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ON THE EXTRAPOLATION ESTIMATES

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Abstract. We present elementary proofs of precise Yano's type extrapolation estimates on infinite measure space recently proved by M. J. Carro.

1. Introduction

We know many operators which are bounded on L^p for every $p > 1$ but unbounded on L^1. In order to treat such operators, in 1951 S. Yano proved the following extrapolation theorem.

THEOREM A ([5]). *Let* (Ω, μ) *and* (Σ, ν) *be finite measure spaces and let* T *be a sublinear operator* (*with respect to* (Ω, μ) *and* (Σ, ν)). *Suppose that*

$$\left[\int_\Sigma |Tf(x)|^p d\nu(x)\right]^{1/p} \leqslant \frac{A}{(p-1)^\alpha}\left[\int_\Omega |f(x)|^p d\mu(x)\right]^{1/p} \tag{1}$$

for all $f \in L^p(\Omega, \mu)$ *and every* $p \in (1, 2]$ *with positive constants* A *and* α *independent of* p *and* f. *Then*

$$\int_\Sigma |Tf(x)| d\nu(x) \leqslant B\int_\Omega |f(x)|(1+\log|f(x)|)^\alpha d\mu(x) + C\mu(\Omega) \tag{2}$$

for all $f \in L(\log L)^\alpha(\Omega, \mu)$.

On infinite measure spaces, such extrapolation theorem has been considered by several authors. For example, the second author has proved:

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THEOREM B ([4] and [3]). *Let* (Ω,μ) *and* (Σ,ν) *be a* σ*-finite measure spaces and let* T *be a subadditive operator* (*with respect to* (Ω,μ) *and* (Σ,ν)). *If* T *satisfies* (1) *for all* $f\in L^p(\Omega,\mu)$ *and every* $p\in(1,2]$, *then*

$$\int_{\{|Tf|\leqslant 1\}}\frac{|Tf(x)|}{(1-\log|Tf(x)|)^{\alpha+\varepsilon}}d\nu(x)+\int_{\{|Tf|>1\}}|Tf(x)|d\nu(x) \tag{3}$$
$$\leqslant C\left[\int_{\{|f|\leqslant 1\}}\frac{|f(x)|}{(1-\log|f(x)|)^{\varepsilon}}d\mu(x)+\int_{\{|f|>1\}}|f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right]$$

for any $\varepsilon>0$ *and all* f *for which the right hand side of* (3) *is finite.*

Yano's result follows from this results but we cannot prove this result for $\varepsilon=0$.

Recently, M. J. Carro ([2]), under a weaker condition on the operator T, proved better estimates for "$\varepsilon\to 0$", which improve Yano's extrapolation theorem.

For simplicity, in this note we assume that T is sublinear operator (with respect to (Ω,μ) and (Σ,ν)) in the sense that $|T(\lambda f)|\leqslant|\lambda||Tf|$ and

$$|T(\sum_{j=0}^{\infty}f_j)(x)|\leqslant\sum_{j=0}^{\infty}|Tf_j(x)|,\quad a.e.\ \ x\in\Sigma.$$

As usual, if T is sublinear in the classical sense, we can adapt our proofs by considering bounded functions first and then extending the result by density argument.

Throughout this paper (Ω,μ) and (Σ,ν) are two σ-finite measure spaces.

Constants such as C will denote universal constants (independent of f,p and α).

We are going to deliver elementary proofs of the following results.

PROPOSITION 1. ([2, Proposition 2.1]) *If a function* f *satisfies* (1) *for all* $f\in L^p(\Omega,\mu)$ *and every* $p\in(1,2]$, *then for every* $r\geqslant e$,

$$\frac{\int_\Sigma(|Tf(x)|-\frac{1}{r})_+d\nu(x)}{(\log r)^{\alpha}}\leqslant C\int_\Omega|f(x)|^{1+1/\log r}d\mu(x). \tag{4}$$

