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A NEW CARLSON TYPE INEQUALITY

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(*communicated by Lars-Erik Persson*)

Abstract. Consider a measure space $(X, d\xi)$ on which weight functions v, v_0 and v_1 are defined, and let $\theta \in (0,1)$ and $p, p_0, p_1 \in \mathbb{R}_+$. We investigate the three-weight Carlson type inequality

$$\|fv\|_{L^p(X,d\xi)} \leqslant A\|fv_0\|^{\theta}_{L^{p_0}(X,d\xi)}\|fv_1\|^{1-\theta}_{L^{p_1}(X,d\xi)}$$

to hold for some constant $A < \infty$ and all measurable functions f. A fairly general inequality of this type is proved. This result may be regarded as a generalization and unification of some other recent results of this type in the literature.

1. Introduction

In 1934, F. Carlson [10] showed that if a_n, $n = 1, 2, \ldots$ are non-negative, real numbers, not all zero, then the inequality

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2 \tag{1}$$

holds, and π^2 is the smallest possible constant.

Carlson also noted in the original paper [10] that the integral companion of the inequality (1), namely

$$\left(\int_0^{\infty} f(x)\,dx\right)^4 \leqslant \pi^2 \int_0^{\infty} f^2(x)\,dx \int_0^{\infty} x^2 f^2(x)\,dx \tag{2}$$

holds; here, as well, the constant π^2 is sharp, and equality is attained precisely when f has the form

$$f(x) = \frac{1}{\alpha + \beta x^2}.$$

Various versions of (1) and (2) will be referred to as *Carlson type inequalities*.

The inequalities (1) and (2) have been generalized, discussed and applied in several texts, see e.g. F. I. Andrianov [1], S. Barza [2], S. Barza, V. Burenkov, J. Pečarić and L.-E. Persson [3], S. Barza, J. Pečarić and L.-E. Persson [4], R. Bellman [5], J. I. Bertolo and D. L. Fernandez [8], W. B. Caton [11], R. M. Gabriel [12], G. H. Hardy [14] A.

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Kamaly [15, 16], V. I. Levin [21, 22], G. M. Pigolkin [26], G.-S. Yang and J.-C. Fang [27] and the books D. S. Mitrinović [23] and D. S. Mitrinović, J. Pečarić and A. M. Fink [24], and the references given there. We remark also that Carlson type inequalities are of crucial importance for some moment problems (see B. Kjellberg [17, 18]), some problems in interpolation theory (see N. Ya. Krugljak, L. Maligranda and L.-E. Persson [19], J. Peetre [25] and J. Gustavsson and J. Peetre [13]), topics in absolutely convergent Fourier transforms (see A. Beurling [9]), and for optimal reconstruction of a sampling signal (see J. Bergh [6]).

Let v, v_0 and v_1 denote weights (positive, measurable functions) on a measure space $(X, d\xi)$, and let $\theta \in (0,1)$ and $p, p_0, p_1 \in \mathbb{R}_+$. In this text, we investigate the general Carlson type inequality

$$\|fv\|_{L^p(X,d\xi)} \leqslant A\|fv_0\|^{\theta}_{L^{p_0}(X,d\xi)}\|fv_1\|^{1-\theta}_{L^{p_1}(X,d\xi)}$$

to hold for some $A < \infty$ and all measurable functions f.

We quote below the main result in S. Barza, V. Burenkov, J. Pečarić and L.-E. Persson [3], one of the most general results of Carlson type in the literature.

Let S be a measurable subset of the unit sphere in $\mathbb{R}^n$, and define the infinite cone Ω by

$$\Omega = \left\{ x \in \mathbb{R}^n; 0 < |x| < \infty, \frac{x}{|x|} \in S \right\}.$$

Suppose that the positive, measurable functions w, w_0 and w_1, defined on Ω, are homogeneous of degrees γ, γ_0 and γ_1, respectively. Thus[1]

$$w_*(x) = |x|^{\gamma_*} w_* \left(\frac{x}{|x|} \right).$$

Suppose that $0 < p < p_0, p_1 < \infty$, and fix $\theta \in (0,1)$. Define

$$d_* = \gamma_* + \frac{n}{p_*}$$

and

$$\frac{1}{q} = \frac{1}{p} - \frac{\theta}{p_0} - \frac{1-\theta}{p_1}.$$

THEOREM 1. *(Barza et. al.[3], Theorem 1) The Carlson type inequality*

$$\|fw\|_{L^p(\Omega,dx)} \leqslant A\|fw_0\|^{\theta}_{L^{p_0}(\Omega,dx)}\|fw_1\|^{1-\theta}_{L^{p_1}(\Omega,dx)} \tag{3}$$

holds for some constant A (independent of f) if and only if

$$d = \theta d_0 + (1-\theta)d_1, \tag{4}$$

$$d_0 \neq d_1, \tag{5}$$

[1]Here and hereafter, the subscript $*$ is used as a wildcard for either no index or one of the indices 0 and 1.

and

$$\left\| \frac{w}{w_0^{\theta} w_1^{1-\theta}} \right\|_{L^q(d\sigma)} < \infty, \tag{6}$$

where $d\sigma$ denotes the surface area measure on S. The best constant $\tilde{A}$ in (3) is given by[2]

$$\tilde{A} = \theta^{-\frac{\theta}{p_0}} \theta^{-\frac{1-\theta}{p_1}} (p_0 p_1)^{-\frac{1}{q}} \left(\frac{B\left(\theta \frac{q}{p_0}, (1-\theta)\frac{q}{p_1}\right)}{|d_0 - d_1|} \right)^{1/q} \times \tag{7}$$

$$\times \left(\frac{1}{p} - \frac{1}{q} \right)^{-\frac{1}{q}} \left\| \frac{w}{w_0^{\theta} w_1^{1-\theta}} \right\|_{L^q(d\sigma)},$$

and the sign of equality holds in (3) with $A = \tilde{A}$ if and only if f satisfies

$$|f(x)| = H\tilde{f}(rx)$$

almost everywhere, with some $H \geqslant 0$, $r > 0$, where

$$\tilde{f} = \left(k \frac{w^p}{w_0^{p_0}} \right)^{\frac{1}{p_0 - p}}$$

and k is defined through the implicit relation

$$\left(k^{1/p_0} \frac{w}{w_0} \right)^{r_0} = \left((1-k)^{1/p_1} \frac{w}{w_1} \right)^{r_1}$$

where

$$\frac{1}{r_i} = \frac{1}{p} - \frac{1}{p_i}, \quad i = 0, 1.$$

□

In the present text, we give sufficient conditions on the weights in order for a Carlson type inequality to hold on a general measure space. In Section 2., we state and prove our main theorems, the first of which is a one-dimensional Carlson type inequality (Theorem 2). We also extend the range of parameters p beyond the restrictive condition $p < p_0, p_1$. From Theorem 2, we can easily deduce a corresponding inequality on a product measure space, such as the cone Ω in Theorem 1. Two versions of the two-dimensional inequality are proved (Theorems 3 and 4). An n-dimensional version is included as well (Theorem 5). In Section 3., some corollaries of the main results are discussed, and Section 4. is reserved for remarks and comments that do not appear to fit into the main text.

This work is part of a thesis of the author, presented at Uppsala University. The reader may consult [20] for more details and further applications of the results presented here.

[2] There was a misprint in the original article [3]. The constant indicated here is the correct one.

