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# AN EXTENSION OF A GEOMETRIC INEQUALITY OF FINITE POINT SET ON A SPHERE IN THE CONSTANT CURVATURE SPACE

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*Abstract.* In this paper, we first prove an algebraic inequality, then use it obtain an extension of a geometirc inequality in the $n$-dimensional constant curvature space.

## 1. Introduction and an algebraic inequality

Let $E^n$ denote an $n$-dimensional Euclidean space, $S_R^{n-1}$ denote $n-1$ dimensional sphere with radius $R$, $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$ $(N > n)$ be a finite point set in $S_R^{n-1} \subset E^n$. The Euclidean distance between two points $A_i$ and $A_j$ will be denoted by $|A_iA_j| = a_{ij}$ $(1 \leqslant i, j \leqslant N)$, and $m_i$ $(1 \leqslant i \leqslant N)$ will be a positive number. The paper [1] proved the following result

$$\sum_{1 \leqslant i < j \leqslant N} m_i m_j (4R^2 - a_{ij}^2) a_{ij}^2 \leqslant \frac{2(n-1)}{n} \Big( \sum_{i=1}^{N} m_i \Big)^2 R^4. \tag{1}$$

To give an extension of (1) in the $n$-dimensional space of constant curvature, we first prove the following algebraic inequality which extends the famous Newton's inequality and Maclaurin's inequality.

THEOREM 1. *Let $p_k$ denote the elementary symmetric mean of degree $k$ of the positive real numbers $a_1, a_2, \dots, a_n$, $\alpha_1, \alpha_2, \dots, \alpha_m$ be non–negative real numbers, and $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$. If for $k_i \in \mathbf{N}$ $(1 \leqslant i \leqslant m)$ we have $\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_m k_m = k_0 \in \mathbf{N}$, then*

$$p_{k_0} \geqslant \prod_{i=1}^{m} p_{k_i}^{\alpha_i}, \tag{2}$$

*where equality is valid if and only if $a_1 = a_2 = \cdots = a_n$.*

*Proof.* By Newton's inequality in the paper [2], we know

$$p_k^2 \geqslant p_{k-1} p_{k+1}, \tag{3}$$

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where equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

So that, from the paper [3] we know that $\{p_k\}$ is logarithmically upper convex sequence, that is $f(k) = \ln p_k$ is upper convex function, hence we have

$$f\Big(\sum_{i=i}^{m} \alpha_i k_i\Big) \geqslant \sum_{i=1}^{m} \alpha_i f(k_i),$$

where equality holds if and only if $k_1 = k_2 = \cdots = k_m$.

Therefore we have

$$\ln p_{k_0} \geqslant \sum_{i=1}^{m} \alpha_i \ln p_{k_i},$$

where equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

The inequality (2) follows immediately. The theorem 1 proved.

COROLLARY 1. *Under the same condition of Theorem 1, if* $\alpha_i = \frac{1}{m}$, $(1 \leqslant i \leqslant m)$, *and* $\sum_{i=1}^{m} k_i = mk$, *then*

$$p_k^m \geqslant \prod_{i=1}^{m} p_{k_i}, \tag{5}$$

*where equality is valid if and only if* $a_1 = a_2 = \cdots = a_n$.

It is not difficult to see that in the case when $m$ is an even number, $a_1, a_2, \ldots, a_n$ can be any real numbers, and when $m = 2$, $k_1 = k-1$, $k_2 = k+1$, $k_0 = k$, (5) is famous Newton's inequality (3).

COROLLARY 2. *Under the same condition of Theorem 1, we have*

$$p_k^{\frac{1}{k}} \geqslant p_l^{\frac{1}{l}}, \quad (1 \leqslant k < l \leqslant n), \tag{7}$$

*where equality holds iff* $a_1 = a_2 = \cdots = a_n$.