THEOREM 2. ([2, Theorem 3.1]) *Let* T *be a subadditive operator satisfying* (1) *for every* $f\in L^p(\Omega,\mu)$ *and all* $p\in(1,2]$, *then for every* $f\in L(\log L)^{\alpha}(\Omega,\mu)$,

$$\sup_{r>0}\frac{\int_\Sigma(|Tf(x)|-\frac{1}{r})_+d\nu(x)}{(1+\log^+r)^{\alpha}}\leqslant C\int_\Omega|f(x)|(1+\log^+|f(x)|)^{\alpha}d\mu(x). \tag{5}$$

THEOREM 3. ([2, Corollary 3.4]) *Let* T *be a sublinear operator such that*

$$\left[\int_\Sigma|T\chi_A(x)|^p d\nu(x)\right]^{1/p}\leqslant\frac{C}{(p-1)^{\alpha}}\mu(A)^{1/p}, \tag{6}$$

for all μ*-measurable subsets* $A\subset\Omega$ *and every* $p\in(1,2]$. *Then* T *satisfies* (5).

THEOREM 4. ([2, Theorem 3.3]) *If T is a sublinear operator satisfying* (6) *for all μ-measurable subsets $A \subset \Omega$ and every $p \in (1,2]$, then*

$$T : L(\log L)^{\alpha}(\Omega,\mu) \to L^1(\Sigma,\nu) + L^{\infty}(\Sigma,\nu)$$

is bounded.

THEOREM 5. ([2, Theorem 3.5]) *Let T be a subadditive operator satisfying* (1) *for all $f \in L^p(\Omega,\mu)$ and every $p \in (1,2]$, or let T be a sublinear operator satisfying* (6) *for all μ-measurable subsets $A \subset \Omega$ and every $p \in (1,2]$. Then, for every $f \in L(\log L)^{\alpha}(\Omega,\mu)$,*

$$\limsup_{r\to\infty} \frac{\int_{\Sigma}(|Tf(x)| - \frac{1}{r})_{+} d\nu(x)}{(\log r)^{\alpha}} \leqslant C\|f\|_1. \tag{7}$$

2. Proofs

Proof of Proposition 1. Using Young's inequality, we get

$$\begin{aligned} |Tf| &= r^{-\frac{1}{1+\log r}} r^{\frac{1}{1+\log r}} |Tf| \\ &\leqslant \frac{r^{-1}}{1+\log r} + \frac{\log r}{1+\log r}\left(r^{\frac{1}{\log r}}|Tf|^{1+\frac{1}{\log r}}\right) \\ &\leqslant \frac{1}{r} + e|Tf|^{1+\frac{1}{\log r}} \end{aligned}$$

for all $r \geqslant e$. Then

$$\begin{aligned} &\int_{\{|Tf|>\frac{1}{r}\}} |Tf(x)| d\nu(x) \\ &\qquad \leqslant \frac{1}{r}\nu\left(\left\{|Tf| > \frac{1}{r}\right\}\right) + e\int_{\{|Tf|>\frac{1}{r}\}} |Tf(x)|^{1+\frac{1}{\log r}} d\nu(x). \end{aligned}$$

Together with (1), this yields

$$\begin{aligned} \int_{\Sigma}\left(|Tf(x)| - \frac{1}{r}\right)_{+} d\nu(x) &\leqslant e\int_{\Sigma} |Tf(x)|^{1+\frac{1}{\log r}} d\nu(x) \\ &\leqslant e(A(\log r)^{\alpha})^{1+\frac{1}{\log r}} \int_{\Omega} |f(x)|^{1+\frac{1}{\log r}} d\mu(x) \\ &\leqslant C(\log r)^{\alpha}\int_{\Omega} |f(x)|^{1+\frac{1}{\log r}} d\mu(x) \end{aligned}$$

and (4) follows for $r \geqslant e$.