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2. Main results

Our first theorem is a one-dimensional Carlson type inequality on a general measure space. From Theorem 2, we can deduce a corresponding inequality on a product measure space. The theorems presented in this section have a wide range of applications, some of which are pointed out in Section 3. (see also L. Larsson [20]). The proofs are postponed to the end of this section, in order not to interrupt the discussion of our inequalities of Carlson type.

Given p, p_0, p_1 and θ, we define, once and for all, the parameter q by the relation

$$\frac{1}{q} = \frac{1}{p} - \frac{\theta}{p_0} - \frac{1-\theta}{p_1}. \tag{8}$$

THEOREM 2. *Let $(Z, d\zeta)$ be a measure space on which measurable functions $\beta \geqslant 0$, $\beta_0 > 0$ and $\beta_1 > 0$ are given. Let $p_0, p_1 \in (0,\infty]$, let $\theta \in (0,1)$ and suppose that $p \in (0,\infty]$ is such that*

$$\frac{1}{p} \geqslant \frac{\theta}{p_0} + \frac{1-\theta}{p_1}. \tag{9}$$

For $m \in \mathbb{Z}$, define

$$Z_m = \left\{ z \in Z; 2^m \leqslant \frac{\beta_0(z)}{\beta_1(z)} < 2^{m+1} \right\},$$

and define the sequence $\{\zeta_m\}_{m\in\mathbb{Z}}$ of non-negative numbers by

$$\zeta_m = \zeta(Z_m), \quad m \in \mathbb{Z}.$$

Suppose that for some constant C we have

$$\zeta_m \leqslant C, \quad m \in \mathbb{Z}, \tag{10}$$

and that there exists a number $s \in (0,\infty]$, satisfying

$$0 \leqslant \frac{1}{s} \leqslant \frac{1}{p} - \frac{\theta}{p_0} - \frac{1-\theta}{p_1},$$

for which

$$\frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}} \in L^s(d\zeta). \tag{11}$$

Then there is a constant A such that

$$\|f\beta\|_{L^p(d\zeta)} \leqslant A\|f\beta_0\|^{\theta}_{L^{p_0}(d\zeta)}\|f\beta_1\|^{1-\theta}_{L^{p_1}(d\zeta)} \tag{12}$$

for all measurable functions f *satisfying* $f\beta_i \in L^{p_i}(d\zeta)$, $i = 0,1$. *We can choose* A *to have the form*

$$A = A_0 \left\| \frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}} \right\|_{L^s(d\zeta)}, \tag{13}$$

where A_0 *does not depend on* β_*.

REMARK 1. As the proof of Theorem 2 will show, the condition (10) is not necessary if (11) holds with $s = q$, where q is as defined in (8). However, there are examples showing that (10) is needed if $\beta/\beta_0^{\theta}\beta_1^{1-\theta}$ is in L^s, where $s > q$, see Remark 6.

REMARK 2. The proof will also show that the constant A depends on C as $C^{1/q-1/s}$. This can also be seen by the renormalization $d\zeta \mapsto C\,d\zeta$. The same dependence on C holds in Theorem 3 below.

Figure 1 shows the region for (s^{-1}, p^{-1}) in which we can show a Carlson type inequality.

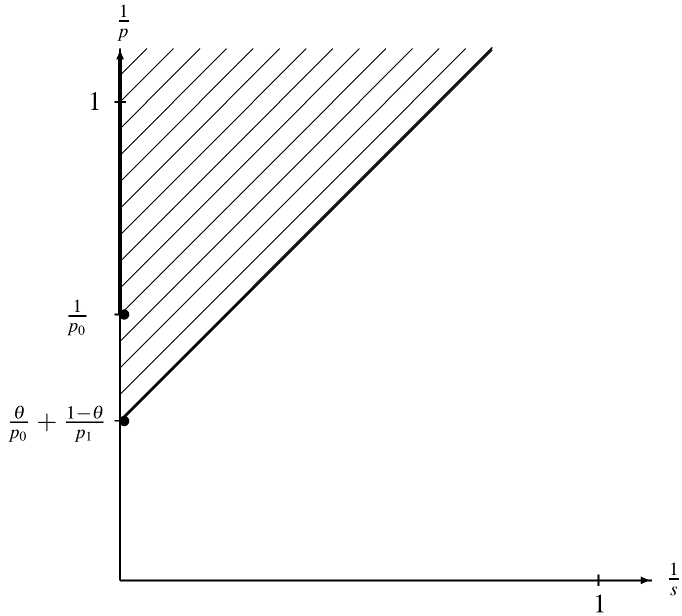

Figure 1. The figure shows the region of admissible parameters in order for a Carlson type inequality to hold.

Once we have the above theorem, a corresponding Carlson type inequality on a product measure space follows easily.

We assume, here and hereafter, that all measure spaces are σ-finite.

THEOREM 3. *Let* $(Y, d\eta)$ *be a measure space. Let weights* $\alpha \geqslant 0$, $\alpha_0 > 0$ *and* $\alpha_1 > 0$ *be given on* Y, *and define* q *by (8). Suppose, in addition to the assumptions in Theorem 2, that*

$$\frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \in L^q(d\eta). \tag{14}$$

Then the Carlson type inequality

$$\|f v\|_{L^p(d\xi)} \leqslant A \|f v_0\|_{L^{p_0}(d\xi)}^\theta \|f v_1\|_{L^{p_1}(d\xi)}^{1-\theta} \tag{15}$$

holds for some constant A *on the product space* $(X, d\xi)$, *where* $X = Y \times Z$ *and* $d\xi = d\eta \times d\zeta$, *and where* $v_* = \alpha_* \beta_*$. A *can be chosen to have the form*

$$A = A_0 \left\| \frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right\|_{L^q(d\eta)} \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^s(d\zeta)}.$$

Theorem 3 is not symmetric, in the sense that we have different conditions on the respective measure spaces. If we impose a condition corresponding to (10) also on the second factor, then we can loosen the condition (14) slightly, and we get a symmetric version of this two-dimensional result.

THEOREM 4. *Suppose that the hypotheses in Theorem 2 hold with* $s = s_Z$. *Suppose, moreover, that there is a constant* C *such that*

$$\eta\left(\{2^m \leqslant \alpha_0/\alpha_1 < 2^{m+1}\}\right) \leqslant C, \quad m \in \mathbb{Z},$$

and that

$$\frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \in L^{s_Y}(d\eta),$$

where

$$0 \leqslant \frac{1}{s_Z} \leqslant \frac{1}{q}, \quad 0 \leqslant \frac{1}{s_Y} \leqslant \frac{1}{q}, \quad \frac{1}{s_Y} + \frac{1}{s_Z} \geqslant \frac{1}{q}. \tag{16}$$

Then there is a constant A *of the form*

$$A = A_0 \left\| \frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right\|_{L^{s_Y}(d\eta)} \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^{s_Z}(d\zeta)}$$

such that (15) holds.

The region (16) is the shaded triangle in Figure 2. In the special case $p_0 = p_1$, it can be shown that we can not go outside this triangle. A proof can be found in Remark 7 in Section 4. The author strongly believes that this is the case also for $p_0 \neq p_1$.

By applying Theorem 3, we get the upper edge of the triangle in Figure 2. Then, letting the measure spaces switch roles, we can apply Theorem 3 to get the right edge. Bilinear interpolation can then be used to get inequality in the whole triangle.

