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ON WEIGHTED WEAK TYPE MAXIMAL INEQUALITIES FOR MARTINGALES

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Dedicated to the memory of Professor Chinami Watari

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Abstract. Let Φ be a Young function and (u, v) a pair of weights on a probability space. We consider the inequality

$$\sup_{\lambda \in (0,\infty)} \Phi(\lambda)\,\mathbb{E}\Big[u : \{Mf > \lambda\}\Big] \leqslant \mathbb{E}\Big[\Phi\Big(C|f_\infty|\Big)\,v\Big]$$

for martingales $f = (f_n)_{n \in \mathbb{Z}_+}$, where $Mf = \sup_{n \in \mathbb{Z}_+} |f_n|$ and $f_\infty = \lim_n f_n$ a.s. We give some necessary and sufficient conditions for this inequality to hold, and extend Uchiyama's result.

1. Introduction

Let u and v be (strictly) positive integrable random variables (r.v.'s) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Such r.v.'s will be called *weights*. Let $1 < p < \infty$ and let $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+}$ be a *filtration*, that is, an increasing sequence of sub-σ-algebras of $\mathcal{A}$. In [5] Uchiyama proved that the inequality

$$\sup_{\lambda \in (0,\infty)} \lambda^p\,\mathbb{E}\big[u : \{Mf > \lambda\}\big] \leqslant C_{p,u,v}\,\mathbb{E}\big[|f_\infty|^p\,v\big]$$

holds for all uniformly integrable $(\mathbb{P}, \mathcal{F})$-martingales $f = (f_n)_{n \in \mathbb{Z}_+}$ if and only if there is a constant $K > 0$ such that almost surely (a.s.)

$$\sup_{n \in \mathbb{Z}_+} \mathbb{E}\big[v^{-1/(p-1)}\,\big|\,\mathcal{F}_n\big]^{p-1}\,\mathbb{E}\big[u\,|\,\mathcal{F}_n\big] \leqslant K, \qquad (A_p)$$

where $Mf = \sup_n |f_n|$, $f_\infty = \lim_n f_n$ a.s., and $\mathbb{E}[u : \Lambda] = \mathbb{E}[u 1_\Lambda]$.

Let Φ be an N-function. In this paper, we give some necessary and sufficient conditions for the inequality

$$\sup_{\lambda \in (0,\infty)} \Phi(\lambda)\,\mathbb{E}\big[u : \{Mf > \lambda\}\big] \leqslant \mathbb{E}\big[\Phi(C|f_\infty|)\,v\big] \qquad (W_\Phi)$$

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to hold, where $f = (f_n)$ is a uniformly integrable $(\mathbb{P}, \mathcal{F})$-martingale and $C > 0$ is a constant independent of $f = (f_n)$. We show that (W_Φ) holds for all $f = (f_n)$ if and only if there is a constant $C' > 0$ such that a.s.

$$\sup_{\substack{\lambda \in (0,\infty) \\ n \in \mathbb{Z}_+}} \frac{1}{\Phi(\lambda)\, \mathbb{E}[\, u \,|\, \mathcal{F}_n\,]} \mathbb{E}\left[\Psi\left(\frac{\Phi(\lambda)\, \mathbb{E}[\, u \,|\, \mathcal{F}_n\,]}{C' v \lambda} \right) v \,\middle|\, \mathcal{F}_n \right] \leqslant 1, \tag{1}$$

where Ψ is the complementary N-function of Φ.

Under the assumption that Φ satisfies the Δ_2- and ∇_2-conditions, we also show that condition (1) holds if and only if there is a constant $K > 0$ such that a.s.

$$\sup_{\substack{\varepsilon \in (0,\infty) \\ n \in \mathbb{Z}_+}} \varepsilon\, \varphi\left(\mathbb{E}\left[\varphi^{-1}\left(\frac{1}{\varepsilon v} \right) \,\middle|\, \mathcal{F}_n \right] \right) \mathbb{E}[\, u \,|\, \mathcal{F}_n\,] \leqslant K, \tag{A_Φ}$$

where φ is the right-derivative of Φ and φ^{-1} is the right-continuous inverse function of φ. Note that if $\Phi(t) = t^p$ $(1 < p < \infty)$, then condition (A_Φ) (or (1)) coincides with (A_p). We also study some norm inequalities in weighted Orlicz spaces.

2. Preliminaries

We shall work with a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+}$. We denote by $\mathcal{M}$ the collection of all uniformly integrable $(\mathbb{P}, \mathcal{F})$-martingales, and by $\mathcal{T}$ the collection of all $\mathcal{F}$-stopping times. For $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}$, we let

$$Mf = \sup_{n \in \mathbb{Z}_+} |f_n| \quad \text{and} \quad f_\infty = \lim_{n \to \infty} f_n \text{ a.s.}$$

If x is a nonnegative r.v. that is not integrable and if $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$, then we define

$$\mathbb{E}[\, x \,|\, \mathcal{B}\,] = \lim_{n \to \infty} \mathbb{E}[\, x \wedge n \,|\, \mathcal{B}\,].$$

If u is a weight, then $\mathbb{P}_u$ denotes the measure defined by

$$\mathbb{P}_u(\Lambda) = \mathbb{E}[\, u : \Lambda\,] = \mathbb{E}[\, u 1_\Lambda] \qquad (\Lambda \in \mathcal{A}).$$

The integral $\int_\Omega x \, d\mathbb{P}_u$ of $x \in L_1(\mathbb{P}_u)$ is denoted by $\mathbb{E}_u[\, x\,]$, that is, $\mathbb{E}_u[\, x\,] = \mathbb{E}[\, xu\,]$.

