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ON HYERS–ULAM–RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the Hyers-Ulam-Rassias stability of a new quadratic functional equation

$$f(x+y+z+w)+2f(x)+2f(y)+2f(z)+2f(w)$$
$$=f(x+y)+f(y+z)+f(z+x)+f(x+w)+f(y+w)+f(z+w).$$

Moreover, the stability results will be applied to the study of an interesting asymptotic property of the quadratic function.

1. Introduction

In 1940, S. M. Ulam [14] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 *be a group and let* G_2 *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta > 0$ *such that if a function* $h : G_1 \to G_2$ *satisfies the inequality* $d(h(xy), h(x)h(y)) < \delta$ *for all* $x, y \in G_1$ *, then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e. if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive functions was solved by D. H. Hyers [5] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized solution to Ulam's problem for approximately linear mappings was given by Th. M. Rassias [12]. The stability problems of several functional equations have been extensively investigated by a number of authors. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [6, 9, 10].

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It is easy to see that the quadratic function $f(x) = cx^2$ is a solution of each of the following functional equations:

$$f(x+y)+f(x-y) = 2f(x)+2f(y), \tag{1}$$
$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(z+x), \tag{2}$$
$$f(x-y-z)+f(x)+f(y)+f(z) = f(x-y)+f(y+z)+f(z-x). \tag{3}$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [1], [10]). The functional equation (2) was solved by Pl. Kannappan. In fact, he proved that a functional on a real vector space is a solution of equation if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x,x) + A(x)$ for any x (see [10]).

A Hyers-Ulam stability theorem for the quadratic functional equation was proved by F. Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [13]). P. W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. In the paper [4], S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Recently S. M. Jung [8] investigated the Hyers-Ulam stability of the equation. Furthermore he proved the Hyers-Ulam-Rassias stability of the equation [9]. K. W. Jun and Y. H. Lee [7] proved the Hyers-Ulam-Rassias stability of the pexiderized quadratic equation.

Now, consider the following functional equation:

$$\begin{aligned} &f(x+y+z+w)+2f(x)+2f(y)+2f(z)+2f(w) \\ &\quad = f(x+y)+f(y+z)+f(z+x)+f(x+w)+f(y+w)+f(z+w). \end{aligned} \tag{4}$$

In this paper, the Hyers-Ulam-Rassias stability of the new equation shall be proved under the approximately even (or odd) condition.

2. Solutions of eq. (4)

In the following theorem, we will find out the general solution of the functional equation (4).

THEOREM 1. *Let X and Y real vector spaces. A function $f : X \to Y$ satisfies the functional equation (4) if and only if there exist a symmetric biadditive function $B : X^2 \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x,x) + A(x)$ for all $x \in X$.*

Proof. We first assume that f is a solution of the functional equation (4). If we put $x = y = z = w = 0$ in (4), $f(0) = 0$. Let $w = 0$ in (4). Then a function f satisfies the functional equation (2). Therefore, according to [10], the assertion is trivial. Conversely, if there exist a symmetric biadditive function $B : X^2 \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x,x) + A(x)$ for all $x \in X$, we may easily check that f satisfies the equation (4). $\square$

3. Approximately even case

In this section, let X and Y be a real normed space and a Banach space, respectively unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers. Let $H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $H(tu,tv,tw,tx) \leqslant t^p H(u,v,w,x)$ for all $t,u,v,w,x \in \mathbb{R}^+$ and for some $p \in \mathbb{R}$. And let E, $O : \mathbb{R}^+ \to \mathbb{R}^+$ be functions satisfying $E(tx) \leqslant t^q E(x)$, $O(tx) \leqslant t^q O(x)$, respectively, for all $t, x \in \mathbb{R}^+$ and for some $q \in \mathbb{R}$. For convenience, we use the following abbreviation:

$$\begin{aligned} Df(x,y,z,w) = & f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) \\ & - f(x+y) - f(y+z) - f(z+x) - f(x+w) \\ & - f(y+w) - f(z+w). \end{aligned}$$

We first prove the following lemma.

