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GEOMETRIC INEQUALITIES INVOLVING TWO SIMPLEXES

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Abstract. We estabish some geometric inequalities on the inradius and circumradius of two n–dimensional simplexes.

1. Some inequalities for two simplexes and applications

Let Ω_n and Ω'_n be two n–dimensional simplexes in the n–dimensional Euclidean space E_n, $\tau\{A_0, A_1, \ldots, A_n\}$ denote the vertex set of Ω_n, V the volume of Ω_n, $a_{ij} = |A_iA_j|$ $(i, j = 0, 1, \ldots, n)$ R and r be the circumradius and inradius of Ω_n, respectively. For $i = 0, 1, \ldots, n$ let m_i be the lengths of the i th median of Ω_n, h_i the altitute of Ω_n from vertex A_i, ρ_i the radius of ith escribed sphere of Ω_n, F_i the area of the i th face $f_i = A_0 \ldots A_{i-1}A_{i+1} \ldots A_n$ of Ω_n. Let θ_{ij} be the dihedral angle formed by two faces f_i and f_j, T_{ij} the area of the bisection plane t_{ij} ($(n-1)$–dimensional simplex) of the dihedral angle θ_{ij} of Ω_n. Let $r_{i_0i_1\ldots i_k}$ be the inradius of k–dimensional simplex $A_{i_0}A_{i_1}\ldots A_{i_k}(0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n)$. For the second simplex Ω'_n, we use the analogous notations, for example, R' and r' are the circumradius and inradius of Ω'_n, respectively.

Recently, Leng Gangsong estabished some inequalities involving two simplexes as follows [1]

$$\frac{R'}{nr} \geqslant \frac{2}{n(n+1)} \sum_{0\leqslant i<j\leqslant n} \frac{a'_{ij}}{a_{ij}}, \tag{1}$$

$$\frac{R'}{nr} \geqslant \binom{n+1}{k+1}^{-1} \sum_{0\leqslant i_0<i_1<\cdots<i_k\leqslant n} \frac{r'_{i_0i_1\ldots i_k}}{r_{i_0i_1\ldots i_k}}. \tag{2}$$

Equality in (1), (2) holds if the simplexes Ω_n and Ω'_n are regular. In this paper, we establish other inequalities involving two simplexes, and give some their applications. Our main results are the following theorems.

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THEOREM 1.

$$\left(\frac{R'}{nr}\right)^{1/2} \geqslant \frac{1}{n+1}\sum_{i=0}^{n}\frac{m_i'^{1/2}}{m_i^{1/2}}, \tag{3}$$

with equality if the simplexes Ω_n and Ω_n' are regular.

THEOREM 2.

$$\left(\frac{R'}{nr}\right)^{1/2} \geqslant \frac{1}{n+1}\sum_{i=0}^{n}\frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} h_j'^{1/2}}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}}, \tag{4}$$

with equality if the simplexes Ω_n and Ω_n' are regular.

THEOREM 3.

$$\frac{R}{nr}\cdot\left(\frac{R'}{nr}\right)^{n-1} \geqslant \frac{2}{n(n+1)}\sum_{0\leqslant i<j\leqslant n}\frac{T_{ij}'}{T_{ij}}. \tag{5}$$

Equality holds if the simplexes Ω_n and Ω_n' are regular.

THEOREM 4.

$$\left(\frac{R'}{nr}\right)^{1/2}\left(\frac{R'}{nr}\right)^{(n^2-n-1)/2} \geqslant \frac{1}{n+1}\sum_{i=0}^{n}\frac{\rho_i'^{1/2}}{\rho_i^{1/2}}. \tag{6}$$

Equality holds if the simplexes Ω_n and Ω_n' are regular.

Now we give some applications of the theorems stated above. If take $\Omega_n' = \Omega_n$, then $R' = R$, $r' = r$, and obtain the following important inequality [2, or 3]

$$R \geqslant nr, \tag{7}$$

with equality if the simplex Ω_n is regular.