Now let Φ be a *Young function* with the right-derivative φ. In other words, let φ be a nonnegative, nondecreasing, and right-continuous function on $\mathbb{R}_+$, and let

$$\Phi(t) = \int_0^t \varphi(s)\, ds \qquad (t \in \mathbb{R}_+).$$

As usual, we call Φ an *N-function* if φ satisfies the following three conditions:

- $\varphi(0) = \lim_{s \to 0+} \varphi(s) = 0$;
- $0 < s < \infty \iff 0 < \varphi(s) < \infty$;
- $\lim_{s \to \infty} \varphi(s) = \infty$.

Note that every N-function is strictly increasing and has the inverse function.

The *right-continuous inverse function* of φ is given by

$$\varphi^{-1}(t) = \sup\{s \in \mathbb{R}_+ : \varphi(s) \leqslant t\} \qquad (t \in \mathbb{R}_+).$$

It is clear that

$$\varphi\big(\tfrac{1}{2}\,\varphi^{-1}(t)\big) \leqslant t \leqslant \varphi\big(\varphi^{-1}(t)\big) \wedge \varphi^{-1}\big(\varphi(t)\big) \qquad (t \in \mathbb{R}_+). \tag{2}$$

The Young function given by

$$\Psi(t) = \int_0^t \varphi^{-1}(s)\,ds \qquad (t \in \mathbb{R}_+)$$

is called the *complementary function* of Φ. Note that Ψ is an N-function if and only if so is Φ. It is well known that

$$st \leqslant \Phi(s) + \Psi(t) \qquad (s,\, t \in \mathbb{R}_+). \tag{3}$$

This is called the *Young inequality* (see e.g. [3, p. 12]). We also use the following inequalities (see [3, p. 13] and [1]):

$$t \leqslant \Phi^{-1}(t)\Psi^{-1}(t) \leqslant 2t \qquad (t \in \mathbb{R}_+); \tag{4}$$

$$\Phi\big(\Psi(t)/t\big) \leqslant \Psi(t), \quad \Psi\big(\Phi(t)/t\big) \leqslant \Phi(t) \qquad (t \in \mathbb{R}_+). \tag{5}$$

We say that Φ satisfies the Δ_2*-condition* and write $\Phi \in \Delta_2$ if there is a constant $c > 0$ such that

$$\Phi(2t) \leqslant c\,\Phi(t) \qquad (t \in \mathbb{R}_+). \tag{6}$$

It is well known that Φ satisfies the Δ_2-condition if and only if there is a constant $c' > 0$ such that

$$\varphi(2t) \leqslant c'\varphi(t) \qquad (t \in \mathbb{R}_+)$$

(cf. [4, p. 211]). We say that Φ satisfies the ∇_2*-condition* and write $\Phi \in \nabla_2$ if Ψ satisfies the Δ_2-condition, namely

$$\Psi(2t) \leqslant d\,\Psi(t) \qquad (t \in \mathbb{R}_+) \tag{7}$$

for some constant $d > 0$.

3. Weak type inequalities

Throughout the rest of the paper, we assume that $\sigma(\mathcal{F}) = \sigma\big(\bigcup_n \mathcal{F}_n\big) = \mathcal{A}$. Our main result is as follows:

THEOREM 1. *Let* (Φ, Ψ) *be a pair of complementary* N*-functions and* (u, v) *a pair of weights. Then the following are equivalent:*

(i) *There is a constant* $C > 0$*, independent of* $f = (f_n) \in \mathcal{M}$*, such that*

$$\sup_{\lambda \in (0, \infty)} \Phi(\lambda)\, \mathbb{P}_u(Mf > \lambda) \leqslant \mathbb{E}_v\big[\,\Phi\big(C|f_\infty|\big)\big].$$

(ii) *There is a constant* $C > 0$*, independent of* $f = (f_n) \in \mathcal{M}$ *and* $n \in \mathbb{Z}_+$*, such that a.s.*

$$\Phi\big(|f_n|\big)\, \mathbb{E}[\,u \mid \mathcal{F}_n\,] \leqslant \mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\, v \,\big|\, \mathcal{F}_n\,\big].$$

(iii) *There is a constant* $C > 0$*, independent of* $f = (f_n) \in \mathcal{M}$ *and* $\tau \in \mathcal{T}$*, such that a.s.*

$$\Phi\big(|f_\tau|\big)\, \mathbb{E}[\,u \mid \mathcal{F}_\tau\,] \leqslant \mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\, v \,\big|\, \mathcal{F}_\tau\,\big].$$

(iv) *There is a constant* $C > 0$*, independent of* $\lambda \in (0, \infty)$ *and* $n \in \mathbb{Z}_+$*, such that a.s.*

$$\mathbb{E}\left[\,\Psi\left(\frac{\Phi(\lambda)\, \mathbb{E}[\,u \mid \mathcal{F}_n\,]}{Cv\lambda}\right) v \,\middle|\, \mathcal{F}_n\right] \leqslant \Phi(\lambda)\, \mathbb{E}[\,u \mid \mathcal{F}_n\,].$$

COROLLARY 1. *Let* (Φ, Ψ) *and* (u, v) *be as in Theorem 1. If* $\Phi \in \Delta_2 \cap \nabla_2$*, then the following are equivalent:*

(i) *There is a constant* $C > 0$*, independent of* $f = (f_n) \in \mathcal{M}$*, such that*

$$\sup_{\lambda \in (0, \infty)} \Phi(\lambda)\, \mathbb{P}_u(Mf > \lambda) \leqslant C\, \mathbb{E}_v\big[\,\Phi\big(|f_\infty|\big)\big].$$

(ii) *There is a constant* $K > 0$ *such that a.s.*

$$\sup_{\substack{\varepsilon \in (0, \infty) \\ n \in \mathbb{Z}_+}} \varepsilon\, \varphi\left(\mathbb{E}\left[\,\varphi^{-1}\left(\frac{1}{\varepsilon v}\right) \,\middle|\, \mathcal{F}_n\,\right]\right) \mathbb{E}[\,u \mid \mathcal{F}_n\,] \leqslant K. \qquad (A_\Phi)$$

Here φ is the right-derivative of Φ and φ^{-1} is the right-continuous inverse function of φ.