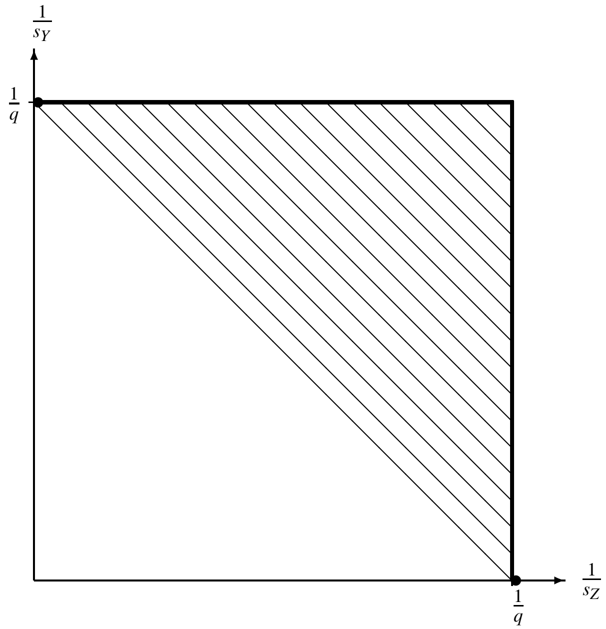

Figure 2. The diagram shows the region for (s_Z^{-1}, s_Y^{-1}) in which a Carlson type inequality holds on a product measure space.

The proof of our last theorem is sketched briefly at the end of this section. It generalizes Theorems 3 and 4 to product spaces with any finite number of factors.

THEOREM 5. *Let $p, p_0, p_1 \in (0,\infty]$ and $\theta \in (0,1)$, and suppose that*

$$\frac{1}{p} \geqslant \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Let $(Z_i, d\zeta^{(i)})$ be measure spaces, on which there are defined measurable functions $\beta^{(i)} \geqslant 0$, $\beta_0^{(i)} > 0$ and $\beta_1^{(i)} > 0$, $i = 1,\dots,n$. Let k be an integer such that $0 \leqslant k \leqslant n$. Define

$$B^{(i)} = \frac{\beta^{(i)}}{(\beta_0^{(i)})^{\theta}(\beta_1^{(i)})^{1-\theta}}, \quad i = 1,\dots,n.$$

Suppose that

$$B^{(i)} \in L^{s_i}(Z_i, d\zeta^{(i)}), \quad i = 1,\dots,k,$$

where

$$0 \leqslant \frac{1}{s_i} \leqslant \frac{1}{q}, \quad \frac{1}{s_1} + \dots + \frac{1}{s_k} \geqslant \frac{k-1}{q}, \tag{17}$$

and

$$B^{(i)} \in L^{q}(Z_i, d\zeta^{(i)}), \quad i = k+1,\dots,n.$$

Suppose, moreover, that for $i = 1,\dots,k$, *the sequences*

$$\{\zeta_m^{(i)}\}_{m\in\mathbb{Z}},$$

defined by

$$\zeta_m^{(i)} = \left\{ z \in Z_i; 2^m \leqslant \frac{\beta_0^{(i)}}{\beta_1^{(i)}} < 2^{m+1} \right\},$$

are bounded. Then there is a constant A *such that the Carlson type inequality*

$$\|f\beta\|_{L^p(Z,d\zeta)} \leqslant A\|f\beta_0\|^{\theta}_{L^{p_0}(Z,d\zeta)}\|f\beta_1\|^{1-\theta}_{L^{p_1}(Z,d\zeta)} \tag{18}$$

holds, where

$$Z = Z_1 \times \cdots \times Z_n,$$
$$d\zeta = d\zeta^{(1)} \times \cdots \times d\zeta^{(n)},$$

and

$$\beta_*(z_1,\dots,z_n) = \beta_*^{(1)}(z_1)\cdots\beta_*^{(n)}(z_n).$$

REMARK 3. If we put $n = k = 1$ in the above theorem, we get Theorem 2. Similarly, with $n = 2$, we get Theorems 3 and 4 by letting $k = 1$ and $k = 2$, respectively. Note also that if we put $n = 1$ and $k = 0$ in Theorem 5, then we get Theorem 2 with $s = q$, without requiring the condition (10) as discussed in Remark 1.

To prove Theorem 2, we employ the following lemma, which is a weak version of Lemma 1 in [3].

LEMMA 1. *Suppose that* $p \leqslant \min\{p_0,p_1\}$. *Define* r_0 *and* r_1 *by*

$$\frac{1}{r_i} = \frac{1}{p} - \frac{1}{p_i}, \quad i = 0,1.$$

If the measurable function $a : Z \to [0,1]$ *is chosen so that*

$$M_0 = \left\| a^{1/p}\frac{\beta}{\beta_0} \right\|_{L^{r_0}(d\zeta)}$$

and

$$M_1 = \left\| (1-a)^{1/p}\frac{\beta}{\beta_1} \right\|_{L^{r_1}(d\zeta)}$$

are both finite, then

$$\|f\beta\|^p_{L^p(d\zeta)} \leqslant M_0^p\|f\beta_0\|^p_{L^{p_0}(d\zeta)} + M_1^p\|f\beta_1\|^p_{L^{p_1}(d\zeta)}. \tag{19}$$

Proof. We write

$$\|f\beta\|^p_{L^p(d\zeta)} = \int |f\beta_0|^p a\left(\frac{\beta}{\beta_0}\right)^p d\zeta + \int |f\beta_1|^p(1-a)\left(\frac{\beta}{\beta_1}\right)^p d\zeta$$

and apply Hölder's inequality with exponents p_0/p and $p_0/(p_0-p)$ in the first integral, and p_1/p and $p_1/(p_1-p)$ in the second. This gives the desired result.

The idea of the proof of Theorem 2 is to define the function a in the above lemma in such a way that we can dominate the M_i by suitable powers of

$$\frac{\|f\beta_1\|_{L^{p_1}(d\zeta)}}{\|f\beta_0\|_{L^{p_0}(d\zeta)}},$$

and in this way get the multiplicative inequality (12) from the additive (19). This will give us a Carlson type inequality in the case $s=\infty$ and $p \leqslant \min\{p_0,p_1\}$. We then prove the result for p satisfying

$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1},$$

and use an interpolation argument to get the desired inequality for intermediate p. Then, the inequality is proved when s satisfies

$$\frac{1}{s}=\frac{1}{p}-\frac{\theta}{p_0}-\frac{1-\theta}{p_1}.$$

A similar interpolation argument is then used to conclude that inequality holds under the correct conditions on the weights for all points $\left(\frac{1}{s},\frac{1}{p}\right)$ in the region shown in Figure 1.