## 2. Several Lemmas

Let $C^n(K)$ denote the $n$-dimensional constant curvature space whose curvature is $K$. We stipulate that $C^n(K)$ denote an $n$-dimensional Euclidean space $E^n$, an $n$-dimensional spherical space $S^n(K)$ and an $n$-dimensional hyperbolic space $H^n(K)$, whenever $K = 0$, $K > 0$ and $K < 0$ respectively. We use $d = (a, b, c)$ to denote, relative to $E^n$, $S^n(K)$, $H^n(K)$ space, $d = a$, $d = b$, $d = c$, respectively.

Let $S^{n-1}$ denote the $(n-1)$-dimensional sphere of radius $R$ in $C^n(K)$, $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ $(N > n)$ be a finite point set, and $\mathcal{A} \subset S^{n-1}$. The Euclidean distance betwen two points $A_i$ and $A_j$ will be denoted by $|A_iA_j|$, the spherical surface distance and the hyperbolic distance betwen two points $A_i$ and $A_j$ will be writed by $\overset{\frown}{A_iA_j}$, put $a_{ij} = (|A_iA_j|, \overset{\frown}{A_iA_j}, \overset{\frown}{A_iA_j})$. In the hyperbolic space $H^n(K)$, suppose that $O'$ is circumcenter of $S^{n-1}$, and write $\measuredangle A_iO'A_j = \alpha_{ij}, q_{ij} = (\cos \frac{a_{ij}}{R}, \cos \sqrt{K} a_{ij}, \cos \alpha_{ij})$.

LEMMA 1. *Let $S^{n-1}$ denote the $(n-1)$-dimensional spherical surface of radius $R$ in Euclidean space $E^n$, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ $(N > n)$ is a finite point set, and $\mathcal{A} \subset S^{n-1}$. Choosing $k+1$ vertices $A_{i_0}$, $A_{i_1}$, ..., $A_{i_k}$ from the set $\mathcal{A}$, denote by $V_{i_0 i_1 \ldots i_k}$ the $k$-dimensional volume of the $k$-dimensional simplex $\mathcal{A}_{i_0 i_1 \ldots i_k}$ spanned by the points $A_{i_0}$, $A_{i_1}$, ..., $A_{i_k}$, and denote the circumradius of $\mathcal{A}_{i_0 i_1 \ldots i_k}$ by $R_{i_0 i_1 \ldots i_k}$. Then we have*

$$\det\left(\cos\frac{a_{i_\alpha i_\beta}}{R}\right)_{\alpha,\beta=0}^{k} = \frac{k!^2}{R^{2(k+1)}} \cdot (R^2 - R^2_{i_0 i_1 \ldots i_k}) V^2_{i_0 i_1 \ldots i_k}. \tag{8}$$

*Proof.* For the $k$-dimensional simplex $\mathcal{A}_{i_0 i_1 \ldots i_k} = \{A_{i_0}, A_{i_1}, \ldots, A_{i_k}\}$ in space $E^n$, we have

$$\det\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{a^2_{i_\alpha i_\beta}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k} = (-1)^k 2^k k!^2 V^2_{i_0 i_1 \ldots i_k};$$

$$\det(a^2_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k} = (-1)^{k+1} 2^{k+1} k!^2 R^2_{i_0 i_1 \ldots i_k} V^2_{i_0 i_1 \ldots i_k},$$

so that from cosine law in $E^2$, we find

$$\begin{aligned}
\det\left(\cos\frac{a_{i_\alpha i_\beta}}{R}\right)_{\alpha,\beta=0}^{k} &= \det\left(1 - \frac{a^2_{i_\alpha j_\beta}}{2R^2}\right)_{\alpha,\beta=0}^{k} \\
&= \det\begin{pmatrix} 1 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{a^2_{i_\alpha i_\beta}}{2R^2}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k} \\
&= \det\begin{pmatrix} 0 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{a^2_{i_\alpha i_\beta}}{2R^2}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k} + \det\left(-\frac{a^2_{i_\alpha i_\beta}}{2R^2}\right)_{\alpha,\beta=0}^{k} \\
&= \frac{(-1)^{k+1}}{(2R^2)^k} \cdot \det\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{a^2_{i_\alpha i_\beta}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k} + \frac{(-1)^{k+1}}{(2R^2)^{k+1}} \cdot \det(a^2_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k} \\
&= \frac{k!^2}{R^{2(k+1)}} \cdot (R^2 - R^2_{i_0 i_1 \ldots i_k}) V^2_{i_0 i_1 \ldots i_k}.
\end{aligned}$$

The proof is finished.