LEMMA 2. *Given $p \in \mathbb{R}$, assume that a mapping $f : X \to Y$ satisfies the following inequality:*

$$\|Df(x,y,z,w)\| \leqslant H(\|x\|, \|y\|, \|z\|, \|w\|)$$

for all $x,y,z,w \in X$ (or $X - \{0\}$). Then it holds that

$$\begin{aligned} & \left\| f(x) - \frac{2^n+1}{2^{2n+1}} f(2^n x) + \frac{2^n-1}{2^{2n+1}} f(-2^n x) \right\| \qquad (5) \\ & \leqslant H(\|x\|, \|x\|, \|x\|, \|x\|) \sum_{k=1}^{n} \frac{2^{(k-1)p}}{2^{k+1}} + 3\|f(0)\| \sum_{k=1}^{n} \frac{1}{2^{k+1}} \end{aligned}$$

for all $x \in X$ (or $X - \{0\}$, respectively) and $n \in \mathbb{N}$.

Proof. We can prove Lemma 2 by using the same argument of Lemma 5 in [8]. $\square$

In the following theorem, we can prove the Hyers-Ulam-Rassias stability of the equation (4) under the approximately even condition.

THEOREM 3. *Let p, $q < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} & \|Df(x,y,z,w)\| \leqslant H(\|x\|, \|y\|, \|z\|, \|w\|), \qquad (6) \\ & \|f(x) - f(-x)\| \leqslant E(\|x\|) \end{aligned}$$

for all $x,y,z,w \in X$. Then there exists a unique quadratic function $g : X \to Y$ which satisfies (4) and the inequality

$$\|f(x) - g(x)\| \leqslant \frac{H(\|x\|, \|x\|, \|x\|, \|x\|)}{4 - 2^{p+1}} + \frac{3}{2}\|f(0)\| \qquad (7)$$

for all $x \in X$. *Moreover, if* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* $g(tx) = t^2 g(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. It follows from (5) and the second condition in (6) that

$$\left\| f(x) - \frac{f(2^n x)}{2^{2n}} \right\| \leqslant H(\|x\|, \|x\|, \|x\|, \|x\|) \sum_{i=1}^{n} \frac{2^{(i-1)p}}{2^{i+1}} + 3\|f(0)\| \sum_{i=1}^{n} \frac{1}{2^{i+1}} + \frac{(2^n - 1)2^{nq}}{2^{2n+1}} E(\|x\|) \tag{8}$$

for all $x \in X$. In order to prove convergence of the sequence $\{g_n(x) = \frac{f(2^n x)}{2^{2n}}\}$, we show that $\{g_n(x)\}$ is a Cauchy sequence in Y. By (8), we have for $n \geqslant m > 0$,

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m} \right\| &= \frac{1}{4^m} \left\| \frac{f(2^{n-m} 2^m x)}{4^{n-m}} - f(2^m x) \right\| \\ &\leqslant \frac{1}{4^m} \Big[H(\|x\|, \|x\|, \|x\|, \|x\|) 2^{mp} \sum_{i=1}^{n-m} \frac{2^{(i-1)p}}{2^{i+1}} + 3\|f(0)\| \sum_{i=1}^{n-m} \frac{1}{2^{i+1}} \\ &\quad + \frac{(2^{n-m} - 1)2^{(n-m)q} 2^{mq}}{2^{2n-2m+1}} E(\|x\|) \Big] \\ &= H(\|x\|, \|x\|, \|x\|, \|x\|) \sum_{i=1}^{n-m} \frac{2^{(m+i-1)p}}{2^{2m+i+1}} + 3\|f(0)\| \sum_{i=1}^{n-m} \frac{1}{2^{2m+i+1}} \\ &\quad + \frac{(2^{n-m} - 1)2^{nq}}{2^{2n+1}} E(\|x\|). \end{aligned}$$

Since the right-hand side of the inequality tends to 0 as m tends to infinity, the sequence $\{g_n(x)\}$ is a Cauchy sequence. Therefore, we may define $g(x) = \lim_{n \to \infty} 2^{-2n} f(2^n x)$ for all $x \in X$; it follows $g(0) = 0$ vacuously. By letting $n \to \infty$ in (8), we arrive at the formula

$$\|f(x) - g(x)\| \leqslant \frac{H(\|x\|, \|x\|, \|x\|, \|x\|)}{4 - 2^{p+1}} + \frac{3}{2}\|f(0)\|$$

for all $x \in X$. To show that g satisfies the equation (4), replace x, y, z and w by $2^n x, 2^n y, 2^n z$ and $2^n w$, respectively in the first condition in (6) and divide by 4^n, then it follows that

$$\begin{aligned} &\|g_n(x + y + z + w) + 2g_n(x) + 2g_n(y) + 2g_n(z) + 2g_n(z) \\ &\quad - g_n(x + y) - g_n(y + z) - g_n(z + x) - g_n(x + w) - g_n(y + w) - g_n(z + w)\| \\ &\quad \leqslant 2^{-2n+np} H(\|x\|, \|y\|, \|z\|, \|w\|). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that g satisfies (4) for all $x, y, z, w \in X$. Analogously, by the second condition in (6), we can show that g is even. Putting $z = -y, w = 0$ and $f = g$ in (4), we see that g is quadratic, i.e.