Let $A(x_i)$ and $G(x_i)$ denote the arithmetic mean and the geometric mean of positive numbers $x_i (i = 1, 2, \ldots, m)$. From the theorems stated above we obtain following inequalities for a simplex.

THEOREM 5.

$$\frac{R}{nr} \geqslant \frac{1}{(n+1)^4}\left[\left(\sum_{i=0}^{n} m_i^{1/2}\right)\left(\sum_{i=0}^{n}\frac{1}{m_i^{1/2}}\right)\right]^2, \tag{8}$$

$$\frac{R}{nr} \geqslant \left[\frac{A(m_i^{1/2})}{B(m_i^{1/2})}\right]^2 \geqslant 1. \tag{9}$$

Equality in (8), (9) holds if the simplex Ω_n is regular.

Proof. Taking $\Omega_n' = \Omega_n$ in Theorem 1, and let $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_n$, we get

$$\left(\frac{R}{nr}\right)^{1/2} \geqslant \frac{1}{(n+1)}\left(\frac{m_0^{1/2}}{m_n^{1/2}} + \frac{m_1^{1/2}}{m_{n-1}^{1/2}} + \cdots + \frac{m_n^{1/2}}{m_0^{1/2}}\right).$$

Applying Chebishev's inequality (see [4, II, 2.5]), we have

$$\left(\frac{R}{nr}\right)^{1/2} \geqslant \frac{1}{(n+1)^2}\Big(\sum_{i=0}^{n} m_i^{1/2}\Big)\Big(\sum_{i=0}^{n}\frac{1}{m_i^{1/2}}\Big). \tag{10}$$

Thus inequality (8) holds. Using inequality (8) and the arithmetic–geometric mean inequality, we derive inequality (9).

THEOREM 6.

$$\frac{R}{nr} \geqslant \frac{1}{(n+1)^{4/n}}\Bigg[\sum_{i=0}^{n}\Big(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}\Big)\Big(\sum_{i=0}^{n}\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}}\Big)\Bigg]^{2/n}, \tag{11}$$

$$\frac{R}{nr} \geqslant \left[\frac{A\Big(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}\Big)}{G\Big(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}\Big)}\right]^{2/n} \geqslant 1. \tag{12}$$

Equality in (11), (12) holds if the simplex Ω_n *is regular.*

Proof. Taking $\Omega_n' = \Omega_n$ in Theorem 2, and let $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_n$. Applying Chebishev's inequality (see [4, II, 2.5]), we have

$$\left(\frac{R}{nr}\right)^{n/2} \geqslant \frac{1}{(n+1)^2}\Big[\sum_{i=0}^{n}\Big(\prod_{\substack{j=0\\ j\neq 1}}^{n} h_j^{1/2}\Big)\Big]\cdot\left[\sum_{i=0}^{n}\Bigg(\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}}\Bigg)\right].$$

Thus inequality (11) holds. Using inequality (11) and the arithmetic–geometric mean inequality, we obtain inequality (12).

THEOREM 7.

$$\frac{R}{nr} \geqslant \Big[\frac{2}{n(n+1)}\Big]^{2/n}\Big[\Big(\sum_{0\leqslant i<j\leqslant n} T_{ij}\Big)\Big(\sum_{0\leqslant i<j\leqslant n}\frac{1}{T_{ij}}\Big)\Big]^{1/n}, \tag{13}$$

$$\frac{R}{nr} \geqslant \Big[\frac{A(T_{ij})}{G(T_{ij})}\Big]^{1/n} \geqslant 1. \tag{14}$$