LEMMA 2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ $(N > n)$ denote a finite point set in an $n$-dimensional spherical space $S^n(K)$. Choosing $k+1$ vertices $A_{i_0}$, $A_{i_1}$, … $A_{i_k}$ from the set $\mathcal{A}$, denote by $\rho_{i_0 i_1 \ldots i_k}$ circumradius of the $k$-dimensional subsimplex $\mathcal{A}_{i_0 i_1 \ldots i_k}$ constitued by the points $A_{i_0}$, $A_{i_1}$, …, $A_{i_k}$, the $k$-dimensional Euclidean space $E^k$, $V_{i_0 i_1 \ldots i_k}$ denote $k$-dimensional volume of $\mathcal{A}_{i_0 i_1 \ldots i_k}$, then*

$$\det(\cos\sqrt{K} a_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k} = \frac{k!^2}{R^{2k}} \cdot \cos^2(\sqrt{K}\rho_{i_0 i_1 \ldots i_k}) \cdot V_{i_0 i_1 \ldots i_k}^2. \tag{9}$$

*Proof.* Let $A_1$, $A_2$, …, $A_{n+1}$ denote $n+1$ points in the $n$-dimensional spherical space $S^n(K)$, $\rho$ is the circumradius of spherical simplex $\{A_1, A_2, \ldots, A_{n+1}\}$, then [4]

$$\cos^2\sqrt{K}\rho = -\frac{\det A}{\det \overline{A}}, \tag{10}$$

where

$$A = (\cos\sqrt{K} a_{ij})_{ij=1}^{n+1}; \quad \overline{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{A} & \\ 1 & & & \end{pmatrix}$$

Consequently, from (8) and (10) we get

$$\begin{aligned}\cos^2\sqrt{K}\rho_{i_0 i_1 \ldots i_k} &= -\frac{\frac{k!^2}{R^{2(k+1)}} \cdot (R^2 - R_{i_0 i_1 \ldots i_k}^2) \cdot V_{i_0 i_1 \ldots i_k}^2}{\frac{(-1)^k}{(2R^2)^k} \cdot (-1)^{k+1} 2^k k!^2 V_{i_0 i_1 \ldots i_k}^2} \\ &= \frac{R^2 - R_{i_0 i_1 \ldots i_k}^2}{R^2}\end{aligned}$$

hence

$$\begin{aligned}\det(\cos\sqrt{K} a_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k} &= -\cos^2\sqrt{K}\rho_{i_0 i_1 \ldots i_k} \cdot \det\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & & & \\ \vdots & & \boxed{\cos\sqrt{K} a_{i_\alpha i_\beta}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k} \\ &= -\cos^2\sqrt{K}\rho_{i_0 i_1 \ldots i_k} \cdot \det\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & & & \\ \vdots & & \boxed{1 - \frac{a_{i_\alpha i_\beta}^2}{2R^2}} & \\ 1 & & & \end{pmatrix}_{\alpha,\beta=0}^{k}\end{aligned}$$

$$=\frac{(-1)^{k+1}\cos^2\sqrt{K}\rho_{i_0i_1\dots i_k}}{(2R^2)^k}\cdot\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{a^2_{i_\alpha i_\beta}} & \\ 1 & & & \end{vmatrix}^k_{\alpha,\beta=0}$$

$$=\frac{(-1)^{k+1}}{(2R^2)^k}\cdot\cos^2\sqrt{K}\rho_{i_0i_1\dots i_k}\cdot(-1)^{k+1}2^k k!^2 V^2_{i_0i_1\dots i_k}$$

$$=\frac{k!^2}{R^{2k}}\cdot\cos^2\sqrt{K}\rho_{i_0i_1\dots i_k}\cdot V^2_{i_0i_1\dots i_k}.$$

The proof is completed.

