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SOME INEQUALITIES FOR GRAPHS DERIVABLE FROM HÖLDER'S INEQUALITY

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Abstract. Inequalities are derived as advertised in the title. For instance, it is shown that for a finite simple graph G, $4m \leqslant n(J_1 + t_1)$, where m is the number of edges of G, n is the number of vertices, J_1 is the average (over the edges) of the sizes of the "joint neighborhoods" of adjacent vertices, and t_1 is the average (over the edges) of the numbers of triangles in which the edges appear; with equality if and only if G is regular.

Throughout, G is a finite graph or multigraph with vertex set V and edge set E, $n = |V|$ and $m = |E|$. For $e \in E$, let $u(e)$, $v(e)$ denote the vertices at the two ends of e (so $u(e) = v(e)$ iff e is a loop). For $w \in V$, led $d(w)$ denote the degree of w in G (the number of edge ends coming into w; or, the number of non-loops incident to w plus twice the number of loops at w). It is elementary that $\sum_{w \in V} d(w) = 2m$. If the degrees of the vertices of G are all the same, G is *regular.* Let $N(w)$ denote the set of neighbors of w in G, i.e., $N(w) = \{x \in V;\text{ for some } e \in E, \{x, w\} = \{u(e), v(e)\}\}$. It is elementary that $|N(w)| = d(w)$ for all $w \in V$ if and only if G is *simple,* i.e., G contains no loops of multiple edges.

For $e \in E$ let $J(e) = |N(u(e)) \cup N(v(e))|$, the size of the "joint neighborhood" of the vertices at the ends of e, and $t(e) = |N(u(e)) \cap N(v(e))|$; if G is simple, $t(e)$ is the number of triangles in G in which e appears. Note that $J(e) + t(e) = |N(u(e))| + |N(v(e))|$, which is equal to $d(u(e)) + d(v(e))$ if G is simple.

For $0 < q < \infty$ we define $d_q(G) = d_q = \left(\frac{1}{n}\sum_{v \in V} d(v)^q\right)^{\frac{1}{q}}$, and if $E \neq \emptyset$, $J_q(G) = J_q = \left(\frac{1}{m}\sum_{e \in E} J(e)^q\right)^{\frac{1}{q}}$, $t_q(G) = t_q = \left(\frac{1}{m}\sum_{e \in E} t(e)^q\right)^{\frac{1}{q}}$, and $e_q(G) = e_q = \left(\frac{1}{2m}\sum_{e \in E}(d(u(e))^q + d(v(e))^q\right)^{\frac{1}{q}}$.

Also $d_\infty(G) = d_\infty = \max[d(v); v \in V]$ and J_∞, t_∞ are defined analogously. (Assuming $E \neq \emptyset$, $e_\infty = d_\infty$.)

Note that $d_1 = \frac{2m}{n}$, the "average degree" of vertices in G, also commonly denoted $\overline{d}$ of $\overline{d}(G)$. Also, d_∞ is elsewhere commonly denoted $\triangle(G)$, or $\triangle$.

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LEMMA. *If* $a_1,\dots,a_k \geqslant 0$ *and* $1 < p < \infty$, *then* $\sum_{j=1}^k a_j^p \geqslant k^{1-p}(\sum_{j=1}^k a_j)^p$, *with equality if and only if* $a_1 = \cdots = a_k$.

Proof. This is a standard application of Hölder's inequality: $\sum_{j=1}^k a_j \leqslant (\sum_{j=1}^k 1)^{\frac{p-1}{p}} (\sum_{j=1}^k a_j^p)^{\frac{1}{p}}$, which, when rearranged, gives the result. The equality condition is the equality condition in Hölder's inequality, applied to this special case. □

COROLLARY 1. *If* $a_1,\dots,a_k \geqslant 0$ *and* $0 < r < q < \infty$ *then* $\left(\frac{1}{k}\sum_{j=1}^k a_j^r\right)^{\frac{1}{r}} \leqslant \left(\frac{1}{k}\sum_{j=1}^k a_j^q\right)^{\frac{1}{q}}$, *with equality if and only if* $a_1 = \cdots = a_k$.

Proof. Apply the Lemma to $a_1^r,\dots,a_k^r$ with $p = q/r$. □

[Of course, Corollary 1 is just a very special case of the well-known fact that for a measurable function f on a probability measure space (X,μ), $||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$ is a non-decreasing function of p. This is not the last time in this note when we will get where we want to go by the most elementary, least general route we can find.]