Equality in (13), (14) holds if the simplex Ω_n *is regular.*

Proof. Taking $\Omega_n' = \Omega_n$ in Theorem 3, and let $T_1, T_2, \ldots, T_{\frac{1}{2}n(n+1)}$ be the sequence of the areas $T_{ij} (0 \leqslant i < j \leqslant n)$ in non–decreasing order, we have

$$\left(\frac{R}{nr}\right)^{n} \geqslant \frac{2}{n(n+1)}\left(\frac{T_1}{T_{\frac{1}{2}n(n+1)}} + \frac{T_2}{T_{\frac{1}{2}n(n+1)-1}} + \cdots + \frac{T_{\frac{1}{2}n(n+1)}}{T_1}\right).$$

Applying Chebishev's inequality, we get

$$\left(\frac{R}{nr}\right)^n \geqslant \left[\frac{2}{n(n+1)}\right]^2 \left(\sum_{i=1}^{\frac{1}{2}n(n+1)} T_i\right)\left(\sum_{i=1}^{\frac{1}{2}n(n+1)} \frac{1}{T_i}\right).$$

Thus inequality (13) holds. Using inequality (13) and the arithmetic–geometric mean inequality, we get inequality (14).

THEOREM 8.

$$\frac{R}{nr} \geqslant \frac{1}{(n+1)^{4/(n^2-n)}}\left[\left(\sum_{i=0}^{n}\rho_i^{1/2}\right)\left(\sum_{i=0}^{n}\frac{1}{\rho_i^{1/2}}\right)\right]^{2/(n^2-n)}, \tag{15}$$

$$\frac{R}{nr} \geqslant \left[\frac{A(\rho_i^{1/2})}{G(\rho_i^{1/2})}\right]^{2/(n^2-n)} \geqslant 1. \tag{16}$$

Equality in (15) and (16) holds if the simplex Ω_n *is regular.*

Proof. Taking $\Omega_n' = \Omega_n$ in Theorem 3, and let $\rho_0 \leqslant \rho_1 \leqslant \cdots \leqslant \rho_n$, we have

$$\left(\frac{R}{nr}\right)^{(n^2-n)/2} \geqslant \frac{1}{(n+1)}\left(\frac{\rho_0^{1/2}}{\rho_n^{1/2}} + \frac{\rho_1^{1/2}}{\rho_{n-1}^{1/2}} + \cdots + \frac{\rho_n^{1/2}}{\rho_0^{1/2}}\right).$$

Applying Chebishev's inequality, we have

$$\left(\frac{R}{nr}\right)^{(n^2-n)/2} \geqslant \frac{1}{(n+1)^2}\left(\sum_{i=0}^{n}\rho_i^{1/2}\right)\left(\sum_{i=0}^{n}\frac{1}{\rho_i^{1/2}}\right).$$

Thus inequality (15) holds. Using inequality (15) and the arithmetic–geometric mean inequality, we derive inequality (16).

2. Proof of theorems

To prove the theorems in previous section, we need some lemmas as follows.

LEMMA 1.

$$\sum_{i=0}^{n}\left(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j\right) \leqslant \frac{(n+1)^{n+1}}{n^n}R^n, \tag{17}$$

with equality if the simplex Ω_n *is regular.*

Proof. For $k = 0, 1, \ldots, n$, let F_k denote the area of the kth face $f_k = A_0 \ldots A_{k-1}A_{k+1} \ldots A_n$ of the simplex Ω_n. The n–dimensional sine of the kth vertex angle of Ω_n was defined by F. Eriksson [5] as follows

$$^{n}\sin\theta_k = \frac{(n!\cdot V)^{n-1}}{\prod\limits_{\substack{j=0\\ j\neq k}}^{n}((n-1)!\cdot F_j)} \quad (k = 0, 1, \ldots, n). \tag{18}$$

Yang and Wang [6] proved an inequality as follows

$$\sum_{i=0}^{n} x_i \cdot {}^n\sin\theta_i \leqslant \left(\frac{n+1}{n^n}\right)^{1/2}\left(\prod_{i=0}^{n} x_i\right)\left(\sum_{i=0}^{n} x_i^{-2}\right)^{n/2}. \tag{19}$$

Where $x \neq 0$ $(i = 0, 1, \ldots, n)$ are arbitrary real numbers. Equality in (19) holds if the simplex Ω_n is regular and $x_0 = x_1 = \cdots = x_n$.