In the hyperbolic space $H^n(K)$, we write

$$\det(\operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta})^k_{\alpha,\beta=0}=(-1)^k k!^2\cdot\operatorname{sh}^2(\sqrt{-K}V_{i_0i_1\dots i_k}),\quad (0\leqslant k\leqslant n). \tag{11}$$

LEMMA 3. *Let* $S_R^{n-1}$ *denote* $(n-1)$*-dimensional sphere with radius* $R$*,* $\mathcal{A}=\{A_1,A_2,\dots,A_N\}$ $(N>n)$ *is a finite point set, and* $\mathcal{A}\subset H^n(K)$*. Choosing* $k+1$ *vertices* $A_{i_0}$*,* $A_{i_1}$*, ...,* $A_{i_k}$ *from the set* $\mathcal{A}$*, denote by* $R_{i_0i_1\dots i_k}$*, the circumradius of the* $k$*-dimensional simplex* $\mathcal{A}_{i_0i_1\dots i_k}$*, constituted by points* $A_{i_0}$*,* $A_{i_1}$*, ...,* $A_{i_k}$*. Then*

$$\det(\cos\alpha_{i_\alpha i_\beta})^k_{\alpha,\beta=0}=\frac{k!^2}{(\operatorname{sh}^2\sqrt{-K}R)^{k+1}}\cdot\frac{\operatorname{ch}^2\sqrt{-K}R-\operatorname{ch}^2\sqrt{-K}R_{i_0i_1\dots i_k}}{\operatorname{ch}^2\sqrt{-K}R_{i_0i_1\dots i_k}}\cdot\operatorname{sh}^2\sqrt{-K}V_{i_0i_1\dots i_k}, \tag{12}$$

*Proof.* Suppose that $\mathcal{A}=\{A_1,A_2,\dots,A_{n+1}\}$ is hyperbolic simplex in the $n$-dimensional hyperbolic space $H^n(K)$, its circumradius is $R$, hyperbolic distance of vertices $A_i$ to $A_j$ is $a_{ij}$. By the paper [5] we have

$$\operatorname{ch}^2\sqrt{-K}R=-\frac{\det B}{\det\overline{B}}. \tag{13}$$

where

$$B=(\operatorname{ch}\sqrt{-K}a_{ij})^{n+1}_{ij},\quad \overline{B}=\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{B} & \\ 1 & & & \end{pmatrix}.$$

According to 2–dimensional cosine law and (13), we get at once

$$\det(\cos_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k} = \det\Big(\frac{\operatorname{ch}^2\sqrt{-K}R - \operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta}}{\operatorname{sh}^2\sqrt{-K}R}\Big)_{\alpha,\beta=0}^{k}$$
$$= \frac{1}{(\operatorname{sh}^2\sqrt{-K}R)^{k+1}}\cdot\det(\operatorname{ch}^2\sqrt{-K}R - \operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k}$$
$$= \frac{1}{(\operatorname{sh}^2\sqrt{-K}R)^{k+1}}\cdot\begin{vmatrix} 1 & -\operatorname{ch}^2\sqrt{-K}R & \cdots & -\operatorname{ch}^2\sqrt{-K}R \\ 1 & & & \\ \vdots & & \boxed{-\operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta}} & \\ 1 & & & \end{vmatrix}_{\alpha,\beta=0}^{k}$$
$$= \frac{(-1)^{k+1}\operatorname{ch}^2\sqrt{-K}R}{(\operatorname{sh}\sqrt{-K}R)^{k+1}}\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{\operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta}} & \\ 1 & & & \end{vmatrix}_{\alpha,\beta=0}^{k}$$
$$+\frac{(-1)^{k+1}}{(\operatorname{sh}^2\sqrt{-K}R)^{k+1}}\cdot\det(\operatorname{ch}\sqrt{-K}a_{i_\alpha i_\beta})_{\alpha,\beta=0}^{k}$$
$$= \frac{k!^2}{(\operatorname{sh}^2\sqrt{-K}R)^{k+1}}\cdot\frac{\operatorname{ch}^2\sqrt{-K}R - \operatorname{ch}^2\sqrt{-K}R_{i_0i_1\ldots i_k}}{\operatorname{ch}^2\sqrt{-K}R_{i_0i_1\ldots i_k}}\cdot\operatorname{sh}^2\sqrt{-K}V_{i_0i_1\ldots i_k}.$$