Proof of Theorem 2. Fix $k_0 \in \mathbb{N}$. For suitable function f, we write

$$f(x) = \sum_{k=k_0}^{\infty} f_k(x),$$

where

$$f_{k_0}(x) = \begin{cases} f(x) & |f(x)| < 2^{k_0} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and, for $k > k_0$,

$$f_k(x) = \begin{cases} f(x) & 2^{k-1} \leqslant |f(x)| < 2^k \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Hence, putting $p_k = 1 + \frac{1}{k}$ and $p'_k = k+1$, using subadditivity, Young's inequality and (1), we obtain

$$\begin{aligned}
\int_{|Tf|>2^{-k_0}} |Tf(x)|d\nu(x) &\leqslant \sum_{k=k_0}^{\infty} \int_{\{|Tf|>2^{-k_0}\}} |Tf_k(x)|d\nu(x) \\
&= \sum_{k=k_0}^{\infty} \int_{\{|Tf|>2^{-k_0}\}} \frac{1}{2} \cdot 2|Tf_k(x)|d\nu(x) \\
&\leqslant \sum_{k=k_0}^{\infty} \int_{\{|Tf|>2^{-k_0}\}} \left(\frac{2^{-p'_k}}{p'_k} + \frac{2^{p_k}|Tf_k(x)|^{p_k}}{p_k} \right) d\nu(x) \\
&\leqslant \nu(\{|Tf| > 2^{-k_0}\}) \sum_{k=k_0}^{\infty} \frac{2^{-k-1}}{k+1} + \sum_{k=k_0}^{\infty} \frac{2^{p_k}}{p_k} \int_{\{|Tf|>2^{-k_0}\}} |Tf_k(x)|^{p_k} d\nu(x) \\
&\leqslant 2^{-k_0}\nu(\{|Tf| > 2^{-k_0}\}) + \sum_{k=k_0}^{\infty} \frac{2^{p_k}}{p_k} \frac{A^{p_k}}{(p_k-1)^{\alpha p_k}} \int_{\Omega} |f_k(x)|^{p_k} d\mu(x) \\
&\leqslant 2^{-k_0}\nu(\{|Tf| > 2^{-k_0}\}) + C_1 \sum_{k=k_0}^{\infty} k^{\alpha} \int_{\Omega} |f_k(x)|^{p_k} d\mu(x) \\
&= 2^{-k_0}\nu(\{|Tf| > 2^{-k_0}\}) + C_1 k_0^{\alpha} \int_{\{|f|<2^{k_0}\}} |f(x)|^{1+\frac{1}{k_0}} d\mu(x) \\
&\qquad + C_1 \sum_{k=k_0+1}^{\infty} k^{\alpha} \int_{\{2^{k-1} \leqslant |f| < 2^k\}} |f(x)|^{1+\frac{1}{k}} d\mu(x) \\
&\leqslant 2^{-k_0}\nu(\{|Tf| > 2^{-k_0}\}) + 2C_1 k_0^{\alpha} \int_{\{|f|<2^{k_0}\}} |f(x)|d\mu(x) \\
&\qquad + 2C_1 \sum_{k=k_0+1}^{\infty} k^{\alpha} \int_{\{2^{k-1} \leqslant |f| < 2^k\}} |f(x)|d\mu(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{\Sigma} (|Tf(x)| - 2^{-k_0})_+ d\nu(x) \\
&\leqslant 2C_1 \left[k_0^{\alpha} \int_{\{|f|<2^{k_0}\}} |f(x)|d\mu(x) + \sum_{k=k_0+1}^{\infty} k^{\alpha} \int_{\{2^{k-1} \leqslant |f| < 2^k\}} |f(x)|d\mu(x) \right]
\end{aligned}$$

$$\leqslant C\left[k_0^\alpha \int_{\{|f|<2^{k_0}\}} |f(x)|d\mu(x) + \sum_{k=k_0+1}^{\infty} \int_{\{2^{k-1}\leqslant|f|<2^k\}} |f(x)|\cdot(\log 2^{k-1})^\alpha d\mu(x)\right]$$

$$\leqslant C\left[k_0^\alpha \int_{\{|f|<2^{k_0}\}} |f(x)|d\mu(x) + \sum_{k=k_0+1}^{\infty} \int_{\{2^{k-1}\leqslant|f|<2^k\}} |f(x)|(\log^+|f(x)|)^\alpha d\mu(x)\right]$$