Proof of Theorem 2. We may assume without loss of generality that $p_0 \leqslant p_1$. Furthermore, by replacing $|f|$ by $|f|^r$ for some suitable r, p_* by p_*/r, s by s/r, and adjusting the weights correspondingly, we may assume that all exponents are $\geqslant 1$ (this is needed when we apply the Riesz-Thorin interpolation theorem below). Suppose first that (11) holds with $s=\infty$, and assume that $p \leqslant p_0$. Define the sequence $\{a_m\}_{m\in\mathbb{Z}}$ by

$$a_m=\begin{cases}1 & \text{if } m \geqslant m_0,\\ 0 & \text{if } m < m_0,\end{cases}$$

where m_0 is to be specified shortly. Let the function a be equal to a_m on each Z_m. By the definition of the Z_m, we have

$$\frac{\beta}{\beta_0} \leqslant \left\|\frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}}\right\|_{L^\infty(d\zeta)} 2^{-m(1-\theta)}$$

and

$$\frac{\beta}{\beta_1} \leqslant \left\|\frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}}\right\|_{L^\infty(d\zeta)} 2^{(m+1)\theta}$$

on Z_m. Thus

$$\begin{aligned} M_0^{r_0} &= \int_Z \left(a^{1/p} \frac{\beta}{\beta_0} \right)^{r_0} d\zeta \leqslant \\ &\leqslant \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)}^{r_0} \sum_{m \in \mathbb{Z}} \int_{Z_m} a_m^{r_0/p} 2^{-m(1-\theta)r_0} \, d\zeta \leqslant \\ &\leqslant C \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)}^{r_0} \sum_{m=m_0}^{\infty} 2^{-m(1-\theta)r_0} = \\ &= C \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)}^{r_0} \frac{2^{-m_0(1-\theta)r_0}}{1 - 2^{-(1-\theta)r_0}}, \end{aligned}$$

where C is the bound on ζ_m in the statement of the theorem. Similarly,

$$M_1^{r_1} \leqslant C \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)}^{r_1} \frac{2^{m_0 \theta r_1}}{1 - 2^{-\theta r_1}}.$$

If the constants D_0 and D_1 are chosen so that

$$\left(D_0 (1 - 2^{-(1-\theta)r_0}) \right)^{1/(1-\theta)r_0} = 2 \left(D_1 (1 - 2^{-\theta r_1}) \right)^{-1/\theta r_1}, \tag{20}$$

then, given $\delta > 0$, we can find m_0 such that

$$\left(D_1 (1 - 2^{-\theta r_1}) \right)^{-1/\theta r_1} \leqslant \frac{2^{-m_0}}{\delta} \leqslant \left(D_0 (1 - 2^{-(1-\theta)r_0}) \right)^{1/(1-\theta)r_0}$$

or

$$\frac{2^{-m_0(1-\theta)r_0}}{1 - 2^{-(1-\theta)r_0}} \leqslant D_0 \delta^{(1-\theta)r_0}$$

and

$$\frac{2^{m_0 \theta r_1}}{1 - 2^{-\theta r_1}} \leqslant D_1 \delta^{-\theta r_1}.$$

Thus, for any $\delta > 0$, we have

$$M_0 \leqslant C^{1/r_0} \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)} D_0^{1/r_0} \delta^{1-\theta} \tag{21}$$

and

$$M_1 \leqslant C^{1/r_1} \left\| \frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)} D_1^{1/r_1} \delta^{-\theta}. \tag{22}$$

Now, put

$$\delta = K \frac{\|f \beta_1\|_{L^{p_1}(d\zeta)}}{\|f \beta_0\|_{L^{p_0}(d\zeta)}},$$

where

$$K=\left(\frac{\theta}{1-\theta}\right)^{1/p}C^{1/r_1-1/r_0}D_0^{-1/r_0}D_1^{1/r_1}.$$

Then (19), (21) and (22) yield, after simplification

$$\|f\beta\|_{L^p(d\zeta)}\leqslant\frac{C^{\theta/r_0+(1-\theta)/r_1}D_0^{\theta/r_0}D_1^{(1-\theta)/r_1}}{\left(\theta^\theta(1-\theta)^{1-\theta}\right)^{1/p}}\times$$

$$\times\left\|\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}}\right\|_{L^\infty(d\zeta)}\|f\beta_0\|^\theta_{L^{p_0}(d\zeta)}\|f\beta_1\|^{1-\theta}_{L^{p_1}(d\zeta)},$$

which proves the Carlson type inequality (12) in this case. We note that (20) can be written as

$$D_0^{\theta/r_0}D_1^{(1-\theta)/r_1}=\frac{2^{\theta(1-\theta)}}{(1-2^{-(1-\theta)r_0})^{\theta/r_0}(1-2^{-\theta r_1})^{(1-\theta)/r_1}},$$

and

$$\frac{\theta}{r_0}+\frac{1-\theta}{r_1}=\frac{1}{q},$$

so that we can choose

$$A=A_0\left\|\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}}\right\|_{L^\infty(d\zeta)}$$

in (12), where in this case

$$A_0= \tag{23}$$

$$=\bar{A}:=\frac{2^{\theta(1-\theta)}C^{1/q}}{(\theta^\theta(1-\theta)^{1-\theta})^{1/p}(1-2^{-(1-\theta)r_0})^{\theta/r_0}(1-2^{-\theta r_1})^{(1-\theta)/r_1}}.$$

Suppose next that p is chosen so that

$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1}. \tag{24}$$

This can be written as

$$\frac{p\theta}{p_0}+\frac{p(1-\theta)}{p_1}=1,$$

and thus $p_0/p\theta$ and $p_1/p(1-\theta)$ are conjugate exponents. It follows by Hölder's inequality that

$$\int_Z|f\beta|^p\,d\zeta\leqslant\left\|\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}}\right\|^p_{L^\infty(d\zeta)}\int_Z|f\beta_0|^{p\theta}|f\beta_1|^{p(1-\theta)}\,d\zeta\leqslant$$

$$\leqslant\left\|\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}}\right\|^p_{L^\infty(d\zeta)}\left(\int_Z|f\beta_0|^{p_0}\,d\zeta\right)^{p\theta/p_0}\left(\int_Z|f\beta_1|^{p_1}\,d\zeta\right)^{p(1-\theta)/p_1},$$

or

$$\|f\beta\|_{L^p(d\zeta)}\leqslant\left\|\frac{\beta}{\beta_0^\theta\beta_1^{1-\theta}}\right\|_{L^\infty(d\zeta)}\|f\beta_0\|^\theta_{L^{p_0}(d\zeta)}\|f\beta_1\|^{1-\theta}_{L^{p_1}(d\zeta)}.$$

This proves (12) when p satisfies (24).

Consider now a fixed f, and also keep the weights β_i and the exponents p_i, $i = 0, 1$, fixed. Define the linear operator T on $L^\infty(d\zeta)$ by

$$Tb = (f\beta_0^\theta \beta_1^{1-\theta})b.$$

We note that the constant in the cases of the inequality proved so far has the form

$$A = A_0 \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)}$$

(in the latter case, $A_0 = 1$). Let

$$\bar{A_0} = \frac{2^{\theta(1-\theta)} C^{(1-\theta)(1/p_0 - 1/p_1)}}{(\theta^\theta(1-\theta)^{1-\theta})^{1/p_0}(1 - 2^{-\theta p_0 p_1/(p_1 - p_0)})^{(1-\theta)(1/p_0 - 1/p_1)}},$$

that is, $\bar{A_0}$ is $\bar{A}$ as defined in (23) where we have replaced p by p_0. Let $\tilde{p}$ be p as defined in (24). Since the modulus of any $b \in L^\infty$ can be written as

$$|b| = \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}}$$

for the correct choice of β, the previously proved cases of our Carlson type inequality says that the operator T is bounded $L^\infty \to L^{p_0}$, with norm not exceeding $\bar{A_0}\|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta}$, and $L^\infty \to L^{\tilde{p}}$ with norm at most $\|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta}$, respectively. By the Riesz-Thorin theorem, we get boundedness of T also for intermediate p. More precisely, if p_σ is defined by

$$\frac{1}{p_\sigma} = \frac{\sigma}{p_0} + \frac{1-\sigma}{\tilde{p}},$$

where σ is any number in $(0,1)$, then the norm of T as an operator $L^\infty \to L^{p_\sigma}$ is at most

$$(\bar{A_0}\|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta})^\sigma (\|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta})^{1-\sigma} =$$
$$= \bar{A_0}^\sigma \|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta}.$$

In other words,

$$\|Tb\|_{p_\sigma} \leqslant \bar{A_0}^\sigma \|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta} \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)},$$

or, re-translated to our original situation

$$\|f\beta\|_{L^{p_\sigma}(d\zeta)} \leqslant \bar{A_0}^\sigma \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^\infty(d\zeta)} \|f\beta_0\|_{L^{p_0}(d\zeta)}^\theta \|f\beta_1\|_{L^{p_1}(d\zeta)}^{1-\theta}.$$

This concludes the proof in the case $s = \infty$.