Lemma 3 is proved.

## 3. Main result and its corollaries

Suppose that the points $A_i \in \mathcal{A}$ are corresponding to the positive real numbers $m$ $(1 \leqslant i \leqslant N)$. Put

$$M_{e,k+1} = \sum_{1\leqslant i_0<i_1<\cdots<i_k\leqslant N}\sum\cdots\sum m_{i_0}m_{i_1}\ldots m_{i_k}(R^2 - R^2_{i_0i_1\ldots i_k})V^2_{i_0i_1\ldots i_k};$$

$$M_{s,k+1} = R^2\sum_{1\leqslant i_0<i_1<\cdots<i_k\leqslant N}\sum\cdots\sum m_{i_0}m_{i_1}\ldots m_{i_k}\cos^2\sqrt{K}\rho_{i_0i_1\ldots i_k}\cdot V^2_{i_0i_1\ldots i_k}; \qquad (K>0),$$

$$M_{h,k+1} = \sum_{1\leqslant i_0<i_1<\cdots<i_k\leqslant N}\sum\cdots\sum m_{i_0}m_{i_1}\ldots m_{i_k}\cdot\frac{\operatorname{ch}^2\sqrt{-K}R - \operatorname{ch}^2\sqrt{-K}R_{i_0i_1\ldots i_k}}{\operatorname{ch}^2\sqrt{-K}R_{i_0i_1\ldots i_k}}\operatorname{sh}^2\sqrt{-K}V_{i_0i_1\ldots i_k},$$

(here $K<0$).

For $k=0$ we have

$$M_{e,1} = R^2\sum_{i=1}^{N}m_1,\quad M_{s,1} = R^2\sum_{i=1}^{N}m_1,\quad M_{h,1} = \operatorname{sh}^2\sqrt{-K}R\cdot\sum_{i=1}^{N}m_i.$$

We write

$$\begin{aligned}
&M_{k+1} = (M_{e,k+1}, M_{s,k+1}, M_{h,k+1}),\\
&f(R) = (R, R, \operatorname{sh}\sqrt{-K}R),\\
&Q = (\sqrt{m_i, m_j} q_{ij})_{N\times N}, \quad \operatorname{rank} Q = r
\end{aligned}$$

THEOREM 2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ $(N > n)$ denotes a finite point set on a sphere in the constant curvature space $C^n(K)$, the points $A_i$ corresponding to the positive real numbers $m_i$ $(1 \leqslant i \leqslant N)$, $\alpha_1$, $\alpha_2$, ..., $\alpha_m$ a set of non–negative real numbers, and $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$. For $k_i \in \mathbf{N}$ $(1 \leqslant i \leqslant m)$, denote $\alpha_1 k_1 + a_2 k_2 + \cdots + \alpha_m k_m = k_0 \in \mathbf{N}$. Then we have*

$$M_{k_0+1} \geqslant \varphi(\alpha, k) \cdot \prod_{i=1}^{m} M_{k_i+1}^{\alpha_i}, \tag{14}$$

*where equality holds if and only if $r$ non–zero eigenvalues of matrix $Q$ are the same where*

$$\varphi(\alpha, k) = \frac{C_r^{k_0+1}}{k_0!^2} \cdot \prod_{i=1}^{m} \left(\frac{k_i!^2}{C_r^{k_i+1}}\right)^{\alpha_i}$$

*and* $C_r^l = \dfrac{r!}{l! \cdot (r-l)!}$, $(l = k_1 + 1, k_2 + 1, \ldots, k_m + 1)$.