REMARK. We cannot replace (A_Φ) by the condition that a.s.

$$\sup_{n \in \mathbb{Z}_+} \varphi\Big(\mathbb{E}\big[\,\varphi^{-1}\big(1/v\big) \mid \mathcal{F}_n\,\big]\Big)\, \mathbb{E}[\,u \mid \mathcal{F}_n\,] \leqslant K.$$

See Appendix.

Before proving these results, we note that if $x > 0$ a.s. and if $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$, then $\mathbb{E}[x \mid \mathcal{B}] > 0$ a.s. Indeed, since $\mathbb{E}\big[x : \{\mathbb{E}[x \mid \mathcal{B}] = 0\}\big] = 0$, we see that $1_{\{\mathbb{E}[x \mid \mathcal{B}] = 0\}} = 0$ a.s.

In order to prove Theorem 1, we need the following lemmas.

LEMMA 1. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$ and C a positive constant. Then the inequality*

$$\mathbb{E}\left[\Psi\left(\frac{\Phi(\lambda)\,\mathbb{E}[u\,|\,\mathcal{B}]}{Cv\lambda}\right)v\,\middle|\,\mathcal{B}\right] \leqslant \Phi(\lambda)\,\mathbb{E}[u\,|\,\mathcal{B}] \tag{8}$$

holds a.s. for all $\lambda \in (0,\,\infty)$ if and only if the inequality

$$\mathbb{E}\left[\Psi\left(\frac{\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}{Cv\eta}\right)v\,\middle|\,\mathcal{B}\right] \leqslant \Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}] \tag{9}$$

holds a.s. for all $\mathcal{B}$-measurable r.v.'s η such that $0<\eta<\infty$ a.s.

Proof. Suppose that (8) holds for all $\lambda \in (0,\,\infty)$. Then it is easy to verify that (9) holds for *simple* $\mathcal{B}$-measurable r.v. η such that $0<\eta<\infty$. If η is any positive *bounded* r.v., then there is a sequence $\{\eta_n\}$ of simple r.v.'s such that $\eta_n \downarrow \eta$. Hence, from Fatou's lemma, we see that (9) holds for such η. Finally, using the monotone convergence theorem, we obtain (9) for any $\mathcal{B}$-measurable r.v. η such that $0<\eta<\infty$ a.s. This completes the proof, since the converse is trivial. □

LEMMA 2. *Let $\mathcal{B}$ and C be as in Lemma 1. If (8) holds a.s. for all $\lambda \in (0,\,\infty)$, then*

$$\Phi\big(\mathbb{E}[x\,|\,\mathcal{B}]\big)\,\mathbb{E}[u\,|\,\mathcal{B}] \leqslant \mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big] \tag{10}$$

a.s. for any nonnegative r.v. x, with $C'=2C$.

Proof. To prove (10), we may assume that $x \in L_\infty$. In view of the dominated convergence theorem, we may assume in addition that $x>0$ a.s.

Let $C'=2C$ and set

$$\eta = \Phi^{-1}\left(\frac{\mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big]}{\mathbb{E}[u\,|\,\mathcal{B}]}\right).$$

Then $\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}] = \mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big]$ a.s. Using the Young inequality (3) and inequality (9) (which is valid by Lemma 1), we find that a.s.

$$\begin{aligned}
\mathbb{E}[x\,|\,\mathcal{B}] &= \frac{\eta}{2\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}\,\mathbb{E}\left[\frac{\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}{Cv\eta}\cdot C'x\cdot v\,\middle|\,\mathcal{B}\right]\\
&\leqslant \frac{\eta}{2\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}\left\{\mathbb{E}\left[\Psi\left(\frac{\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}{Cv\eta}\right)v\,\middle|\,\mathcal{B}\right]+\mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big]\right\}\\
&\leqslant \frac{\eta}{2\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]}\Big\{\Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}]+\mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big]\Big\}\\
&= \eta,
\end{aligned}$$

so that a.s.

$$\Phi\big(\mathbb{E}[x\,|\,\mathcal{B}]\big)\,\mathbb{E}[u\,|\,\mathcal{B}] \leqslant \Phi(\eta)\,\mathbb{E}[u\,|\,\mathcal{B}] = \mathbb{E}\big[\Phi(C'x)\,v\,\big|\,\mathcal{B}\big].$$

This completes the proof. □

As the following lemma shows, the converse of Lemma 2 is true.

LEMMA 3. *Let* $\mathcal{B}$ *be a sub-* σ *-algebra of* $\mathcal{A}$ *and* C' *a positive constant. If* (10) *holds a.s. for all nonnegative r.v.'s* x, *then* (8) *holds a.s. for all* $\lambda \in (0, \infty)$, *with* $C = 2C'$.

