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EULER–MACLAURIN FORMULAE

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Abstract. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions in L_p-spaces, is proved by applying the Euler-Maclaurin formulae. The results are applied to obtain some error estimates for the Maclaurin quadrature rules.

1. Introduction

Some of the most elementary quadrature rules are the Simpson rule based on the Simpson formula [3, p. 45]

$$\int_a^b f(t)dt = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \tag{1.1}$$

where $a \leqslant \xi \leqslant b$, and the Maclaurin rule based on the Maclaurin formula [10, p. 88]

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] \\ &\quad + \frac{7(b-a)^5}{51\,840} f^{(4)}(\eta),\end{aligned} \tag{1.2}$$

where $a \leqslant \eta \leqslant b$. Formulae (1.1) and (1.2) are valid for any function f with continuous fourth derivative $f^{(4)}$ on $[a,b]$. In the recent paper [4] the following two identities, named the extended Euler formulae, have been proved:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_n(x) + R_n^1(x) \tag{1.3}$$

and

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_{n-1}(x) + R_n^2(x), \tag{1.4}$$

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where $n \geqslant 1,\ T_0(x)=0$ and

$$T_n(x)=\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right], \tag{1.5}$$

while

$$R_n^1(x)=-\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}B_n^*\left(\frac{x-t}{b-a}\right)df^{(n-1)}(t)$$

and

$$R_n^2(x)=-\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right]df^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_{[a,b]}g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi:[a,b]\to\mathbf{R}$ of bounded variation, and $\int_a^b g(t)dt$ for the Riemann integral. The identities (1.3) and (1.4) extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials [11, p. 17]. They hold for every function $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n\geqslant 1$, and for every $x\in[a,b]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k=B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k\geqslant 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t)=B_k(t),\ 0\leqslant t<1,\quad \text{and}\quad B_k^*(t+1)=B_k^*(t),\ t\in\mathbf{R}.$$

The Bernoulli polynomials $B_k(t)$, $k\geqslant 0$ are uniquely determined by the following identities

$$B_k'(t)=kB_{k-1}(t),\ k\geqslant 1;\ \ B_0(t)=1 \tag{1.6}$$

and

$$B_k(t+1)-B_k(t)=kt^{k-1},\ k\geqslant 0. \tag{1.7}$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have

$$B_0(t)=1,\ B_1(t)=t-\frac{1}{2},\ B_2(t)=t^2-t+\frac{1}{6},\ B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t, \tag{1.8}$$

so that $B_0^*(t)=1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. From (1.7) it follows that $B_k(1)=B_k(0)=B_k$ for $k\geqslant 2$, so that $B_k^*(t)$ are continuous functions for $k\geqslant 2$. Moreover, using (1.6) we get

$$B_k^{*\prime}(t)=kB_{k-1}^*(t),\ k\geqslant 1 \tag{1.9}$$

for every $t\in\mathbf{R}$ when $k\geqslant 3$, and for every $t\in\mathbf{R}\setminus\mathbf{Z}$ when $k=1,2$.

Recently, a number of results related to the Simpson formula (1.1) have been obtained (see [5], [6–9] and [13]). The aim of this paper is to establish generalizations of Maclaurin formula (1.2) and give various error estimates for the quadrature rules based on such generalizations.

In Section 2 we use the extended Euler formulae (1.3) and (1.4) to obtain two new integral identities. We call these new identities the Euler-Maclaurin formulae, since they generalize Maclaurin formula (1.2).

In Section 3 we prove a number of inequalities related to the Euler-Maclaurin formulae, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions from the L_p -spaces.

Finally, in Section 4, we consider the repeated Euler-Maclaurin quadrature rule and the repeated modified Euler-Maclaurin quadrature rule based on the Euler-Maclaurin formulae. We give some error estimates for these quadrature rules applied to functions of various classes.