$$\leqslant C\left[k_0^\alpha \int_\Omega |f(x)|d\mu(x) + \int_\Omega |f(x)|(\log^+|f(x)|)^\alpha d\mu(x)\right],$$

and we obtain (5) for $r = 2^{k_0}$. Let us now prove (5) for any $r > 0$. If $r \leqslant 1$, then

$$\int_\Sigma \left(|Tf(x)| - \frac{1}{r}\right)_+ d\nu(x) \leqslant \int_\Sigma (|Tf(x)| - 1)_+ d\nu(x)$$
$$\leqslant C\int_\Omega |f(x)|(1+\log^+|f(x)|)^\alpha d\mu(x).$$

If $r > 1$ and $2^{k_0-1} < r \leqslant 2^{k_0}$ for some $k_0 \in \mathbb{N}$, then

$$\int_\Sigma \left(|Tf(x)| - \frac{1}{r}\right)_+ d\nu(x) \leqslant \int_\Sigma (|Tf(x)| - 2^{-k_0})_+ d\nu(x) \tag{10}$$
$$\leqslant Ck_0^\alpha \int_\Omega |f(x)|\mu(x) + C\int_\Omega |f(x)|(\log^+|f(x)|)^\alpha d\mu(x)$$
$$\leqslant C(1+\log r)^\alpha \int_\Omega |f(x)|\mu(x) + C\int_\Omega |f(x)|(\log^+|f(x)|)^\alpha d\mu(x).$$

Therefore, for all $r > 0$,

$$\int_\Sigma \left(|Tf(x)| - \frac{1}{r}\right)_+ d\nu(x)$$
$$\leqslant C(1+\log^+ r)^\alpha \int_\Omega |f(x)|(1+\log^+|f(x)|)^\alpha d\mu(x),$$

and Theorem 2 follows.

To prove Theorem 3, we shall need the following definition and lemma.

DEFINITION. We say that a function d is a dyadic function and write $d \in D$ if $d = \sum_{k\in\mathbb{Z}} 2^k \chi_{A_k}$, where A_k are pairwise disjoint measurable sets.

LEMMA 6. ([2, Lemma 2.4]) *If f is a positive function, then $f = \sum_{j=0}^{\infty} d_j$ with $d_j \in D$ and $d_j \leqslant f/2^j$.*

Now we show that estimate (6) implies that the inequality

$$\left(\int_\Sigma |Tf(x)|^p d\nu(x)(x)\right)^{1/p} \leqslant \frac{C}{(p-1)^\alpha}\int_0^\infty (\mu(\{x : |f(x)| > \lambda\}))^{1/p} d\lambda \tag{11}$$

holds for every μ-measurable function f.

First we prove estimate (11) for dyadic functions and then for every μ-measurable functions using Lemma 6.

If $d \in D$ with $d = \sum_{k\in\mathbb{Z}} 2^k \chi_{A_k}$, then, by (6),

$$\begin{aligned}
\left(\int_\Sigma |Td(x)|^p d\nu(x)\right)^{1/p} &\leqslant \sum_{k\in\mathbb{Z}} 2^k \left(\int_\Sigma |T(\chi_{A_k})|^p d\nu(x)\right)^{1/p} \\
&\leqslant \frac{C}{(p-1)^\alpha} \sum_{k\in\mathbb{Z}} 2^k (\mu(A_k))^{1/p} \\
&\leqslant \frac{2C}{(p-1)^\alpha} \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} (\mu(\{x : d(x) > \lambda\}))^{1/p} d\lambda \\
&\leqslant \frac{2C}{(p-1)^\alpha} \int_0^\infty (\mu(\{x : d(x) > \lambda\}))^{1/p} d\lambda,
\end{aligned}$$

and (11) follows for dyadic functions.