Taking $x_0 = x_1 = \cdots = x_n$ in inequality (19), we get

$$\sum_{i=0}^{n} {}^n\sin\theta_i \leqslant \frac{(n+1)^{(n+1)/2}}{n^{n/2}}. \tag{20}$$

Substituting $F_j = nV \cdot h_j^{-1} (j = 0, 1, \ldots, n)$ into (18), we get

$${}^n\sin\theta_i = (n! \cdot V)^{-1} \prod_{\substack{j=0\\ j\neq i}}^{n} h_j \quad (i = 0, 1, \ldots, n). \tag{21}$$

Combining (20) with (21), we get

$$\sum_{i=0}^{n}\left(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j\right) \leqslant \frac{n! \cdot (n+1)^{(n+1)/2}}{n^{n/2}} V. \tag{22}$$

Using inequality (22) and the known inequality [3]

$$V \leqslant \frac{(n+1)^{(n+1)/2}}{n! \cdot n^{n/2}} R^n, \tag{23}$$

we obtain inequality (17). It is easy to know that equality in (17) holds if the simplex Ω_n is regular.

LEMMA 2.

$$\sum_{i=0}^{n} m_i \leqslant \frac{(n+1)^2}{n} R,$$

with equality if the simplex Ω_n *is regular.*

Proof. Applying the know inequality [3], we have

$$\sum_{i=0}^{n} m_i \leqslant (n+1)^{1/2}\left(\sum_{i=0}^{n} m_i^2\right)^{1/2}. \tag{25}$$

Combining this with the known result [3]

$$\sum_{i=0}^{n} m_i^2 = \frac{n+1}{n^2} \sum_{0\leqslant i<j\leqslant n} a_{ij}^2,$$

we get

$$\sum_{i=0}^{n} m_i \leqslant \frac{n+1}{n}\Big(\sum_{0\leqslant i<j\leqslant n} a_{ij}^2\Big)^{1/2}. \tag{26}$$

Using inequality (26) and an inequality [3]

$$\sum_{0\leqslant i<j\leqslant n} a_{ij}^2 \leqslant (n+1)^2R^2, \tag{27}$$

we get inequality (24). It is easy to see that equality in (24) holds if the simplex Ω_n is regular.

LEMMA 3.

$$V \geqslant \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!}r^n. \tag{28}$$

Equality holds if and only if the simplex Ω_n *is regular.*

For the proof of Lemma 3, the reader is referred to [3].

LEMMA 4.

$$\sum_{i=0}^{n} F_i^2 \leqslant \frac{(n+1)^n}{(n!)^2n^{n-4}}R^2(n-1), \tag{29}$$

with equality if the simplex Ω_n *is regular.*

Proof. Using the known inequality [1]

$$\sum_{i=0}^{n} F_i^2 \leqslant \frac{1}{(n-1)!^2[n(n+1)]^{n-2}}\Big(\sum_{0\leqslant i<j\leqslant n} a_{ij}^2\Big)^{n-1}$$

and inequality (27), we get inequality (29). It is easy to know that equality holds if the simplex Ω_n is regular.

LEMMA 5.

$$\sum_{0\leqslant i<j\leqslant n} T_{ij}^2 \leqslant \frac{n+1}{4}\sum_{i=0}^{n} F_i^2. \tag{30}$$

Equality holds if the simplex Ω_n *is regular.*

For the proof of Lemma 5, the reader is referred to [7].

LEMMA 6. *Put* $F=\sum_{j=0}^{n} F_j$*, then we have*

$$\prod_{i=0}^{n}(F-2F_i) \geqslant \Big[\frac{(n-1)n^{3/2}}{(n!)^{1/n}(n+1)^{(n-1)/2n}}\Big]^{n+1} V^{(n^2-1)/n}. \tag{31}$$

Equality holds if and only if the simplex Ω_n *is regular.*

For the proof of Lemma 6, the reader is referred to [8].