Proof. Given $\lambda \in (0, \infty)$ and $k \in (0, \infty)$, we let

$$\eta = \frac{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}{2\lambda} \quad \text{and} \quad x = \frac{v}{\eta}\,\Psi\Big(\frac{\eta}{C'v}\Big)\,1_\Lambda,$$

where $\Lambda = \{\eta \leqslant kv\}$. Then by (10),

$$\Phi\left(\frac{2\lambda\,\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\right) = \Phi\big(\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big) \leqslant \frac{\mathbb{E}\big[\,\Phi(C'x)\,v\,\big|\,\mathcal{B}\,\big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]}. \tag{11}$$

On the other hand, by (5),

$$\Phi(C'x) = \Phi\Big(\frac{C'v}{\eta}\,\Psi\Big(\frac{\eta}{C'v}\Big)\,1_\Lambda\Big) \leqslant \Psi\Big(\frac{\eta}{C'v}\Big)\,1_\Lambda. \tag{12}$$

Combining (11) and (12), we find that

$$\frac{2\lambda\,\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]} \leqslant \Phi^{-1}\left(\frac{\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\right).$$

This, together with (4), implies that

$$\frac{2\lambda\,\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\,\Psi^{-1}\left(\frac{\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\right)$$

$$\leqslant \frac{2\,\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]}.$$

Since $\mathbb{E}\big[\Psi\big(\frac{\eta}{C'v}\big)\,v\,1_\Lambda\,\big|\,\mathcal{B}\big] \leqslant \Psi\big(\frac{k}{C'}\big)\,\mathbb{E}[\,v\,|\,\mathcal{B}\,] < \infty$ a.s.,

$$\Psi^{-1}\left(\frac{\mathbb{E}\big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\right) \leqslant \frac{\Phi(\lambda)}{\lambda},$$

and hence by (5),

$$\frac{\mathbb{E}\Big[\Psi\Big(\frac{\eta}{C'v}\Big)\,v\,1_\Lambda\,\Big|\,\mathcal{B}\Big]}{\mathbb{E}[\,u\,|\,\mathcal{B}\,]} \leqslant \Psi\Big(\frac{\Phi(\lambda)}{\lambda}\Big) \leqslant \Phi(\lambda).$$

From the definition of η, we conclude that

$$\mathbb{E}\left[\Psi\left(\frac{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}{2\,C'v\lambda}\right) v\,1_{\{\eta\leqslant kv\}}\,\middle|\,\mathcal{B}\right] \leqslant \Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,].$$

Letting $k\to\infty$, we obtain (8) with $C=2C'$. $\square$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. (ii)$\Leftrightarrow$(iii). It is obvious that (iii)$\Rightarrow$(ii). To prove the converse, suppose that (ii) holds. Setting $f_\infty\equiv\lambda$ $(\lambda\in\mathbb{R}_+)$ and then letting $n\to\infty$, we see that

$$\Phi(\lambda)\,u\leqslant\Phi(C\lambda)\,v \qquad (\lambda\in\mathbb{R}_+).$$

Therefore, if $\tau\in\mathcal{T}$, then

$$\begin{aligned}\Phi\big(|f_\tau|\big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_\tau\,] &= \sum_{n=0}^{\infty}\Phi\big(|f_n|\big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,]\,1_{\{\tau=n\}}+\Phi\big(|f_\infty|\big)\,u\,1_{\{\tau=\infty\}}\\ &\leqslant \sum_{n=0}^{\infty}\mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\,v\,\big|\,\mathcal{F}_n\,\big]\,1_{\{\tau=n\}}+\Phi\big(C|f_\infty|\big)\,v\,1_{\{\tau=\infty\}}\\ &= \mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\,v\,\big|\,\mathcal{F}_\tau\,\big],\end{aligned}$$

as was to be shown.

(ii)$\Leftrightarrow$(iv). Lemma 2 yields that (iv) implies (ii), and Lemma 3 yields that (ii) implies (iv).

(iii)$\Rightarrow$(i). Let $\lambda\in(0,\infty)$, and define

$$\tau=\inf\big\{n\in\mathbb{Z}_+ : |f_n|>\lambda\big\}\in\mathcal{T}.$$

Here and elsewhere we follow the convention that $\inf\emptyset=\infty$. Then $\{\tau<\infty\}=\{Mf>\lambda\}$ and $|f_\tau|>\lambda$ on $\{\tau<\infty\}$. If (iii) holds, then

$$\begin{aligned}\Phi(\lambda)\,\mathbb{P}_u(Mf>\lambda) &\leqslant \mathbb{E}\big[\,\Phi\big(|f_\tau|\big)\,u:\{\tau<\infty\}\big]\\ &= \mathbb{E}\big[\,\Phi\big(|f_\tau|\big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_\tau\,]:\{\tau<\infty\}\big]\\ &\leqslant \mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\,v:\{\tau<\infty\}\big]\\ &\leqslant \mathbb{E}_v\big[\,\Phi\big(C|f_\infty|\big)\,\big].\end{aligned}$$

Thus (i) follows from (iii).

(i)$\Rightarrow$(ii). Suppose that (i) holds, and let $f=(f_n)\in\mathcal{M}$. If $\Lambda\in\mathcal{F}_n$ and if $\lambda\in(0,\infty)$, then

$$\begin{aligned}\Phi(\lambda)\,\mathbb{E}\big[\,1_{\{|f_n|>\lambda\}}\,u:\Lambda\big] &= \Phi(\lambda)\,\mathbb{P}_u\Big(\big\{|f_n|>\lambda\big\}\cap\Lambda\Big)\\ &\leqslant \Phi(\lambda)\,\mathbb{P}_u\Big(\big\{\mathbb{E}\big[|f_\infty|\,\big|\,\mathcal{F}_n\big]>\lambda\big\}\cap\Lambda\Big)\\ &\leqslant \Phi(\lambda)\,\mathbb{P}_u\Big(\sup_n\mathbb{E}\big[|f_\infty|\,1_\Lambda\,\big|\,\mathcal{F}_n\big]>\lambda\Big)\\ &\leqslant \mathbb{E}_v\big[\,\Phi\big(C|f_\infty|\,1_\Lambda\big)\,\big]\\ &= \mathbb{E}\big[\,\Phi\big(C|f_\infty|\big)\,v:\Lambda\big].\end{aligned}$$