2. Euler-Maclaurin formulae

For $k \geqslant 1$ define the functions $G_k(t)$ and $F_k(t)$ as

$$G_k(t) = 3B_k^*\left(\frac{1}{6} - t\right) + 2B_k^*\left(\frac{1}{2} - t\right) + 3B_k^*\left(\frac{5}{6} - t\right),\ t \in \mathbf{R}$$

and

$$F_k(t) = G_k(t) - \tilde{B}_k,\ k \geqslant 1$$

where

$$\tilde{B}_k = G_k(0) = 3B_k\left(\frac{1}{6}\right) + 2B_k\left(\frac{1}{2}\right) + 3B_k\left(\frac{5}{6}\right),\ k \geqslant 1.$$

Specially, we have $\tilde{B}_1 = \tilde{B}_2 = \tilde{B}_3 = 0$. Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k \geqslant 2$. Thus, it is enough to know the behavior of these functions on the interval $[0,1]$. We shall investigate this behavior in the next section.

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geqslant 1$. We introduce the following notation

$$D(a,b) = \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right].$$

Further, we define $\tilde{T}_0(a,b) = 0$ and, for $1 \leqslant m \leqslant n$,

$$\tilde{T}_m(a,b) = \frac{b-a}{8}\left[3T_m\left(\frac{5a+b}{6}\right) + 2T_m\left(\frac{a+b}{2}\right) + 3T_m\left(\frac{a+5b}{6}\right)\right],$$

where $T_m(x)$ is given by (1.5). It is easy to see that $\tilde{T}_1(a,b) = \tilde{T}_2(a,b) = \tilde{T}_3(a,b) = 0$ and for $m \geqslant 4$

$$\tilde{T}_m(a,b) = \frac{1}{8}\sum_{k=4}^{m}\frac{(b-a)^k}{k!}\tilde{B}_k\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]. \tag{2.1}$$

In the next theorem we establish two formulae which play the key role in this paper. We call them the Euler-Maclaurin formulae.

THEOREM 1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geqslant 1$. Then*

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_n(a,b) + \tilde{R}_n^1(a,b), \tag{2.2}$$

and

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_{n-1}(a,b) + \tilde{R}_n^2(a,b), \tag{2.3}$$

where

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{8n!} \int_{[a,b]} G_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t),$$

and

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{8n!} \int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t).$$

Proof. Put

$$x = \frac{5a+b}{6},\ \frac{a+b}{2},\ \frac{a+5b}{6}$$

in formula (1.3) to get three new formulae. Then multiply these new formulae by

$$\frac{3(b-a)}{8},\ \frac{2(b-a)}{8},\ \frac{3(b-a)}{8}$$

respectively, and add. The result is formula (2.2). Formula (2.3) is obtained from (1.4) by the same procedure. □

REMARK 1. Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[a,b]$, for some $n \geqslant 1$. In this case (2.2) holds with

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{8n!} \int_a^b G_n\left(\frac{t-a}{b-a}\right) f^{(n)}(t)dt,$$

while (2.3) holds with

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{8n!} \int_a^b F_n\left(\frac{t-a}{b-a}\right) f^{(n)}(t)dt.$$

By direct calculation

$$F_1(t) = G_1(t) = \begin{cases} -8t, & 0 \leqslant t \leqslant 1/6 \\ -8t+3, & 1/6 < t \leqslant 1/2 \\ -8t+5, & 1/2 < t \leqslant 5/6 \\ -8t+8, & 5/6 < t \leqslant 1 \end{cases}, \tag{2.4}$$

$$F_2(t) = G_2(t) = \begin{cases} 8t^2, & 0 \leqslant t \leqslant 1/6 \\ 8t^2-6t+1, & 1/6 \leqslant t \leqslant 1/2 \\ 8t^2-10t+3, & 1/2 \leqslant t \leqslant 5/6 \\ 8t^2-16t+8, & 5/6 \leqslant t \leqslant 1 \end{cases} \tag{2.5}$$

and

$$F_3(t) = G_3(t) = \begin{cases} -8t^3, & 0 \leqslant t \leqslant 1/6 \\ -8t^3 + 9t^2 - 3t + 1/4, & 1/6 \leqslant t \leqslant 1/2 \\ -8t^3 + 15t^2 - 9t + 7/4, & 1/2 \leqslant t \leqslant 5/6 \\ -8t^3 + 24t^2 - 24t + 8, & 5/6 \leqslant t \leqslant 1 \end{cases} . \tag{2.6}$$

