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HEINZ AND MCINTOSH INEQUALITIES, ALUTHGE TRANSFORMATION AND THE SPECTRAL RADIUS

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Abstract. Employing Heinz and McIntosh inequalities, this paper presents a simplified proof of Yamazaki's characterization of the spectral radius: If T_n is the n-th Aluthge transformation of a bounded linear operator T, then the sequence $\{||T_n||\}_{n=0}^{\infty}$ converges to the spectral radius of T.

1. Introduction

Let T be a bounded linear operator on a Hilbert space with spectrum $\sigma(T)$. The spectral radius $r(T)$ of T is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

It is well known that the spectral radius may be characterized as

$$r(T) = \lim_{k\to\infty} ||T^k||^{1/k}. \tag{1}$$

Employing a norm inequality of Heinz and a laborious scheme, Yamazaki [6] recently obtained a new characterization (Theorem 3 below) of the spectral radius as the limit of the norm of the n-th Aluthge transformation of T. In this paper we will further employ a norm inequality due to McIntosh to give a simplified proof of Yamazaki's characterization.

2. Preliminaries

For a bounded linear operator T, we will write $T_0 = T$, and throughout the discussion, T and T_0 will be used interchangeably. Let $T = T_0 = U_0|T_0|$ be the polar decomposition of T. Following [1], we define

$$T_1 = |T_0|^{1/2}U_0|T_0|^{1/2}.$$

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The operator T_1 is known as the Aluthge transformation, or first Aluthge transformation of T. Let $T_1 = U_1|T_1|$ be the polar decomposition of T_1. The second Aluthge transformation T_2 of T is defined by

$$T_2 = |T_1|^{1/2}U_1|T_1|^{1/2}.$$

Inductively, if $T_n = U_n|T_n|$ is the polar decomposition of the n-th Aluthge transformation, one defines the $(n+1)$-st Aluthge transformation as

$$T_{n+1} = |T_n|^{1/2}U_n|T_n|^{1/2}.$$

Yamazaki's characterization of the spectral radius is

$$\lim_{n\to\infty} ||T_n|| = r(T). \tag{2}$$

3. The Result

Our proof, by Lemmas 1 – 4 and Theorem 3 below, employs the following two thoerems. The first theorem is the McIntosh inequality, the second, the Heinz inequality.

THEOREM 1 ([5], [2, Theorem 1]). *For bounded linear operators A, B and X,*

$$\|A^*XB\| \leqslant \|AA^*X\|^{1/2}\|XBB^*\|^{1/2}.$$

THEOREM 2 ([3], [4]). *For positive operators A and B, and bounded linear operator X,*

$$\|A^\alpha XB^\alpha\| \leqslant \|AXB\|^\alpha \|X\|^{1-\alpha},$$

for all $0 \leqslant \alpha \leqslant 1$.

For the Aluthge transformations defined above, it is apparent that $\|T_{n+1}\| \leqslant \|T_n\|$ for all $n \geqslant 0$. Moreover, it is known that $\sigma(T_n) = \sigma(T)$ for all $n \geqslant 0$. Consequently, the sequence $\{\|T_n\|\}_{n=0}^\infty$ is a decreasing sequence which is bounded below by $r(T)$. This yields our first lemma.

LEMMA 1. *There is an $s \geqslant r(T)$ for which $\lim_{n\to\infty} ||T_n|| = s$.*

To prove (2), we need only show that $s = r(T)$. Our next lemma shows that for any positive integer k, the sequence $\{\|T_n^k\|\}_{n=0}^\infty$ is decreasing.

LEMMA 2. *For any positive integer k,*

$$\|T_{n+1}^k\| \leqslant \|T_n^k\|$$

for all $n \geqslant 0$. Consequently, the decreasing sequence $\{\|T_n^k\|\}_{n=0}^\infty$ is convergent.

Proof. By Theorem 1, we have

$$\begin{aligned}\|T_{n+1}^k\| &= \||T_n|^{1/2}T_n^{k-1}U_n|T_n|^{1/2}\| \\ &\leqslant \||T_n|T_n^{k-1}U_n\|^{1/2}\|T_n^{k-1}U_n|T_n|\|^{1/2} \\ &\leqslant \|T_n^k\|.\end{aligned}$$

□

Using Theorem 2, Lemma 3 was essentially proven by Yamazaki [6, Lemma 3]. For the sake of completeness, we reproduce the proof here.

LEMMA 3. *For any positive integer* k,

$$\|T_{n+1}^k\| \leqslant \|T_n^{k+1}\|^{1/2}\|T_n^{k-1}\|^{1/2},$$

for all $n \geqslant 0$.

Proof. By Theorem 2, we have

$$\begin{aligned} \|T_{n+1}^k\| &= \||T_n|^{1/2}T_n^{k-1}U_n|T_n|^{1/2}\| \\ &\leqslant \||T_n|T_n^{k-1}U_n|T_n|\|^{1/2}\|T_n^{k-1}U_n\|^{1/2} \\ &\leqslant \|T_n^{k+1}\|^{1/2}\|T_n^{k-1}\|^{1/2}. \end{aligned}$$

□

The next lemma shows that the decreasing sequence $\{\|T_n^k\|\}_{n=0}^{\infty}$ converges to s^k, where $s = \lim_{n\to\infty} \|T_n\|$ is as in Lemma 1.

LEMMA 4. *For any positive integer* k, $\lim_{n\to\infty} \|T_n^k\| = s^k$.

Proof. We will prove the lemma by induction. Since $\lim_{n\to\infty} \|T_n\| = s$ by Lemma 1, the lemma is proven for $k = 1$. Assume the lemma is proven for $1 \leqslant k \leqslant m$. By Lemma 3,

$$\|T_{n+1}^m\| \leqslant \|T_n^{m+1}\|^{1/2}\|T_n^{m-1}\|^{1/2} \leqslant \|T_n^m\|^{1/2}\|T_n\|^{1/2}\|T_n^{m-1}\|^{1/2}. \tag{3}$$

Let $\lim_{n\to\infty} \|T_n^{m+1}\| = t$. The existence of the limit follows from Lemma 2. Taking limits, the induction hypothesis and (3) show that

$$s^m \leqslant t^{\frac{1}{2}} s^{\frac{m-1}{2}} \leqslant s^{\frac{m}{2}+\frac{1}{2}+\frac{m-1}{2}} = s^m.$$

It follows that $t = s^{m+1}$, and the proof is complete. □

We are now ready to prove Yamazaki's characterization of the spectral radius.

THEOREM 3 ([6]). *For any bounded linear operator* T,

$$\lim_{n\to\infty} \|T_n\| = r(T).$$

Proof. It follows from Lemmas 2 and 4 that, for each positive integer k, the decreasing sequence $\{\|T_n^k\|^{1/k}\}_{n=0}^{\infty}$ converges to s. Therefore,

$$s \leqslant \|T_n^k\|^{1/k} \tag{4}$$

for all n and k. Now fix an n. If $r(T) < s$, then (1) would imply that

$$\|T_n^k\|^{1/k} < s$$

for sufficiently large k. Clearly this is a contradiction to (4). Therefore, we must have $s = r(T)$, and the result follows from Lemma 1. □

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