Therefore we have:

$$\Phi(\lambda)\, 1_{\{|f_n|>\lambda\}}\, \mathbb{E}[\, u \,|\, \mathcal{F}_n \,] \leqslant \mathbb{E}\big[\, \Phi(C|f_\infty|)\, v \,\big|\, \mathcal{F}_n \,\big] \quad \text{a.s.}$$

This implies that

$$\begin{aligned}\Phi\big(|f_n|\big)\, \mathbb{E}[\, u \,|\, \mathcal{F}_n \,] &= \sup_{\lambda\in(0,\,\infty)} \Phi(\lambda)\, 1_{\{|f_n|>\lambda\}}\, \mathbb{E}[\, u \,|\, \mathcal{F}_n \,] \\ &\leqslant \mathbb{E}\big[\, \Phi(C|f_\infty|)\, v \,\big|\, \mathcal{F}_n \,\big] \qquad \text{a.s.,}\end{aligned}$$

which proves (ii). □

We now turn to the proof of Corollary 1. Note first that if $\Phi \in \Delta_2$, then (i) of Theorem 1 and (i) of Corollary 1 are equivalent. Hence it suffices to prove the following lemma.

LEMMA 4. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$ and suppose that $\Phi \in \Delta_2 \cap \nabla_2$. Then* (8) *holds a.s. for all $\lambda \in (0,\, \infty)$ if and only if*

$$\sup_{\varepsilon\in(0,\,\infty)} \varepsilon\, \varphi\left(\mathbb{E}\left[\, \varphi^{-1}\left(\frac{1}{\varepsilon v}\right) \,\middle|\, \mathcal{B} \,\right]\right) \mathbb{E}[\, u \,|\, \mathcal{B} \,] \leqslant K \tag{13}$$

a.s. for some constant $K > 0$.

Proof. Suppose that inequality (13) holds a.s. Then, by (2),

$$\mathbb{E}\left[\, \varphi^{-1}\left(\frac{\eta}{v}\right) \,\middle|\, \mathcal{B} \,\right] \leqslant \varphi^{-1}\left(\frac{K\eta}{\mathbb{E}[\, u \,|\, \mathcal{B} \,]}\right) \qquad \big(\eta \in (0,\, \infty)\big).$$

As in the proof of Lemma 1, we can show that the same inequality holds for any nonnegative $\mathcal{B}$-measurable r.v. η. Note that, since $\Phi \in \nabla_2$,

$$\Psi(t) \leqslant t\varphi^{-1}(t) \leqslant \Psi(2t) \leqslant d\,\Psi(t),$$

where $d > 0$ is the constant in (7). Hence, if η is nonnegative and $\mathcal{B}$-measurable, then

$$\begin{aligned}\mathbb{E}\left[\, \Psi\left(\frac{\eta}{v}\right) v \,\middle|\, \mathcal{B} \,\right] &\leqslant \mathbb{E}\left[\, \eta\, \varphi^{-1}\left(\frac{\eta}{v}\right) \,\middle|\, \mathcal{B} \,\right] \leqslant \eta\, \varphi^{-1}\left(\frac{K\eta}{\mathbb{E}[\, u \,|\, \mathcal{B} \,]}\right) \\ &\leqslant \frac{d}{K}\, \Psi\left(\frac{K\eta}{\mathbb{E}[\, u \,|\, \mathcal{B} \,]}\right) \mathbb{E}[\, u \,|\, \mathcal{B} \,] \\ &\leqslant \Psi\left(\frac{C\eta}{\mathbb{E}[\, u \,|\, \mathcal{B} \,]}\right) \mathbb{E}[\, u \,|\, \mathcal{B} \,],\end{aligned}$$

where $C = \left(\frac{d}{K} \vee 1\right) K = d \vee K$. Setting $\eta = (C\lambda)^{-1}\Phi(\lambda)\, \mathbb{E}[\, u \,|\, \mathcal{B} \,]$ and using (5), we see that

$$\mathbb{E}\left[\, \Psi\left(\frac{\Phi(\lambda)\, \mathbb{E}[\, u \,|\, \mathcal{B} \,]}{Cv\lambda}\right) v \,\middle|\, \mathcal{B} \,\right] \leqslant \Psi\left(\frac{\Phi(\lambda)}{\lambda}\right) \mathbb{E}[\, u \,|\, \mathcal{B} \,] \leqslant \Phi(\lambda)\, \mathbb{E}[\, u \,|\, \mathcal{B} \,]$$

for all $\lambda \in (0,\, \infty)$. Thus (8) holds a.s. with $C = d \vee K$.

Suppose conversely that (8) holds a.s. for all $\lambda \in (0, \infty)$. Then (9) holds a.s. for $\mathcal{B}$-measurable r.v.'s η such that $0 < \eta < \infty$ a.s. Given $\varepsilon \in (0, \infty)$, we let

$$\eta = 2\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big).$$

Then, by (2) and the inequality $t\varphi(t) \leqslant \Phi(2t)$, we have that

$$\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]} \leqslant \varphi\Big(\frac{\eta}{2}\Big) \leqslant \frac{2}{\eta}\,\Phi(\eta)$$

and hence that

$$\frac{2}{\varepsilon v} \leqslant \frac{\Phi(\eta)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}{Cv\eta}. \tag{14}$$

On the other hand, using (2), (6), and the inequality $\Phi(t) \leqslant t\varphi(t)$, we find:

$$\begin{aligned}\Phi(\eta) &\leqslant c^2\,\Phi\Big(\frac{\eta}{4}\Big) \leqslant c^2\,\frac{\eta}{4}\,\varphi\Big(\frac{\eta}{4}\Big)\\ &\leqslant \frac{c^2}{2}\,\varphi\bigg(\frac{1}{2}\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big)\bigg)\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big)\\ &\leqslant \frac{2Cc^2}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big).\end{aligned} \tag{15}$$