Suppose now that p satisfies

$$\frac{1}{p} \geqslant \frac{\theta}{p_0} + \frac{1-\theta}{p_1}. \tag{25}$$

Suppose also that $s = q$, where q is defined by (8). This can be written as

$$\frac{p}{q} + \frac{p\theta}{p_0} + \frac{p(1-\theta)}{p_1} = 1,$$

and hence we can apply Hölder's inequality with three factors, using the exponents

$$\frac{q}{p}, \frac{p_0}{p\theta}, \frac{p_1}{p(1-\theta)},$$

which yields

$$\begin{aligned}
\|f\beta\|^p_{L^p(d\zeta)} &= \int_Z |f\beta|^p \, d\zeta = \\
&= \int_Z \left(\frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}}\right)^p |f\beta_0|^{p\theta} |f\beta_1|^{p(1-\theta)} \, d\zeta \leqslant \\
&\leqslant \left(\int_Z \left(\frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}}\right)^q d\zeta\right)^{p/q} \times \\
&\times \left(\int_Z |f\beta_0|^{p_0} \, d\zeta\right)^{p\theta/p_0} \left(\int_Z |f\beta_1|^{p_1} \, d\zeta\right)^{p(1-\theta)/p_1} = \\
&= \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|^p_{L^q(d\zeta)} \|f\beta_0\|^{p\theta}_{L^{p_0}(d\zeta)} \|f\beta_1\|^{p(1-\theta)}_{L^{p_1}(d\zeta)}.
\end{aligned}$$

As before, we apply the Riesz-Thorin interpolation theorem to get our result for the remaining range of parameters. We now consider T as an operator $L^s \to L^p$. If $0 < \sigma < 1$, let

$$\frac{1}{s_\sigma} = \frac{\sigma}{\infty} + \frac{1-\sigma}{q} = \frac{1-\sigma}{q}.$$

The norm of $T : L^\infty \to L^p$ is at most

$$\bar{A} \|f\beta_0\|^\theta_{p_0} \|f\beta_1\|^{1-\theta}_{p_1},$$

where $\bar{A}$ is given by (23), and $T : L^q \to L^p$ has norm at most

$$\|f\beta_0\|^\theta_{p_0} \|f\beta_1\|^{1-\theta}_{p_1}.$$

The Riesz-Thorin theorem now implies that the norm of T as an operator $L^{s_\sigma} \to L^p$ does not exceed

$$\bar{A}^\sigma \|f\beta_0\|^\theta_{p_0} \|f\beta_1\|^{1-\theta}_{p_1}.$$

We finally note that $\sigma/q = 1/q - 1/s$, and that $\bar{A}$ depends on C as $C^{1/q}$ by (23), and thus $\bar{A}^\sigma$ has a factor

$$C^{1/q-1/s},$$

as indicated in Remark 2. The proof is complete.

Proof of Theorem 3. If $p, p_0, p_1 < \infty$, then the condition (8) can be written as

$$\frac{p}{q} + \frac{p\theta}{p_0} + \frac{p(1-\theta)}{p_1} = 1,$$

and thus we can apply Hölder's inequality with three factors, using the exponents

$$\frac{q}{p}, \frac{p_0}{p\theta}, \frac{p_1}{p(1-\theta)}.$$

Assuming that (12) holds with constant

$$A = \bar{A} = A_0 \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^s(d\zeta)},$$

we thus get, by also using Fubini's theorem

$$\begin{aligned}
\|f v\|_{L^p(d\xi)}^p = \int_Y \alpha^p \int_Z |f\beta|^p \, d\zeta \, d\eta \leqslant \\
&\leqslant \bar{A}^p \int_Y \left(\frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right)^p \cdots \\
&\quad \cdots \left(\int_Z |f\alpha_0\beta_0|^{p_0} \, d\zeta \right)^{p\theta/p_0} \left(\int_Z |f\alpha_1\beta_1|^{p_1} \, d\zeta \right)^{p(1-\theta)/p_1} \, d\eta \leqslant \\
&\leqslant \bar{A}^p \left(\int_Y \left(\frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right)^q d\eta \right)^{p/q} \times \\
&\times \left(\int_Y \int_Z |f\alpha_0\beta_0|^{p_0} \, d\zeta \, d\eta \right)^{p\theta/p_0} \times \\
&\times \left(\int_Y \int_Z |f\alpha_1\beta_1|^{p_1} \, d\zeta \, d\eta \right)^{p(1-\theta)/p_1} = \\
&= \bar{A}^p \left\| \frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right\|_{L^q(d\eta)}^p \|f v_0\|_{L^{p_0}(d\xi)}^{p\theta} \|f v_1\|_{L^{p_1}(d\xi)}^{p(1-\theta)}.
\end{aligned}$$

Taking p th roots, we get (15) with

$$A = A_0 \left\| \frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right\|_{L^q(d\eta)} \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^s(d\zeta)}.$$

If $p = \infty$, then by (8) and (9) we also have $p_0 = p_1 = q = \infty$, and

$$|f v| \leqslant \left\| \frac{\alpha}{\alpha_0^\theta \alpha_1^{1-\theta}} \right\|_{L^\infty(Y)} \left\| \frac{\beta}{\beta_0^\theta \beta_1^{1-\theta}} \right\|_{L^\infty(Z)} |f\alpha_0\beta_0|^\theta |f\alpha_1\beta_1|^{1-\theta}.$$

By taking suprema, the desired result follows. If $p < \infty$ but p_0 or p_1 is infinite, the inequality follows similarly.

Proof of Theorem 4. As in the proof of Theorem 2, we may assume that all exponents are $\geqslant 1$. Theorem 3 applied first as-is, and then with the factors interchanged, gives the result when (s_Z^{-1}, s_Y^{-1}) is situated along two of the edges of the triangle shown in Figure 2. The plan is to use bilinear interpolation to get inequality in the convex hull of the two faces, namely in the whole triangle. Keep everything but α and β fixed. Define the bilinear operator T on

$$L^{s_Y}(Y, d\eta) \times L^{s_Z}(Z, d\zeta)$$

by

$$T(a,b) = (f(\alpha_0\beta_0)^{\theta}(\alpha_1\beta_1)^{1-\theta})ab.$$

Then the inequality (15), with

$$A = A_0 \left\| \frac{\alpha}{\alpha_0^{\theta}\alpha_1^{1-\theta}} \right\|_{s_Y} \left\| \frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}} \right\|_{s_Z},$$

says that T is bounded as an operator

$$L^{s_Y}(d\eta) \times L^{s_Z}(d\zeta) \to L^p(d\xi),$$

where either $s_Y = q$ and $q \leqslant s_Z \leqslant \infty$ or $s_Z = q$ and $q \leqslant s_Y \leqslant \infty$. By applying a multilinear interpolation theorem (see J. Bergh and J. Löfström [7], Theorem 4.4.1), we conclude that whenever (s_Y^{-1}, s_Z^{-1}) is in the triangle (16), we have

$$\|T(a,b)\|_p \leqslant A_0 \|f\alpha_0\beta_0\|_{p_0}^{\theta} \|f\alpha_1\beta_1\|_{p_1}^{1-\theta} \left\| \frac{\alpha}{\alpha_0^{\theta}\alpha_1^{1-\theta}} \right\|_{s_Y} \left\| \frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}} \right\|_{s_Z}$$

for some constant A_0, and this is precisely (15).