*Proof.* It is easy to see that $Q = (\sqrt{m_i m_j} q_{ij})_{N\times N}$ is a semi–positive matrix and $\operatorname{rank} Q = r = \operatorname{rank}(n, n+1, n+1)$. Let $I$ be $N \times N$ unit matrix, then the characteristic equation of $Q$ is $\det(Q - xI) = 0$ we obtain by expanding it

$$x^N - q_1 x^{N-1} + \cdots + (-1)^k q_k x^{N-k} + \cdots + (-1)^N q_N = 0,$$

for this equation, deleting $N - r$ zero roots, we obtain the equation

$$x^r - a_1 x^{r-1} + \cdots + (-1)^k a_k x^{r-k} + \cdots + (-1)^r a_r = 0.$$

By the relation principal minors of matrix and coefficient of characteristic equation and lemma 1 to 3 can get

$$a_k = \frac{(k-1)!^2}{[f(R)]^{2k}} \cdot M_k, \quad (1 \leqslant k \leqslant r),$$

consequently

$$a_{k+1} = \frac{k!^2}{[f(R)]^{2(k+1)}} \cdot M_{k+1}, \quad (0 \leqslant k \leqslant r-1). \tag{15}$$

By Vieta's theorem and Theorem 1 we yields

$$\frac{\frac{k_0!^2}{[f(R)]^{2(k_0+1)}} \cdot M_{k_0+1}}{C_r^{k_0+1}} \geqslant \prod_{i=1}^{m} \left[\frac{\frac{k_i!^2}{[f(R)]^{2(k_i+1)}} \cdot M_{k_i+1}}{C_r^{k_i+1}}\right]^{\alpha_i}, \tag{16}$$

substituting (15) into (16) we can obtain (14). The necessary and sufficient condition of the equality is not difficult to see. Theorem 2 is proved.

COROLLARY 3. *Under the same condition of Theorem 2, if* $\alpha_i = \frac{1}{m}$, $(1 \leqslant i \leqslant m)$, *and* $\sum_{i=1}^{m} k_i = mk$, *then*

$$M_{k+1}^m \geqslant \varphi(k) \cdot \prod_{i=1}^{m} M_{k_i+1}. \tag{17}$$

*where equality holds if and only if* $r$ *nonzero eigenvalues of matrix* $Q$ *are the same, where*

$$\varphi(k) = \left(\frac{C_r^{k+1}}{k!^2}\right)^m \cdot \left(\prod_{i=1}^{m} \frac{k_i!^2}{C_r^{k_i+1}}\right).$$

In $(17)$, when $m$ is even number, from the paper [6] we know that $m_i$ $(1 \leqslant i \leqslant N)$ may be arbitrary real numbers in matrix $Q = (\sqrt{m_i m_j} q_{ij})_{N\times N}$.

COROLLARY 4. *Under the hypotheses in Theorem 2 and Corollary 2, we have*

$$\frac{M_{k+1}^{l+1}}{M_{l+1}^{k+1}} \geqslant \left(\frac{l!^2}{C_r^{l+1}}\right)^{k+1} \left(\frac{C_r^{k+1}}{k!^2}\right)^{l+1}, \quad (0 \leqslant k < l \leqslant r-1). \tag{18}$$

*where equality holds iff all nonzero eigenvalues of matrix* $Q$ *are equal.*

Suppose that in $(18)$ $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$ $(N > n)$ is a finite point set on a sphere in the $n$-dimensional Euclidean space $E^n$. For $k = 0$, $l = 1$, taking into account $R_{ij} = \dfrac{a_{ij}}{2}$, we can obtain immediately $(1)$. Therefore, $(18)$ is an extension of $(1)$ in the $n$-dimensional space of constant curvature.

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