Therefore, by (9), (14), and (15),

$$\begin{aligned}\mathbb{E}\Big[\,\Psi\Big(\frac{2}{\varepsilon v}\Big)\,v\,\Big|\,\mathcal{B}\,\Big] &\leqslant \mathbb{E}\Big[\,\Psi\Big(\frac{\Phi(\eta)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}{Cv\eta}\Big)\,v\,\Big|\,\mathcal{B}\,\Big]\\ &\leqslant \Phi(\eta)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]\\ &\leqslant \frac{2Cc^2}{\varepsilon}\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big).\end{aligned}$$

Since $\frac{1}{\varepsilon v}\varphi^{-1}\big(\frac{1}{\varepsilon v}\big) \leqslant \Psi\big(\frac{2}{\varepsilon v}\big)$, we obtain that

$$\mathbb{E}\Big[\,\varphi^{-1}\Big(\frac{1}{\varepsilon v}\Big)\,\Big|\,\mathcal{B}\,\Big] \leqslant 2Cc^2\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big).$$

Since $\Phi \in \Delta_2$, there is a constant $k > 0$ such that $\varphi(4Cc^2t) \leqslant k\varphi(t)$ for all $t \in \mathbb{R}_+$. Therefore by (2),

$$\begin{aligned}\varphi\bigg(\mathbb{E}\Big[\,\varphi^{-1}\Big(\frac{1}{\varepsilon v}\Big)\,\Big|\,\mathcal{B}\,\Big]\bigg) &\leqslant \varphi\bigg(2Cc^2\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big)\bigg)\\ &\leqslant k\,\varphi\bigg(\frac{1}{2}\,\varphi^{-1}\Big(\frac{4C}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}\Big)\bigg) \leqslant \frac{4kC}{\varepsilon\,\mathbb{E}[\,u\,|\,\mathcal{B}\,]}.\end{aligned}$$

Thus (13) holds a.s. with $K = 4kC$. □

4. Norm inequalities

In this section we study some weighted norm inequalities. Let Φ be an N-function. Given a weight u, we consider the *Orlicz space* $L_\Phi(u)$ consisting of r.v.'s x such that $\mathbb{E}\big[\,\Phi\big(\lambda|x|\big)u\big] < \infty$ for some $\lambda \in (0,\,\infty)$. For each $\alpha \in (0,\,\infty)$, we define a norm on $L_\Phi(u)$ by setting

$$\|x\|_{\Phi,\,u,\,\alpha} = \inf\big\{\lambda \in (0,\,\infty) : \mathbb{E}\big[\,\Phi\big(\lambda^{-1}|x|\big)u\big] \leqslant \alpha\big\} \qquad \big(x \in L_\Phi(u)\big).$$

Note that the norms $\|\cdot\|_{\Phi,\,u,\,\alpha}$ and $\|\cdot\|_{\Phi,\,u,\,1}$ are equivalent. Indeed, using the elementary inequality $\Phi(at) \leqslant a\Phi(t)$ $(0 < a \leqslant 1,\ t \in \mathbb{R}_+)$, we see that

$$(\alpha \wedge 1)\,\|\cdot\|_{\Phi,\,u,\,\alpha} \leqslant \|\cdot\|_{\Phi,\,u,\,1} \leqslant (\alpha \vee 1)\,\|\cdot\|_{\Phi,\,u,\,\alpha}.$$

For convenience, we follow the convention that $\|x\|_{\Phi,\,u,\,\alpha} = \infty$ unless $x \in L_\Phi(u)$. Our aim here is to prove:

THEOREM 2. *Let* $(\Phi,\,\Psi)$ *be a pair of complementary* N*-functions and* $(u,\,v)$ *a pair of weights. Then the following are equivalent:*

(i) *There is a constant* $C > 0$*, independent of* $f = (f_n) \in \mathcal{M}$ *and* $\alpha \in (0,\,\infty)$*, such that*

$$\sup_{\lambda \in (0,\,\infty)} \lambda\,\|1_{\{Mf>\lambda\}}\|_{\Phi,\,u,\,\alpha} \leqslant C\|f_\infty\|_{\Phi,\,v,\,\alpha}.$$

(ii) *There is a constant* $C > 0$*, independent of* $x \in L_\Phi(v)$*,* $n \in \mathbb{Z}_+$*, and* $\alpha \in (0,\,\infty)$*, such that*

$$\big\|\mathbb{E}[\,x\,|\,\mathcal{F}_n\,]\big\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha}.$$

(iii) *There is a constant* $C > 0$*, independent of* $x \in L_\Phi(v)$*,* $\tau \in \mathcal{T}$*, and* $\alpha \in (0,\,\infty)$*, such that*

$$\big\|\mathbb{E}[\,x\,|\,\mathcal{F}_\tau\,]\big\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha}.$$

(iv) *There is a constant* $C > 0$*, independent of* $\lambda \in (0,\,\infty)$ *and* $n \in \mathbb{Z}_+$*, such that a.s.*

$$\mathbb{E}\left[\,\Psi\!\left(\frac{\Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,]}{Cv\lambda}\right) v \,\middle|\, \mathcal{F}_n\right] \leqslant \Phi(\lambda)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,].$$

Once we establish the following lemma, the proof of Theorem 2 is easy.