Proof of Theorem 5. We note first that if we can get the inequality on the product space made up from the first k factors, then the result will follow by an application of Hölders inequality, just as in the proof of Theorem 3. We thus concentrate of the factors $(Z_i, d\zeta^{(i)})$, $i = 1, \dots, k$. Fix such an index i. Let Y be the product of the other $k-1$ factors. By Theorem 3, we get inequality with $s_i \in [q, \infty]$ and $s_j = q$, $j \neq i$. Repeating this procedure for all $i = 1, \dots, k$, the inequality is seen to hold when $(s_1^{-1}, \dots, s_k^{-1})$ is situated along the lines joining the point $(q^{-1}, \dots, q^{-1})$ to the points $(q^{-1}, \dots, q^{-1}, 0, q^{-1}, \dots, q^{-1})$. These line segments make up a skeleton, whose convex hull is all of the simplex defined in (17). Thus, assuming as above that all exponents are $\geqslant 1$, the result follows by using n-linear interpolation.

3. Applications and further results

In this section, we discuss some of the applications of our results, beginning with Carlson's inequality (1).

(a) Let $Z = \{1, 2, \ldots\}$, and define the measure $d\zeta$ on Z by

$$\zeta(\{k\}) = \frac{1}{k}, \quad k \in Z.$$

Let $\beta(k) = k$, $\beta_0(k) = k^{1/2}$ and $\beta_1(k) = k^{3/2}$. Define the parameters p_* as $p = 1$ and $p_0 = p_1 = 2$, and put $\theta = 1/2$. If $a_k = |f(k)|$, $k \in Z$, we have

$$\int_Z |f\beta|^p \, d\zeta = \sum_{k=1}^{\infty} a_k,$$

$$\int_Z |f\beta_0|^{p_0} \, d\zeta = \sum_{k=1}^{\infty} a_k^2$$

and

$$\int_Z |f\beta_1|^{p_1} \, d\zeta = \sum_{k=1}^{\infty} k^2 a_k^2.$$

The reader can verify that the weights satisfy the requirements (10) and (11) of Theorem 2, with $C = 1$ and $s = \infty$, respectively. Except that we do not get the sharp constant, Theorem 2 now implies (1).

(b) Let us also show how (4), (5) and (6) in Theorem 1 imply (3), although not with the best constant. Let $Z = (0, \infty)$, equipped with the homogeneous measure

$$d\zeta(\rho) = \rho^{-1} \, d\rho,$$

where $d\rho$ denotes Lebesgue measure. Let $Y = S$, where S is the subset of the unit sphere in $\mathbb{R}^n$ defining the cone Ω, and let $d\eta$ be its surface area measure $d\sigma$. Define

$$\alpha_*(s) = w_*(s), \quad s \in S,$$

and put

$$\beta_*(\rho) = \rho^{d_*}, \quad \rho \in (0, \infty).$$

The condition (6) is precisely (14). By (4), it follows that

$$\frac{\beta}{\beta_0^{\theta} \beta_1^{1-\theta}} \equiv 1,$$

so that (11) certainly holds with $s = \infty$. Let $\tau = d_0 - d_1$. Then τ is non-zero, by the assumption (5). Let

$$V(\rho) = \frac{\beta_0(\rho)}{\beta_1(\rho)} = \rho^{\tau}.$$

Then

$$\zeta\left(\{2^m \leqslant V < 2^{m+1}\}\right) \leqslant \frac{1}{|\tau|} \log 2, \quad m \in \mathbb{Z},$$

so (10) is satisfied as well. We note that

$$\begin{aligned}\int_\Omega |f w_*|^{p_*}\,dx &= \int_S \int_0^\infty |f(s,\rho)w_*(s)\rho^{\gamma_*}|^{p_*}\rho^{n-1}\,d\rho\,d\sigma = \\ &= \int_S \int_0^\infty |f(s,\rho)w_*(s)\rho^{d_*}|^{p_*}\rho^{-1}\,d\rho\,d\sigma = \\ &= \int_Y \int_Z |f\alpha_*\beta_*|^{p_*}\,d\zeta\,d\eta = \int_X |f v_*|^{p_*}\,d\xi.\end{aligned}$$

(3) now follows from Theorem 3.

Note that, in Theorem 3, we do not restrict ourselves to $p < p_0, p_1$, but allow a wider range of p. We thus get the following proposition, which does *not* follow from Theorem 1.

PROPOSITION 1. *Suppose that* $0 < \theta < 1$, *and that the positive numbers* p, p_0 *and* p_1 *satisfy*

$$\frac{1}{p} \geqslant \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

If $d = \theta d_0 + (1-\theta)d_1$ *and* $d_0 \neq d_1$, *and if*

$$\frac{w}{w_0^\theta w_1^{1-\theta}} \in L^q,$$

where the d_* *and the* w_* *are as in Theorem 1, then there is a constant* A *such that*

$$\|f w\|_{L^p(\Omega,dx)} \leqslant A\|f w_0\|^{\theta}_{L^{p_0}(\Omega,dx)}\|f w_1\|^{1-\theta}_{L^{p_1}(\Omega,dx)}$$

holds for all measurable functions f.

(c) In the following application, we control the integrability of the Fourier transform of a function by integral norms of the function itself and its derivatives.

Fix an integer $n \geqslant 1$. Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is integrable. We define the Fourier transform of f by

$$\hat{f}(z) = \int_{\mathbb{R}^n} f(x)e^{-iz\cdot x}\,dx, \quad z \in \mathbb{R}^n,$$

where $z \cdot x$ denotes the Euclidian inner product in $\mathbb{R}^n$.

PROPOSITION 2. *Suppose that* $1 < r_0, r_1 \leqslant 2$, *and that* α *is a positive integer satisfying*

$$\alpha > \frac{n}{r_0}. \tag{26}$$

Define

$$\theta = \frac{n/r_1}{\alpha + n\left(\frac{1}{r_1} - \frac{1}{r_0}\right)}.$$

Then there is a constant C such that

$$\|\hat{f}\|_{L^1(\mathbb{R}^n)} \leqslant C\left(\sum_{|\gamma|=\alpha}\|D^\gamma f\|_{L^{r_0}(\mathbb{R}^n)}\right)^\theta \|f\|^{1-\theta}_{L^{r_1}(\mathbb{R}^n)}.$$

Here, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_+$, and standard multi-index notation is used. $D^\gamma f$ denotes the derivative

$$\frac{\partial^{|\gamma|} f}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}.$$