Applying (2.2) with $n = 1,2,3$ we get the identities

$$\begin{aligned} \int_a^b f(t)dt - D(a,b) &= \frac{b-a}{8} \int_{[a,b]} G_1\left(\frac{t-a}{b-a}\right) df(t) \\ &= \frac{(b-a)^2}{16} \int_{[a,b]} G_2\left(\frac{t-a}{b-a}\right) df'(t) \\ &= \frac{(b-a)^3}{48} \int_{[a,b]} G_3\left(\frac{t-a}{b-a}\right) df''(t). \end{aligned}$$

The same identities are obtained from (2.3) with $n = 1,2,3$, since $\tilde{T}_0(a,b) = \tilde{T}_1(a,b) = \tilde{T}_2(a,b) = 0$ and $F_k(t) = G_k(t)$ for $k = 1,2,3$, while (2.3) with $n = 4$ yields the identity

$$\int_a^b f(t)dt - D(a,b) = \frac{(b-a)^4}{192} \int_{[a,b]} F_4\left(\frac{t-a}{b-a}\right) df'''(t).$$

3. Some inequalities related to Euler-Maclaurin formulae

In this section we use the Euler-Maclaurin formulae established in Theorem 1 to prove a number of inequalities for various classes of functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$ defined in the previous section. As we noted earlier, it is enough to know the behavior of these functions on the interval $[0,1]$.

The Bernoulli polynomials are symmetric with respect to $\frac{1}{2}$, that is [1, 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t),\ 0 \leqslant t \leqslant 1,\ k \geqslant 1. \tag{3.1}$$

Setting $t = \frac{1}{6}$ in (3.1) we get

$$B_k\left(\frac{5}{6}\right) = (-1)^k B_k\left(\frac{1}{6}\right)$$

Also, we have

$$B_k(1) = B_k(0) = B_k,\ k \geqslant 2,\ \ B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2j-1} = 0,\ j \geqslant 2.$$

This implies

$$\tilde{B}_{2k-1} = 3B_{2k-1}\left(\frac{1}{6}\right) + 2B_{2k-1}\left(\frac{1}{2}\right) + 3B_{2k-1}\left(\frac{5}{6}\right) = 0 \tag{3.2}$$

for $k \geqslant 1$, and

$$\tilde{B}_{2k} = 3B_{2k}\left(\frac{1}{6}\right) + 2B_{2k}\left(\frac{1}{2}\right) + 3B_{2k}\left(\frac{5}{6}\right) = 2B_{2k}\left(\frac{1}{2}\right) + 6B_{2k}\left(\frac{5}{6}\right).$$

Also, we have [1, 23.1]

$$B_{2k}\left(\frac{1}{2}\right) = -\left(1 - 2^{1-2k}\right) B_{2k},\ B_{2k}\left(\frac{1}{3}\right) = -\frac{1}{2}\left(1 - 3^{1-2k}\right) B_{2k},$$

and

$$B_{2k}\left(\frac{1}{6}\right) = \frac{1}{2}\left(1 - 2^{1-2k}\right)\left(1 - 3^{1-2k}\right) B_{2k},$$

which gives the formula

$$\tilde{B}_{2k} = \left(1 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right) B_{2k}, \tag{3.3}$$

for $k \geqslant 1$. Now, by (3.2) we have

$$F_{2k-1}(t) = G_{2k-1}(t) \tag{3.4}$$

and

$$F_{2k}(t) = G_{2k}(t) - \tilde{B}_{2k}, \tag{3.5}$$

for $k \geqslant 1$. Further, the points 0 and 1 are the zeros of $F_n(t)$, that is

$$F_n(0) = F_n(1) = 0,\ n \geqslant 1.$$

As we shall see below, 0 and 1 are the only zeros of $F_n(t)$ for $n = 2k,\ k \geqslant 2$, while for $n = 2k-1,\ k \geqslant 2$ we have