LEMMA 5. *Let* $\mathcal{B}$ *be a sub-*σ*-algebra of* $\mathcal{A}$ *and* C *a positive constant. Then the following are equivalent:*

(i) *If* x *is a nonnegative r.v. in* $L_\Phi(v)$*, then*

$$\sup_{\lambda \in (0,\,\infty)} \lambda\,\|1_{\{\mathbb{E}[x\,|\,\mathcal{B}]>\lambda\}}\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha} \ \textit{for all}\ \alpha \in (0,\,\infty).$$

(ii) *If* $x \in L_\Phi(v)$*, then*

$$\big\|\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha} \ \textit{for all}\ \alpha \in (0,\,\infty).$$

(iii) *If* x *is a nonnegative r.v., then*

$$\Phi\big(\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big)\,\mathbb{E}[\,u\,|\,\mathcal{B}\,] \leqslant \mathbb{E}\big[\,\Phi(Cx)\,v\,\big|\,\mathcal{B}\,\big] \ \textit{a.s.}$$

Proof. (i) $\Rightarrow$ (iii). Let x be a nonnegative r.v. and suppose that (i) holds. To prove the inequality in (iii), we may assume that $\mathbb{E}\big[\,\Phi(Cx)\,v\,\big] < \infty$. If we set $\alpha = \mathbb{E}\big[\,\Phi(Cx)\,v\,\big]$, then

$$\frac{\lambda}{\Phi^{-1}\left(\dfrac{\alpha}{\mathbb{E}\big[\,u : \{\mathbb{E}[\,x\,|\,\mathcal{B}\,] > \lambda\}\big]}\right)} = \lambda\|1_{\{\mathbb{E}[\,x\,|\,\mathcal{B}\,]>\lambda\}}\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha} \leqslant 1,$$

provided that $\mathbb{E}\big[\,u : \{\mathbb{E}[\,x\,|\,\mathcal{B}\,] > \lambda\}\big] \neq 0$. Thus

$$\Phi(\lambda)\,\mathbb{E}\big[\,u : \{\mathbb{E}[\,x\,|\,\mathcal{B}\,] > \lambda\}\big] \leqslant \alpha = \mathbb{E}\big[\,\Phi(Cx)\,v\,\big]$$

for any $\lambda \in (0,\,\infty)$. Replacing x by $x1_\Lambda$ with $\Lambda \in \mathcal{B}$ yields that

$$\Phi(\lambda)1_{\{\mathbb{E}[\,x\,|\,\mathcal{B}\,]>\lambda\}}\,\mathbb{E}[\,u\,|\,\mathcal{B}\,] \leqslant \mathbb{E}\big[\,\Phi(Cx)\,v\,\big|\,\mathcal{B}\,\big] \quad \text{a.s.}$$

Taking the supremum over $\lambda \in (0,\,\infty)$, we conclude that the inequality in (iii) holds a.s.

(iii) $\Rightarrow$ (ii). Let $x \in L_\Phi(v)$ and $\alpha \in (0,\,\infty)$. If we set $\beta = \|x\|_{\Phi,\,v,\,\alpha}$, then a.s.

$$\Phi\left(\frac{\big|\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big|}{C\beta}\right)\mathbb{E}[\,u\,|\,\mathcal{B}\,] \leqslant \mathbb{E}\big[\,\Phi\big(\beta^{-1}|x|\big)\,v\,\big|\,\mathcal{B}\,\big]$$

by the inequality in (iii).

Therefore

$$\mathbb{E}\left[\,\Phi\left(\frac{\big|\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big|}{C\beta}\right)u\,\right] \leqslant \mathbb{E}\big[\,\Phi\big(\beta^{-1}|x|\big)\,v\,\big] \leqslant \alpha,$$

whence

$$\big\|\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big\|_{\Phi,\,u,\,\alpha} \leqslant C\beta = C\|x\|_{\Phi,\,v,\,\alpha}.$$

(ii) $\Rightarrow$ (i). Let x be a nonnegative r.v. in $L_\Phi(v)$. If (ii) holds, then for all $\lambda \in (0,\,\infty)$,

$$\lambda\|1_{\{\mathbb{E}[\,x\,|\,\mathcal{B}\,]>\lambda\}}\|_{\Phi,\,u,\,\alpha} \leqslant \big\|\mathbb{E}[\,x\,|\,\mathcal{B}\,]\big\|_{\Phi,\,u,\,\alpha} \leqslant C\|x\|_{\Phi,\,v,\,\alpha},$$

which proves (i). □

Proof of Theorem 2. The equivalence of (ii) and (iii) follows from Theorem 1 and Lemma 5. The equivalence of (ii) and (iv) follows from Lemmas 2, 3, and 5. It only remains to prove the equivalence of (i) and (iii).

(iii) $\Rightarrow$ (i). Let $f = (f_n) \in \mathcal{M}$. Given $\lambda > 0$, we define

$$\tau = \inf\big\{n \in \mathbb{Z}_+ : |f_n| > \lambda\big\} \in \mathcal{T}.$$

Since $\{Mf > \lambda\} = \{\tau < \infty\} \in \mathcal{F}_\tau$ and $|f_\tau| > \lambda$ on $\{\tau < \infty\}$, we see that

$$\lambda 1_{\{Mf>\lambda\}} \leqslant |f_\tau|1_{\{\tau<\infty\}} \leqslant \mathbb{E}\big[\,|f_\infty|\,\big|\,\mathcal{F}_\tau\,\big].$$

Hence, if (iii) holds, then

$$\lambda \|1_{\{Mf>\lambda\}}\|_{\Phi,u,\alpha} \leqslant \|\mathbb{E}\big[\,|f_\infty|\,\big|\,\mathcal{F}_\tau\big]\|_{\Phi,u,\alpha} \leqslant C\|f_\infty\|_{\Phi,v,\alpha}$$

for all $\lambda \in (0, \infty)$ (with the convention that $\|f_\infty\|_{\Phi,v,\alpha} = \infty$ unless $f_\infty \in L_\Phi(v)$).