Proof of Theorem 1. Using Cauchy's inequality, Lemma 2 and $m_i \geqslant h_i$ $(i = 0, 1, \ldots, n)$, we have

$$\sum_{i=0}^{n} \frac{m_i'^{1/2}}{m_i^{1/2}} \leqslant \Big(\sum_{i=0}^{n} m_i'\Big)^{1/2} \Big(\sum_{i=0}^{n} \frac{1}{m_i}\Big)^{1/2} \leqslant \frac{(n+1)}{n^{1/2}} R'^{1/2} \Big(\sum_{i=0}^{n} \frac{1}{h_i}\Big)^{1/2}. \tag{32}$$

Substituing $\sum_{i=0}^{n} h_i^{-1} = r^{-1}$ into (32), we get inequality (3). It is easy to see that equality in (3) holds if the simplexes Ω_n and Ω_n' are regular.

Proof of Theorem 2. Using Cauchy's inequality and Lemma 1, we have

$$\sum_{i=0}^{n} \frac{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j'^{1/2}}{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}} \leqslant \Big[\sum_{i=0}^{n} \Big(\prod_{\substack{j=0\\ j\neq i}}^{n} h_j'\Big)\Big]^{1/2} \Big[\sum_{i=0}^{n} \Big(\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j'}\Big)\Big]^{1/2}$$

$$\leqslant \frac{(n+1)^{(n+1)/2}}{n^{n/2}} R'^{n/2} \Big[\sum_{i=0}^{n} \Big(\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{1}{h_j}\Big)\Big]^{1/2}. \tag{33}$$

Using (33) and Maclaurin's inequality [3], and $\sum_{i=0}^{n} h_i^{-1} = r^{-1}$, we get

$$\sum_{i=0}^{n} \frac{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j'^{1/2}}{\prod_{\substack{j=0\\ j\neq i}}^{n} h_j^{1/2}} \leqslant \frac{(n+1)^{(n+1)/2}}{n^{n/2}} R'^{n/2} \cdot \frac{1}{(n+1)^{(n-1)/2}} \Big(\sum_{i=0}^{n} \frac{1}{h_i}\Big)^{n/2}$$

$$= (n+1)\Big(\frac{R'}{nr}\Big)^{n/2}.$$

Thus inequality (4) holds. It is easy to prove that equality in (4) holds if the simplex Ω_n is regular.

Proof of Theorem 3. For the simplex $\Omega_n = A_0 A_1 \ldots A_n$, let d_i be the distance from the point A_i to the $(n-1)$–dimensional plane σ_{ij} containing the bisection plane t_{ij} of the dihedral angle θ_{ij} formed by two faces f_i and f_i of Ω_n, and d_j the distance from the point A_j to the plane σ_{ij}, then

$$nV = (d_i + d_j)T_{ij} \leqslant a_{ij} T_{ij} \quad (0 \leqslant i < j \leqslant n). \tag{34}$$

Thus

$$\sum_{0 \leqslant i<j \leqslant n} \frac{1}{T_{ij}^2} \leqslant \frac{1}{n^2 V^2} \sum_{0 \leqslant i<j \leqslant n} a_{ij}^2. \tag{35}$$

Using (35), (27) and (28), we get

$$\sum_{0\leqslant i<j\leqslant n}\frac{1}{T_{ij}^2}\leqslant\frac{(n!)^2}{n^{n+2}(n+1)^{n-1}}\cdot\frac{R^2}{r^{2n}}. \tag{36}$$