$$F_{2k-1}\left(\frac{1}{2}\right) = G_{2k-1}\left(\frac{1}{2}\right) = 0.$$

We shall see that 0, $\frac{1}{2}$ and 1 are the only zeros of $F_{2k-1}(t) = G_{2k-1}(t)$, for $k \geqslant 2$. Also, note that for $n = 2k,\ k \geqslant 1$ we have

$$G_{2k}(0) = G_{2k}(1) = \tilde{B}_{2k} = \left(1 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right) B_{2k}$$

and

$$G_{2k}\left(\frac{1}{2}\right) = 2B_{2k} + 6B_{2k}\left(\frac{1}{3}\right) = -\left(1 - 3^{2-2k}\right) B_{2k},$$

while

$$F_{2k}\left(\frac{1}{2}\right) = G_{2k}\left(\frac{1}{2}\right) - \tilde{B}_{2k} = -\left(2 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right) B_{2k}. \tag{3.6}$$

LEMMA 1. *For $k \geqslant 2$ we have*

$$G_k(1-t) = (-1)^k G_k(t),\ 0 \leqslant t \leqslant 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t),\ 0 \leqslant t \leqslant 1.$$

Proof. As we noted in Introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geqslant 2$. Therefore, for $k \geqslant 2$ and $0 \leqslant t \leqslant 1$ we have

$$B_k^*\left(\frac{1}{6}-t\right) = \begin{cases} B_k\left(\frac{1}{6}-t\right), & 0 \leqslant t \leqslant 1/6 \\ B_k\left(\frac{7}{6}-t\right), & 1/6 \leqslant t \leqslant 1, \end{cases}$$

and, using (3.1),

$$\begin{aligned} B_k^*\left(\frac{5}{6}+t\right) &= \begin{cases} B_k\left(\frac{5}{6}+t\right), & 0 \leqslant t \leqslant 1/6 \\ B_k\left(-\frac{1}{6}+t\right), & 1/6 \leqslant t \leqslant 1 \end{cases} \\ &= \begin{cases} (-1)^k B_k\left(\frac{1}{6}-t\right), & 0 \leqslant t \leqslant 1/6 \\ (-1)^k B_k\left(\frac{7}{6}-t\right), & 1/6 \leqslant t \leqslant 1. \end{cases} \end{aligned}$$

Comparing the above equalities, we see that

$$B_k^*\left(\frac{5}{6}+t\right) = (-1)^k B_k^*\left(\frac{1}{6}-t\right), \; 0 \leqslant t \leqslant 1.$$

By a similar observation we get

$$B_k^*\left(\frac{1}{6}+t\right) = (-1)^k B_k^*\left(\frac{5}{6}-t\right), \; 0 \leqslant t \leqslant 1.$$

Using these identities, we get

$$\begin{aligned} G_k(1-t) &= 3B_k^*\left(-\frac{5}{6}+t\right) + 2B_k^*\left(-\frac{1}{2}+t\right) + 3B_k^*\left(-\frac{1}{6}+t\right) \\ &= 3B_k^*\left(\frac{1}{6}+t\right) + 2B_k^*\left(\frac{1}{2}+t\right) + 3B_k^*\left(\frac{5}{6}+t\right) \\ &= (-1)^k \left[3B_k^*\left(\frac{1}{6}-t\right) + 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right)\right] \\ &= (-1)^k G_k(t), \end{aligned}$$

which proves the first identity. Further, we have $\tilde{B}_k = (-1)^k \tilde{B}_k$, since (3.2) holds, so that

$$F_k(1-t) = G_k(1-t) - \tilde{B}_k = (-1)^k \left[G_k(t) - \tilde{B}_k\right] = (-1)^k F_k(t),$$

which proves the second identity. □

Note that the identities established in Lemma 1 are valid for $k = 1$, too, except at the points $1/6$, $1/2$ and $5/6$ of discontinuity of $F_1(t) = G_1(t)$.