$$g(x + y) + g(x - y) - 2g(x) - 2g(y) = 0.$$

Now, let $T : X \to Y$ be another quadratic mapping which satisfies (4) and the inequality (7). Obviously, we have $g(2^n x) = 4^n g(x)$ and $T(2^n x) = 4^n T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (7) that

$$\begin{aligned}\|g(x) - T(x)\| &= 4^{-n}\|g(2^n x) - T(2^n x)\| \\ &\leqslant 4^{-n}(\|g(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\|) \\ &\leqslant 4^{-n}\left[\frac{2^{np}H(\|x\|,\|x\|,\|x\|,\|x\|)}{2-2^p} + 3\|f(0)\|\right]\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of g. The proof of the last assertion in the theorem goes through in the same way as that of [4]. This completes the proof. $\square$

Define functions $H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $E : \mathbb{R}^+ \to \mathbb{R}^+$ by $H(x,y,z,w) = (x^p + y^p + z^p + w^p)\varepsilon$ and $E(x) = x^q\theta$, where $\varepsilon,\ \theta \geqslant 0$ and $p,\ q \in \mathbb{R}$. Then we have the following corollary.

COROLLARY 4. *Let $\varepsilon \geqslant 0$, $\theta \geqslant 0$ and $p,\ q < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned}&\|Df(x,y,z,w)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \\ &\|f(x) - f(-x)\| \leqslant \theta\|x\|^q\end{aligned} \tag{9}$$

for all $x,y,z,w \in X$. Then there exists a unique quadratic function $g : X \to Y$ which satisfies (4) and the inequality

$$\|f(x) - g(x)\| \leqslant \frac{2\varepsilon\|x\|^p}{2-2^p} + \frac{3}{2}\|f(0)\| \tag{10}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $g(tx) = t^2 g(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

COROLLARY 5. *Assume that a function $f : X \to Y$ satisfies*

$$\begin{aligned}&\|Df(x,y,z,w)\| \leqslant \varepsilon, \\ &\|f(x) - f(-x)\| \leqslant \theta\end{aligned}$$

for some $\varepsilon,\ \theta \geqslant 0$ and for all $x,y,z,w \in X$. Then there exists a unique quadratic function $g : X \to Y$ which satisfies (4) and the inequality

$$\|f(x) - g(x)\| \leqslant \varepsilon$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $g(tx) = t^2 g(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. In similar fashion of Lemma 2, we note that

$$\left\|f(x) - \frac{2^n+1}{2^{2n+1}} f(2^n x) + \frac{2^n-1}{2^{2n+1}} f(-2^n x)\right\| \leqslant \sum_{k=1}^{n} \frac{1}{2^k}\varepsilon.$$

The arguments used in Theorem 3 carry over almost verbatim. □

4. Approximately odd case

THEOREM 6. *Let $p,\ q < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|Df(x,y,z,w)\| \leqslant H(\|x\|, \|y\|, \|z\|, \|w\|), \\ &\|f(x) + f(-x)\| \leqslant O(\|x\|) \end{aligned} \tag{11}$$

for all $x,y,z,w \in X$. Then there exists a unique additive function $A : X \to Y$ which satisfies (4) and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{H(\|x\|, \|x\|, \|x\|, \|x\|)}{4 - 2^{p+1}} + \frac{3}{2}\|f(0)\| \tag{12}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Proof. Using Lemma 2 and the second condition in (11), we get

$$\begin{aligned} \left\|f(x) - \frac{f(2^n x)}{2^n}\right\| &\leqslant H(\|x\|, \|x\|, \|x\|, \|x\|) \sum_{i=1}^{n} \frac{2^{(i-1)p}}{2^{i+1}} \\ &\quad + 3\|f(0)\| \sum_{i=1}^{n} \frac{1}{2^{i+1}} + \frac{2^n-1}{2^{2n+1}} O(2^n\|x\|) \end{aligned} \tag{13}$$