Now we show that (11) holds for every μ-measurable function f. To this end, we observe that it is enough to prove (11) for positive μ-measurable function f. If f is positive μ-measurable function, then, by Lemma 6, $f = \sum_{j=0}^{\infty} d_j$ with $d \in D$ and $d_j \leqslant f/2^j$. Consequently,

$$\begin{aligned}
\left(\int_\Sigma |Tf(x)|^p d\nu(x)\right)^{1/p} &\leqslant \sum_{j=0}^{\infty} \left(\int_\Sigma |T(d_j)|^p d\nu(x)\right)^{1/p} \\
&\leqslant \frac{C}{(p-1)^\alpha} \sum_{j=0}^{\infty} \int_0^\infty (\mu(\{x : d_j(x) > \lambda\}))^{1/p} d\lambda \\
&\leqslant \frac{C}{(p-1)^\alpha} \sum_{j=0}^{\infty} 2^{-j} \int_0^\infty (\mu(\{x : f(x) > \lambda\}))^{1/p} d\lambda \\
&\leqslant \frac{2C}{(p-1)^\alpha} \int_0^\infty (\mu(\{x : f(x) > \lambda\}))^{1/p} d\lambda,
\end{aligned}$$

and (11) follows.

Proof of Theorem 3. Note that in the proof of Theorem 2 we have only used condition (1) for the function f_k given by (8) and (9). Applying the fact that $|f_k| \leqslant 2^k$, Hölder's inequality and ([1, Chapter 2, Proposition 2.8]), we obtain that

$$\begin{aligned}
\left(\int_0^\infty (\mu(\{x : |f_k(x)| > \lambda\}))^{1/p_k} d\lambda\right)^{p_k} &= \left(\int_0^{2^k} (\mu(\{x : |f_k(x)| > \lambda\}))^{1/p_k} d\lambda\right)^{p_k} \\
&\leqslant 2^{k(p_k-1)} \int_0^{2^k} \mu(\{x : |f_k(x)| > \lambda\}) d\lambda
\end{aligned}$$

$$= 2 \int_\Omega |f_k(x)| d\mu(x).$$

Together with (11), this yields

$$\int_\Sigma |Tf_k(x)|^{p_k} d\nu(x) \leqslant \frac{2C^{p_k}}{(p_k - 1)^{\alpha p_k}} \int_\Omega |f_k(x)| d\mu(x)$$

and the same argument as that used in the proof of Theorem 2 gives the result.

Proof of Theorem 4. By Theorem 3, $(|Tf(x)| - 1)_+ \in L^1(\Sigma, \nu)$ and

$$\|(Tf - 1)_+\|_{L^1(\Sigma,\nu)} \leqslant C \int_\Omega |f(x)|(1 + \log^+ |f(x)|)^\alpha d\mu(x).$$

Since, moreover,

$$\begin{aligned} |Tf(x)| &= (|Tf(x)|\chi_{\{|Tf(x)|<1\}} + \chi_{\{|Tf(x)|\geqslant 1\}}) \\ &\quad + (|Tf(x)| - 1)\chi_{\{|Tf(x)|\geqslant 1\}} \in L^\infty(\Sigma, \nu) + L^1(\Sigma, \nu), \end{aligned}$$

we obtain that

$$\|Tf\|_{L^\infty(\Sigma,\nu)+L^1(\Sigma,\nu)} \leqslant 1 + C \int_\Omega |f(x)|(1 + \log^+ |f(x)|)^\alpha d\mu(x),$$

and Theorem 4 follows.

Proof of Theorem 5. If T is subadditive operator and satisfies (1) we have got (cf. (10))

$$\begin{aligned} &\int_\Sigma \left(|Tf(x)| - \frac{1}{r}\right)_+ d\nu(x) \qquad (12) \\ &\leqslant C(1 + \log r)^\alpha \int_\Omega |f(x)|\mu(x) + C \int_\Omega |f(x)|(\log^+ |f(x)|)^\alpha d\mu(x). \end{aligned}$$

If T is sublinear operator, we have got estimate (12) under condition (6) (cf. the proof of Theorem 3). Dividing the both sides of the estimate (12) by $(\log r)^\alpha$ and letting $r \to \infty$, we obtain (7).

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