(i) $\Rightarrow$ (iii). Let y be a nonnegative r.v. in $L_\Phi(v)$ and let $\tau \in \mathcal{T}$. If $y \in L_1(\mathbb{P})$, then we can consider the martingale $f = \big(\mathbb{E}[\,y\,|\,\mathcal{F}_n\,]\big)_{n\in\mathbb{Z}_+} \in \mathcal{M}$ and apply (i) to conclude that

$$\lambda \|1_{\{\mathbb{E}[\,y\,|\,\mathcal{F}_\tau\,]>\lambda\}}\|_{\Phi,u,\alpha} \leqslant C\|y\|_{\Phi,v,\alpha} \qquad \big(\lambda,\ \alpha \in (0, \infty)\big). \tag{16}$$

This inequality remains valid for any nonnegative $y \in L_\Phi(v) \setminus L_1(\mathbb{P})$. Indeed, setting $y_k = y \wedge k$, we see that

$$\lambda \|1_{\{\mathbb{E}[\,y_k\,|\,\mathcal{F}_\tau\,]>\lambda\}}\|_{\Phi,u,\alpha} \leqslant C\|y_k\|_{\Phi,v,\alpha} \leqslant C\|y\|_{\Phi,v,\alpha},$$

which implies (16). Thus (i) of Lemma 5 holds with $\mathcal{B} = \mathcal{F}_\tau$. Therefore, for any $x \in L_\Phi(v)$,

$$\big\|\mathbb{E}[\,x\,|\,\mathcal{F}_\tau\,]\big\|_{\Phi,u,\alpha} \leqslant C\|x\|_{\Phi,v,\alpha}$$

by Lemma 5. This completes the proof. □

Appendix

Recall condition (A_Φ) in Corollary 1:

$$\sup_{\substack{\varepsilon\in(0,\infty)\\ n\in\mathbb{Z}_+}} \varepsilon\,\varphi\Big(\mathbb{E}\Big[\,\varphi^{-1}\Big(\frac{1}{\varepsilon v}\Big)\,\Big|\,\mathcal{F}_n\Big]\Big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \leqslant K. \tag{A_Φ}$$

If $\varphi(t) = t^{p-1}$, then (A_Φ) is equivalent to the condition that a.s.

$$\sup_{n\in\mathbb{Z}_+} \varphi\Big(\mathbb{E}\big[\,\varphi^{-1}(1/v)\,|\,\mathcal{F}_n\,\big]\Big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \leqslant K. \tag{17}$$

In general, however, (A_Φ) cannot be replaced by (17) as shown below.

According to Bonami and Lépingle [2, Section 3], for any $p \in (1, \infty)$, there exists a weight u (on a suitable probability space) satisfying the following conditions:

(i) $u \in L_\infty(\mathbb{P})$;

(ii) $\sup_{n\in\mathbb{Z}_+} \mathbb{E}\big[\,u^{-1/(p-1)}\,\big|\,\mathcal{F}_n\big]^{p-1}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \in L_\infty(\mathbb{P})$;

(iii) $\sup_{n\in\mathbb{Z}_+} \mathbb{E}\big[\,u^{-1/(q-1)}\,\big|\,\mathcal{F}_n\big]^{q-1}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \notin L_\infty(\mathbb{P})$ for any $q \in (1, p)$.

Let $1 < q < p < \infty$ and let Φ be the N-function defined by

$$\Phi(t) = \int_0^t \varphi(s)\,ds, \qquad \text{where } \varphi(s) = \begin{cases} s^{q-1} & \text{if } s \in [0,\,1), \\ s^{p-1} & \text{if } s \in [1,\,\infty). \end{cases}$$

Suppose that u satisfies (i) – (iii) above. Then

$$\begin{aligned} &\sup_{\substack{\varepsilon\in(0,\infty)\\ n\in\mathbb{Z}_+}} \varepsilon\,\varphi\Big(\mathbb{E}\Big[\,\varphi^{-1}\Big(\frac{1}{\varepsilon u}\Big)\,\Big|\,\mathcal{F}_n\Big]\Big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \\ &\qquad \geqslant \sup_{\substack{\varepsilon\in(0,\infty)\\ n\in\mathbb{Z}_+}} \varepsilon\,\mathbb{E}\Big[\Big(\frac{1}{\varepsilon u}\Big)^{1/(q-1)} 1_{\{\varepsilon u>1\}}\,\Big|\,\mathcal{F}_n\Big]^{q-1}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \\ &\qquad = \sup_{n\in\mathbb{Z}_+} \mathbb{E}\big[\,u^{-1/(q-1)}\,\big|\,\mathcal{F}_n\,\big]^{q-1}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,]. \end{aligned}$$

By (iii) the right-hand side does not belong to $L_\infty(\mathbb{P})$. On the other hand, since $\varphi^{-1}(1/u) \leqslant 1 + u^{-1/(p-1)}$, we find that

$$\begin{aligned} &\sup_{n\in\mathbb{Z}_+} \varphi\Big(\mathbb{E}\big[\,\varphi^{-1}(1/u)\,\big|\,\mathcal{F}_n\,\big]\Big)\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \\ &\qquad \leqslant \sup_{n\in\mathbb{Z}_+} \Big\{1 + \mathbb{E}\big[\,u^{-1/(p-1)}\,\big|\,\mathcal{F}_n\,\big]\Big\}^{p-1}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,] \\ &\qquad \leqslant 2^{p-1} \sup_{n\in\mathbb{Z}_+} \Big\{1 + \mathbb{E}\big[\,u^{-1/(p-1)}\,\big|\,\mathcal{F}_n\,\big]^{p-1}\Big\}\,\mathbb{E}[\,u\,|\,\mathcal{F}_n\,]. \end{aligned}$$

The right-hand side belongs to $L_\infty(\mathbb{P})$ by (i) and (ii). This shows that (17) does not imply (A_Φ) even in the case where $v = u$.

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