CORORLLARY 2. *If* $0 < r < q \leqslant \infty$ *and* $E \neq \emptyset$ *then*
(a) $d_r \leqslant d_q$ *with equality if and only if* G *is regular;*
(b) $J_r \leqslant J_q$ *with equality if and only if* J *is constant, on* E*;*
(c) $t_r \leqslant t_q$ *with equality if and only if* t *is constant, on* E*; and*
(d) if G *has no isolated vertices, then* $e_r \leqslant e_q$ *with equality if and only if* G *is regular.*

When $q < \infty$, the proof is straightforward from Corollary 1. The case $q = \infty$ is straightforward. The requirement that G has no isolated vertices, i.e. vertices with degree zero, in (d), ensures that every vertex's degree appears at least once in the list $[d(u(e)), d(v(e)); e \in E]$, which is necessary to infer regularity from equality. In fact, the inequality $e_r \leqslant e_q$ holds whether G has isolates or not.

THEOREM 1. *Suppose* $E \neq \emptyset$. *For all* $q > 0$, $e_q \geqslant d_1$, *with equality if and only if* G *is regular.*

Proof. The case $q = \infty$ is trivial, so assume $0 < q < \infty$. We have

$$\begin{aligned} 2me_q^q &= \sum_{e\in E}\Big(d(u(e))^q + d(v(e))^q\Big) \\ &= \sum_{w\in V}(d(w)^{q+1} \\ &\geqslant n^{-q}(\sum_{w\in V} d(w))^{q+1} \quad \text{(by the Lemma)} \\ &= n^{-q}(2m)^{q+1} \end{aligned}$$

from which $e_q \geqslant d_1$ follows, since $d_1 = 2m/n$. The condition for equality follows from the Lemma. □

COROLLARY 3. *Suppose* $E \neq \emptyset$. *If* $q > 0$ *and* $0 < r < 1$ *then* $e_q \geqslant d_r$ *with equality if and only if* G *is regular.*

This result follows immediately from Corollary 2(a) and Theorem 1.

THEOREM 2. *Suppose* G *is simple, and* $E \neq \emptyset$. *Then* $d_1 \leqslant \frac{J_1 + t_1}{2}$, *with equality if and only if* G *is regular.*

Proof.

$$m(J_1 + t_1) = \sum_{e \in E} \Big(J(e) + t(e) \Big) = \sum_{e \in E} \Big(d(u(e)) + d(u(e)) \Big) \quad (G \quad \text{is simple})$$
$$= 2me_1 \geqslant 2md_1,$$

with equality if and only if G is regular, by Theorem 1, with $q = 1$. $\square$

COROLLARY 4. *Suppose* G *is simple,* $E \neq \emptyset$, $1 \leqslant q$, $r \leqslant \infty$, *and* $q + r > 2$. *Then* $d_1 \leqslant \frac{J_q + t_r}{2}$ *with equality if and only if* G *is regular and* t *is constant on* E.

Proof. The inequality follows from Theorem 2 and Corollary 2, and the equality condition likewise, when $r > 1$. If $r = 1$ and $q > 1$, equality implies regularity and the constancy of J. But since $J(e) + t(e) = d(u(e)) + d(v(e))$ for each $e \in E$, when G is simple, as previously noted, if follows that when G is regular and simple the constancy of J is equivalent to the constancy of t. $\square$

The inequality $d_1 \leqslant \frac{J_1 + t_1}{2}$ of Theorem 2 can be rewritten as $m \leqslant \frac{n(J_1 + t_1)}{4}$, and in this form can be seen as a generalization of the well-known inequality of Mantel (see[5]), that for G triangle-free, $m \leqslant \frac{n^2}{4}$. (Of course, Mantel's inequality, together with the condition for equality, that $G = K_{\frac{n}{2},\frac{n}{2}}$, is a special case of Turán's Theorem. See [5].) Theorem 2 and Corollary 4 sharpen two previous generalizations of Mantel's inequality, in [1] and [2]. The result in [2] is the special case of Corollary 4 in which $\{q, r\} = \{1, \infty\}$. In [2] the inequality was just the start of an inquiry into graph structure; still, we wonder how we managed not to notice the inequality of Theorem 2.

Theorem 1 is also a consequence of the main result of [3]. We are not treating it as such here because we feel that in the area of inequalities, there is a place for minimalism, for proofs more elementary and/or accessible than others.

The graphs for which equality holds in Corollary 4 form an interesting class. They might be called the nearly strongly regular graphs. All they lack to be strongly regular is the existence of an integer μ such that $|N(u) \cap N(v)| = \mu$ for all $u, v \in V$ with u and v not adjacent. The study of the structure of these graphs is begun in [1],[2], and [4].

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