Applying Cauchy's inequality, inequality (36), lemmas 5 and 4, we get

$$\begin{aligned}\sum_{0\leqslant i<j\leqslant n}\frac{T'_{ij}}{T_{ij}}&\leqslant\Big(\sum_{0\leqslant i<j\leqslant n}T_{ij}'^{2}\Big)^{1/2}\Big(\sum_{0\leqslant i\leqslant n}\frac{1}{T_{ij}^2}\Big)^{1/2}\\&\leqslant\frac{n!}{n^{(n+2)/2}(n+1)^{(n-1)/2}}\cdot\frac{R^2}{r^n}\cdot\Big(\frac{n+1}{4}\sum_{i=0}^{n}F_i'^{2}\Big)^{1/2}\\&\leqslant\frac{n(n+1)}{2}\frac{R}{nr}\cdot\Big(\frac{R'}{nr}\Big)^{n-1}.\end{aligned}$$

Thus inequality (5) holds. It is easy to see that equality in (5) holds if the simplex Ω_n is regular.

Proof of Theorem 4. Applying Cauchy's inequality and the known formulas [3]

$$\sum_{i=0}^{n}\frac{1}{\rho_i}=\frac{n-1}{r},\quad \rho_i'=\frac{nV'}{F'-2F_i'}\quad\Big(F'=\sum_{i=0}^{n}F_i'\Big),$$

we have

$$\begin{aligned}\Big(\sum_{i=0}^{n}\frac{\rho_i'^{1/2}}{\rho_i^{1/2}}\Big)&\leqslant\Big(\sum_{i=0}^{n}\rho_i'\Big)\Big(\sum_{i=0}^{n}\frac{1}{\rho_i}\Big)=\frac{n-1}{r}\sum_{i=0}^{n}\frac{nV'}{F'-2F_i'}\\&=\frac{n-1}{r}\cdot\frac{nV'}{\prod_{i=0}^{n}(F'-2F_i')}\cdot\sum_{i=0}^{n}\prod_{\substack{j=0\\j\neq i}}^{n}(F'-2F_j').\end{aligned} \tag{37}$$

Using (37), Maclaurin's inequality [3], lemmas 6 and 3, we get

$$\begin{aligned}\Big(\sum_{i=0}^{n}\frac{\rho_i'^{1/2}}{\rho_i^{1/2}}\Big)^2&\leqslant\frac{n(n-1)}{r}\cdot\Big[\frac{(n!)^{1/n}(n+1)^{(n-1)/2n}}{(n-1)n^{3/2}}\Big]^{n+1}\times\\&\quad\times V'^{-(n^2-n-1)/n}\cdot\frac{1}{(n+1)^{(n-1)}}\Big[\sum_{i=0}^{n}(F'-2F_i')\Big]^n\\&\leqslant\frac{(n!)^n}{(n-1)^n n^{n(n+2)/2}(n+1)^{(n^2+n-4)/2}}\cdot\frac{1}{r\cdot r'^{n^2-n-1}}\Big[\sum_{i=0}^{n}(F'-2F_i')\Big]^n\\&=\frac{(n!)^n}{n^{n(n+2)/2}(n+1)^{(n^2+n-4)/2}}\cdot\frac{1}{rr'^{n^2-n-1}}\Big(\sum_{i=0}^{n}F_i'\Big)^n.\end{aligned} \tag{38}$$

Applying the known inequality [3] and inequality (29), we have

$$\left(\sum_{i=0}^{n} F_i'\right)^n \leqslant \left[(n+1)\sum_{i=0}^{n} F_i'^2\right]^{\frac{1}{2}\cdot n} \leqslant \frac{(n+1)^{(n^2+n)/2}}{(n!)^n \cdot n^{(n^2-4n)/2}} R'^{(n^2-n)}. \tag{39}$$

Combining (38) with (39), we get

$$\left(\sum_{i=0}^{n} \frac{\rho_i'^{1/2}}{\rho_i^{1/2}}\right)^2 \leqslant (n+1)^2 \frac{R'}{nr} \cdot \left(\frac{R'}{nr'}\right)^{n^2-n-1}.$$

Thus inequality (6) holds. It is easy to prove that equality in (6) holds if the simplexes Ω_n and Ω_n' are regular.

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