LEMMA 2. *For $k \geqslant 2$ the function $G_{2k-1}(t)$ has no zeros in the interval $\left(0, \frac{1}{2}\right)$. The sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}(t) > 0, \; 0 < t < \frac{1}{2}.$$

Proof. For $k=2$, $G_3(t)$ is given by (2.6) and it is easy to see that

$$-G_3(t)>0,\ 0<t<\frac{1}{2}.$$

Thus, our assertion is true for $k=2$. Now, assume that $k\geqslant 3$. Then $2k-1\geqslant 5$ and $G_{2k-1}(t)$ is continuous and twice differentiable function. Using (1.9) we get

$$G'_{2k-1}(t)=-(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t)=(2k-1)(2k-2)G_{2k-3}(t).$$

We know that 0 and $\frac{1}{2}$ are the zeros of $G_{2k-1}(t)$. Let us suppose that some α, $0<\alpha<\frac{1}{2}$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $\left(\alpha,\frac{1}{2}\right)$ the derivative $G'_{2k-1}(t)$ must have at least one zero, say β_1, $0<\beta_1<\alpha$ and β_2, $\alpha<\beta_2<\frac{1}{2}$. Therefore, the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1,β_2). Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $\left(0,\frac{1}{2}\right)$, it follows that $(2k-1)(2k-2)G_{2k-3}(t)$ also has a zero inside this interval. From this it follows that $G_3(t)$ would have a zero inside the interval $\left(0,\frac{1}{2}\right)$, which is not true. Thus, $G_{2k-1}(t)$ can not have a zero inside the interval $\left(0,\frac{1}{2}\right)$. To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}\left(\frac{1}{6}\right)=-B_{2k-1}\left(\frac{1}{3}\right).$$

We have [1, 23.1.14]

$$(-1)^kB_{2k-1}(t)>0,\ 0<t<\frac{1}{2},$$

which implies

$$(-1)^{k-1}G_{2k-1}\left(\frac{1}{6}\right)=(-1)^kB_{2k-1}\left(\frac{1}{3}\right)>0.$$

Consequently, we have

$$(-1)^{k-1}G_{2k-1}(t)>0,\ 0<t<\frac{1}{2}. \qquad \square$$

COROLLARY 1. *For $k\geqslant 2$ the functions $(-1)^kF_{2k}(t)$ and $(-1)^kG_{2k}(t)$ are strictly increasing on the interval $\left(0,\frac{1}{2}\right)$, and strictly decreasing on the interval $\left(\frac{1}{2},1\right)$. Further, for $k\geqslant 2$, we have*

$$\max_{t\in[0,1]}|F_{2k}(t)|=\left(2-2^{1-2k}\right)\left(1-3^{2-2k}\right)|B_{2k}|,$$

and

$$\max_{t\in[0,1]}|G_{2k}(t)|=\left(1-3^{2-2k}\right)|B_{2k}|.$$

Proof. Using (1.9) we get

$$\left[(-1)^k F_{2k}(t)\right]' = \left[(-1)^k G_{2k}(t)\right]' = 2k(-1)^{k-1} G_{2k-1}(t)$$

and $(-1)^{k-1} G_{2k-1}(t) > 0$ for $0 < t < \frac{1}{2}$, by Lemma 2. Thus, $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{2}\right)$. Also, by Lemma 1, we have $F_{2k}(1-t) = F_{2k}(t),\ 0 \leqslant t \leqslant 1$ and $G_{2k}(1-t) = G_{2k}(t),\ 0 \leqslant t \leqslant 1$, which implies that $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly decreasing on the interval $\left(\frac{1}{2}, 1\right)$. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = \frac{1}{2}$, that is

$$\max_{t \in [0,1]} |F_{2k}(t)| = \left| F_{2k}\left(\frac{1}{2}\right) \right| = \left(2 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right)|B_{2k}|.$$

Also

$$\begin{aligned}
\max_{t \in [0,1]} |G_{2k}(t)| &= \max\left\{ |G_{2k}(0)|, \left| G_{2k}\left(\frac{1}{2}\right) \right| \right\} \\
&= \left| G_{2k}\left(\frac{1}{2}\right) \right| \\
&= \left(1 - 3^{2-2k}\right)|B_{2k}|,
\end{aligned}$$

which completes the proof. □

COROLLARY 2. *For $k \geqslant 2$, we have*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{1}{k}\left(2 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right)|B_{2k}|.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,dt = \left|\tilde{B}_{2k}\right| = \left(1 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right) B_{2k}$$

and

$$\int_0^1 |G_{2k}(t)|\,dt \leqslant 2\left|\tilde{B}_{2k}\right| = 2\left(1 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right) B_{2k}.$$

