3).

Inequalities (3.3) and (3.4) are equivalent. We may show that the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ in (3.4) is the best possible, by using (3.3) and (3.9).

The theorem is proved.

REMARK 12. Inequality (3.3) realtes to (2.4) with the same best constant factor; so does (3.4) to (2.10). They are all new results.

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