for all $x \in X$. We will show that $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence in Y. By (13), we have, for $n \geqslant m > 0$,

$$\begin{aligned} &\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| = \frac{1}{2^m}\left\|\frac{f(2^{n-m} 2^m x)}{2^{n-m}} - f(2^m x)\right\| \\ &\leqslant \frac{1}{2^m}\Bigg[H(\|x\|, \|x\|, \|x\|, \|x\|) 2^{mp} \sum_{i=1}^{n-m} \frac{2^{(i-1)p}}{2^{i+1}} + 3\|f(0)\| \sum_{i=1}^{n-m} \frac{1}{2^{i+1}} \\ &\qquad + \frac{(2^{n-m}-1) 2^{(n-m)q} 2^{mq}}{2^{2n-2m+1}} O(\|x\|)\Bigg] \\ &= H(\|x\|, \|x\|, \|x\|, \|x\|) \sum_{i=1}^{n-m} \frac{2^{(m+i-1)p}}{2^{m+i+1}} + 3\|f(0)\| \sum_{i=1}^{n-m} \frac{1}{2^{m+i+1}} \\ &\qquad + \frac{(2^{n-m}-1) 2^{nq}}{2^{2n-m+1}} O(\|x\|), \end{aligned}$$

which becomes arbitrarily small as $m \to \infty$. Now, we can define $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ for all $x \in X$; it follows $A(0) = 0$. Similarly, as in the proof of Theorem 3, due to (11), we see that the mapping A satisfies the equation (4) and is odd. Let $z = -y, w = 0$ and $f = A$ in (4). Since A is odd and $A(0) = 0$, we get

$$2A(x) = A(x+y) + A(x-y). \tag{14}$$

In particular, putting $y = x$ in the above relation (14), we obtain $A(2x) = 2A(x)$. If set $u = x + y$ and $v = x - y$ in (14), we have

$$2A(\frac{u+v}{2}) = A(u) + A(v).$$

By replacing u, v and $2x, 2y$ in the preceding relation and dividing by 2, we note that the mapping A is additive mapping. The validity of the inequality (12) follows directly from (13) and the definition of A.

Now, $G : X \to Y$ be another additive mapping which satisfies (12). It then follows from (12) that

$$\begin{aligned} \|A(x) - G(x)\| &= 2^{-n}\|A(2^n x) - G(2^n x)\| \\ &\leqslant 2^{-n}(\|A(2^n x) - f(2^n x)\| + \|f(2^n x) - G(2^n x)\|) \\ &\leqslant 2^{-n}\left[\frac{2^{np}H(\|x\|, \|x\|, \|x\|, \|x\|)}{2-2^p} + 3\|f(0)\|\right] \end{aligned}$$

for all $x \in X$. This implies the uniqueness of the A. The proof of the linearity of A needs no essential alternations in comparison with the case [12]. □

Now as Corollary 4 in section 3, we can define functions H, O. Then we obtain the following Corollary.

COROLLARY 7. *Let $\varepsilon \geqslant 0$, $\theta \geqslant 0$ and p, $q < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|Df(x,y,z,w)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \\ &\|f(x) + f(-x)\| \leqslant \theta\|x\|^q \end{aligned}$$

for all $x, y, z, w \in X$. Then there exists a unique additive function $A : X \to Y$ which satisfies (4) and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{2\varepsilon\|x\|^p}{2-2^p} + \frac{3}{2}\|f(0)\|$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

COROLLARY 8. *Assume that a function* $f : X \to Y$ *satisfies*

$$\|Df(x,y,z,w)\| \leqslant \varepsilon,$$
$$\|f(x)+f(-x)\| \leqslant \theta$$

for some $\varepsilon,\ \theta \geqslant 0$ *and for all* $x,y,z,w \in X$. *Then there exists a unique additive function* $A : X \to Y$ *which satisfies (4) and the inequality*

$$\|f(x) - A(x)\| \leqslant \varepsilon$$

for all $x \in X$. *Moreover , if* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* A *is linear.*

Proof. By same method of Lemma 2, we get

$$\left\|f(x) - \frac{2^n+1}{2^{2n+1}}f(2^n x) + \frac{2^n-1}{2^{2n+1}}f(-2^n x)\right\| \leqslant \sum_{k=1}^{n} \frac{1}{2^k}\varepsilon.$$

The proof is similar to the theorem 6. □

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