Proof. Define, on $\mathbb{R}^n$, the measure

$$d\zeta = \frac{dz}{|z|^n}$$

and the weights

$$\beta(z) = |z|^n,$$
$$\beta_0(z) = |z|^{n/r_0'} \sum_{|\gamma|=\alpha} |z^\gamma|,$$

and

$$\beta_1(z) = |z|^{n/r_1'}.$$

Put $p = 1$, $p_0 = r_0'$ and $p_1 = r_1'$. Let

$$B(z) = \frac{\beta(z)}{\beta_0^\theta(z)\beta_1^{1-\theta}(z)} =$$

$$= |z|^{n/q}\left(\sum_{|\gamma|=\alpha} |z^\gamma|\right)^{-\theta},$$

where q is defined by

$$\frac{1}{q} = 1 - \frac{\theta}{p_0} - \frac{1-\theta}{p_1}.$$

Then B is homogeneous of degree

$$\frac{n}{q} - \theta\alpha = 0,$$

and thus constant along rays from the origin. In particular, since B is continuous, as is easily seen, B is bounded on $\mathbb{R}^n$. This gives the condition (11) of Theorem 2 with $s = \infty$. Moreover, if V is defined by

$$V(z) = \frac{\beta_0(z)}{\beta_1(z)} =$$

$$= |z|^{n\left(\frac{1}{r_1} - \frac{1}{r_0}\right)} \sum_{|\gamma|=\alpha} |z^\gamma|,$$

then V is homogeneous of degree

$$\tau = \alpha + n\left(\frac{1}{r_1} - \frac{1}{r_0}\right),$$

and $\tau > 0$ by the assumption (26). There is a constant C_0 such that

$$\zeta\left(\{2^m \leqslant V < 2^{m+1}\}\right) \leqslant \frac{C_0}{\tau}, \quad m \in \mathbb{Z},$$

that is, (10) holds. Thus, by Theorem 2, we have

$$\begin{aligned}
&\|\hat{f}\|_{L^1(\mathbb{R}^n)} \leqslant \\
&\leqslant C\left(\int_{\mathbb{R}^n} |\hat{f}(z)\beta_0(z)|^{p_0}\, d\zeta(z)\right)^{\theta/p_0}\left(\int_{\mathbb{R}^n} |\hat{f}(z)\beta_1(z)|^{p_1}\, d\zeta(z)\right)^{(1-\theta)/p_1} = \\
&= C\left(\int_{\mathbb{R}^n}\left|\hat{f}(z)\sum_{|\gamma|=\alpha}|z^\gamma|\right|^{r_0'} dz\right)^{\theta/r_0'}\left(\int_{\mathbb{R}^n}|\hat{f}(z)|^{r_1'}\, dz\right)^{(1-\theta)/r_1'} = \\
&= C\left(\int_{\mathbb{R}^n}\left(\sum_{|\gamma|=\alpha}|\hat{f}(z)z^\gamma|\right)^{r_0'} dz\right)^{\theta/r_0'}\left(\int_{\mathbb{R}^n}|\hat{f}(z)|^{r_1'}\, dz\right)^{(1-\theta)/r_1'} = \\
&= C\left(\int_{\mathbb{R}^n}\left(\sum_{|\gamma|=\alpha}|\widehat{D^\gamma f}(z)|\right)^{r_0'} dz\right)^{\theta/r_0'}\left(\int_{\mathbb{R}^n}|\hat{f}(z)|^{r_1'}\, dz\right)^{(1-\theta)/r_1'} = \\
&= C\left\|\sum_{|\gamma|=\alpha}|\widehat{D^\gamma f}|\right\|_{L^{r_0'}(\mathbb{R}^n)}^{\theta}\|\hat{f}\|_{L^{r_1'}(\mathbb{R}^n)}^{1-\theta} \leqslant \\
&\leqslant C\left(\sum_{|\gamma|=\alpha}\|\widehat{D^\gamma f}\|_{L^{r_0'}(\mathbb{R}^n)}\right)^{\theta}\|\hat{f}\|_{L^{r_1'}(\mathbb{R}^n)}^{1-\theta},
\end{aligned}$$

where we have used the triangle inequality. We can now apply the Hausdorff-Young inequality (this is where we need $1 < r_i \leqslant 2$) to each term of the sum, and to the rightmost norm, and we get

$$\|\hat{f}\|_{L^1(\mathbb{R}^n)} \leqslant C\left(\sum_{|\gamma|=\alpha}\|D^\gamma f\|_{L^{r_0}(\mathbb{R}^n)}\right)^{\theta}\|f\|_{L^{r_1}(\mathbb{R}^n)}^{1-\theta},$$

which is what we wanted.

In analogy with Proposition 2, we get the corresponding discrete version, except that we have to treat the 0th Fourier coefficient with a little extra care.

If $f : \mathbb{T}^n \to \mathbb{C}$ is integrable, where $\mathbb{T}^n$ is the n-dimensional torus

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n,$$

define the mth Fourier coefficient of f by

$$\hat{f}(m) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i m \cdot x}\, dx, \quad m \in \mathbb{Z}^n.$$

$A(\mathbb{T}^n)$ is, by definition, the vector space of all integrable functions having absolutely convergent Fourier series. We equip $A(\mathbb{T}^n)$ with the norm

$$\|f\|_{A(\mathbb{T}^n)} = \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)|,$$

under which it becomes a Banach space.

PROPOSITION 3. *Suppose that* $1 < r_0, r_1 \leqslant 2$*, and that* α *is a positive integer satisfying*

$$\alpha > \frac{n}{r_0}.$$

Define

$$\theta = \frac{n/r_1}{\alpha + n\left(\frac{1}{r_1} - \frac{1}{r_0}\right)}.$$

Then there is a constant C *such that*

$$\|f\|_{A(\mathbb{T}^n)} \leqslant |\hat{f}(0)| + C \left(\sum_{|\gamma| = \alpha} \|D^\gamma f\|_{L^{r_0}(\mathbb{T}^n)} \right)^{\theta} \|f\|_{L^{r_1}(\mathbb{T}^n)}^{1-\theta}. \tag{27}$$

Proof. Assume first that $\hat{f}(0) = 0$. We can then proceed as in the proof of the previous proposition, but define the weights β_* and the functions B and V only on the integer lattice. The general case follows easily from this.

If we let $r_0 = r_1 = r$ in Proposition 3, then

$$\theta = \frac{n}{r\alpha}.$$

If we also keep in mind that $|\hat{f}(0)| \leqslant \|f\|_{L^1(\mathbb{T}^n)}$, we get the following special case.

COROLLARY 1. *(A. Kamaly [15], Theorem 1) Let* $f \in A(\mathbb{T}^n)$ *and* $\hat{f}(0) = 0$*. Let* $1 < r \leqslant 2$*, and let the positive integer* α *be such that* $\alpha > \frac{n}{r}$*. Then we get*

$$\|f\|_{A(\mathbb{T}^n)} \leqslant C \|f\|_r^{1 - \frac{n}{r\alpha}} \left(\sum_{|\gamma| = \alpha} \|D^\gamma f\|_r \right)^{\frac{n}{r\alpha}}.$$

In the case $\hat{f}(0) \neq 0$*, we obtain*

$$\|f\|_{A(\mathbb{T}^n)} \leqslant \|f\|_1 + C\|f\|_r^{1-\frac{n}{r\alpha}} \left(\sum_{|\gamma|=\alpha} \|D^\gamma f\|_r \right)^{\frac{n}{r\alpha}}.$$

The constant C *depends only on* α*,* n *and* r*.*

For more applications of the main result, see [20].

4. Concluding remarks

REMARK 4. Although we lose track of best constants and cases of equality in our very general setting, we can still compare the asymptotic behaviour of constants in the cases where the best constants are known. Consider, for example, the best constant in Theorem 1, given by 7). It has a factor $|d_0 - d_1|^{-1/q}$. Applying Theorem 3, with $s = \infty$, to the case of homogeneous weights on the cone Ω, the constant A has a factor $C^{1/q-1/s} = C^{1/q}$ (see Remark 2), where

$$C = \frac{1}{|d_0 - d_1|} \log 2$$

(see Section 3.). Thus we do, indeed, have the correct asymptotic behaviour of our constant when $|d_0 - d_1| \to 0$.

Consider now what happens if we let $\theta \to 0$ in (7). Since $t^{-t} \to 1$ if $t \to 0$, the only factor which blows up is

$$\left(B\left(\theta \frac{q}{p_0}, (1-\theta)\frac{q}{p_1} \right) \right)^{1/q},$$

or even

$$\left(\Gamma\left(\theta \frac{q}{p_0} \right) \right)^{1/q}.$$

It is well-known that

$$\Gamma(t) \sim \frac{1}{t} \quad \text{as} \quad t \to 0,$$

so that $\tilde{A} \sim \theta^{-1/q}$. The critical factor in our constant A is

$$(1 - 2^{-\theta r_1})^{-(1-\theta)/r_1},$$

see (23). This behaves like θ^{-1/r_1} as $\theta \to 0$. Now,

$$\left. \frac{1}{q} \right|_{\theta=0} = \frac{1}{p} - \frac{1}{p_1} = \frac{1}{r_1}.$$

Thus, we have the correct qualitative constant also in this limiting case. The behaviour when $\theta \to 1$ can be treated analogously.

REMARK 5. We show by an example that the condition

$$\frac{1}{p} \geqslant \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \tag{28}$$

is, in general, necessary for Theorem 2 to hold. Consider $Z = [0,1]$ equipped with Lebesgue measure. Fix $\theta \in (0,1)$ and $p_0, p_1 > 0$. Let $\beta = \beta_0 = \beta_1 = 1$. Since the measure is finite, the condition (10) is trivially satisfied. Also,

$$B = \frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}} \equiv 1,$$

so $B \in L^s$ for any s. If $0 < \epsilon < 1$, let f_ϵ be the characteristic function of the interval $[0,\epsilon]$. Then

$$\|f_\epsilon \beta_*\|_{p_*} = \epsilon^{1/p_*},$$

and hence

$$\frac{\|f_\epsilon \beta\|_p}{\|f_\epsilon \beta_0\|_{p_0}^{\theta}\|f_\epsilon \beta_1\|_{p_1}^{1-\theta}} = \epsilon^{\frac{1}{p} - \frac{\theta}{p_0} - \frac{1-\theta}{p_1}}.$$

Thus, if (28) is violated, this quotient can be made as large as we want by choosing ϵ small, that is, there is no finite constant such that (12) holds.

REMARK 6. Let us show that the condition (10) can not be relaxed in the case $s > q$, in order for (12) to hold in general. We consider the interval $Z = (1,\infty)$ and we let $d\zeta$ be Lebesgue measure. Let

$$\beta_*(z) = z^{-1/p_*}.$$

Then

$$\frac{\beta(z)}{\beta_0^{\theta}(z)\beta_1^{1-\theta}(z)} = z^{-1/q}.$$

This quotient is in L^s precisely when $s > q$. For $R > 1$, let f_R be the characteristic function of the interval $(1,R)$. Then

$$\int_Z |f_R \beta_*|^{p_*}\, d\zeta = \int_1^R \frac{dz}{z} = \log R,$$

so that

$$\frac{\left(\int_Z |f_R \beta|^p\, d\zeta\right)^{1/p}}{\left(\int_Z |f_R \beta_0|^{p_0}\, d\zeta\right)^{\theta/p_0}\left(\int_Z |f_R \beta_1|^{p_1}\, d\zeta\right)^{(1-\theta)/p_1}} = (\log R)^{1/q}.$$

This clearly tends to infinity as $R \to \infty$. Thus, even though the condition (11) in Theorem 2 is satisfied, there is no finite constant A such that (12) holds.

REMARK 7. The triangle (16) of Theorem 4 is, at least in the case $p_0 = p_1$, in the following sense, the largest possible region in which we have an inequality of Carlson type on a product space. It suffices to show failure on the diagonal $s_Z = s_Y = s$, $s > 2q$. Let $Z = Y = (2,\infty)$ and consider the measures

$$d\zeta(z) = \frac{dz}{z}, \quad d\eta(y) = \frac{dy}{y}.$$

Define

$$\beta_0(z) = z^{1/p_0}, \quad \beta_1(z) = z^{1+1/p_0},$$
$$\alpha_0(y) = y^{1+1/p_0}, \quad \alpha_1(y) = y^{1/p_0}.$$

Also, let

$$\beta = g\beta_0^{\theta}\beta_1^{1-\theta}$$

and

$$\alpha = g\alpha_0^{\theta}\alpha_1^{1-\theta},$$

where

$$g(t) = (\log t)^{-1/2q}.$$

Then

$$A = \frac{\alpha}{\alpha_0^{\theta}\alpha_1^{1-\theta}}$$

and

$$B = \frac{\beta}{\beta_0^{\theta}\beta_1^{1-\theta}}$$

are both in L^s if and only if $s > 2q$. The condition (10) is then satisfied on both spaces. Thus Theorem 2 guarantees that the Carlson type inequality holds for both factors. Consider, however, the functions f_R, defined on $Z \times Y$ by

$$f_R(z,y) = (g(z)g(y))^{q/p_0}\sqrt{z^2+y^2}^{\,-1-2/p_0}K_R(z,y),$$

where K_R is the characteristic function of the set

$$P_R = \left\{(z,y) \in (2,\infty)^2; r_0 \leqslant \sqrt{z^2+y^2} \leqslant R, \frac{\pi}{8} \leqslant \arctan\frac{y}{z} \leqslant \frac{3\pi}{8}\right\},$$

and

$$r_0^2 = \frac{8\sqrt{2}}{\sqrt{2}-1}.$$

It can be shown that there are constants c and C such that for all $(z,y) \in P_R$, we have

$$c(\log\sqrt{z^2+y^2})^2 \leqslant (\log z)(\log y) \leqslant C(\log\sqrt{z^2+y^2})^2.$$

Thus, if $d\xi$ denotes the product measure, we have

$$\int |f_R\alpha\beta|^p\,d\xi \geqslant k\int_{r_0}^{R}\frac{dr}{r\log r}$$

and

$$\left(\int |f_R\alpha_0\beta_0|^{p_0}\,d\xi\right)^{\theta/p_0}\left(\int |f_R\alpha_1\beta_1|^{p_0}\,d\xi\right)^{(1-\theta)/p_0} \leqslant$$
$$\leqslant K\left(\int_{r_0}^{R}\frac{dr}{r\log r}\right)^{1/p_0}.$$

Hence

$$\frac{\|f_R\alpha\beta\|_{L^p(d\xi)}}{\|f_R\alpha_0\beta_0\|^{\theta}_{L^{p_0}(d\xi)}\|f_R\alpha_1\beta_1\|^{1-\theta}_{L^{p_0}(d\xi)}} \geqslant$$

$$\geqslant H\left(\int_{r_0}^{R}\frac{dr}{r\log r}\right)^{1/q} \to \infty, \quad R\to\infty,$$

where H is a positive constant, and we see that a Carlson type inequality can not hold on the product space.

REMARK 8. The author conjectures that the simplex (17) in Theorem 5 is the largest possible parameter region in which we can prove the Carlson type inequality (18). The counter-example in Remark 7 above will only generalize to a smaller set than the complement of (17) if $k > 2$.

REMARK 9. We recall that Theorem 1 gives necessary and sufficient conditions for the Carlson type inequality (3) to hold for special parameters and the specific product measure space. It remains an open question whether there are similar necessary and sufficient conditions in more general cases – or even in the general situation discussed in Theorems 2 to 5.

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