Proof. Using (1.9) it is easy to see that

$$G'_m(t) = -mG_{m-1}(t),\ m \geqslant 3. \tag{3.7}$$

Now, using Lemma 1, Lemma 2 and (3.7) we get

$$\begin{aligned}
\int_0^1 |G_{2k-1}(t)|\,dt &= 2\left| \int_0^{\frac{1}{2}} G_{2k-1}(t)dt \right| \\
&= 2\left| -\frac{1}{2k}\, G_{2k}(t)\Big|_0^{\frac{1}{2}} \right| \\
&= \frac{1}{k}\left| G_{2k}\left(\frac{1}{2}\right) - G_{2k}(0) \right| \\
&= \frac{1}{k}\left(2 - 2^{1-2k}\right)\left(1 - 3^{2-2k}\right)|B_{2k}|,
\end{aligned}$$

which proves the first assertion. By Corollary 1, $F_{2k}(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (3.5) and (3.7), we get

$$\begin{aligned}\int_0^1 |F_{2k}(t)|\,dt &= \left|\int_0^1 F_{2k}(t)dt\right| = \left|\int_0^1 \left[G_{2k}(t) - \tilde{B}_{2k}\right] dt\right| \\ &= \left|-\frac{1}{2k+1}\, G_{2k+1}(t)|_0^1 - \tilde{B}_{2k}\right| = \left|\tilde{B}_{2k}\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (3.5) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,dt = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right| dt \leqslant \int_0^1 |F_{2k}(t)|\,dt + \left|\tilde{B}_{2k}\right| = 2\left|\tilde{B}_{2k}\right|,$$

which proves the third assertion. □

THEOREM 2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left|\int_a^b f(t)dt - D(a,b) + \tilde{T}_{n-1}(a,b)\right| \leqslant \frac{(b-a)^{n+1}}{8n!}\int_0^1 |F_n(t)|\,dt \cdot L \tag{3.8}$$

and

$$\left|\int_a^b f(t)dt - D(a,b) + \tilde{T}_n(a,b)\right| \leqslant \frac{(b-a)^{n+1}}{8n!}\int_0^1 |G_n(t)|\,dt \cdot L. \tag{3.9}$$

Proof. For any integrable function $\Phi : [a,b] \to \mathbf{R}$ we have

$$\left|\int_{[a,b]} \Phi(t)df^{(n-1)}(t)\right| \leqslant \int_a^b |\Phi(t)|\,dt \cdot L, \tag{3.10}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (3.10) with $\Phi(t) = F_n\left(\frac{t-a}{b-a}\right)$, we get

$$\begin{aligned}&\left|\frac{(b-a)^n}{8n!}\int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t)\right| \\ \leqslant\ & \frac{(b-a)^n}{8n!}\int_a^b \left|F_n\left(\frac{t-a}{b-a}\right)\right| dt \cdot L \\ =\ & \frac{(b-a)^{n+1}}{8n!}\int_0^1 |F_n(t)|\,dt \cdot L.\end{aligned}$$

Applying the above inequality, we get inequality (3.8) from identity (2.3). Similarly, we can apply inequality (3.10) with $\Phi(t) = G_n\left(\frac{t-a}{b-a}\right)$, and then use identity (2.2), to obtain inequality (3.9). □

COROLLARY 3. *Let $f : [a,b] \to \mathbf{R}$ be given function.*
If f is L-Lipschitzian on $[a,b]$, then

$$\left| \int_a^b f(t)dt - D(a,b) \right| \leqslant \frac{25}{288}(b-a)^2 \cdot L.$$

If f' is L-Lipschitzian on $[a,b]$, then

$$\left| \int_a^b f(t)dt - D(a,b) \right| \leqslant \frac{1}{192}(b-a)^3 \cdot L.$$

If f'' is L-Lipschitzian on $[a,b]$, then

$$\left| \int_a^b f(t)dt - D(a,b) \right| \leqslant \frac{1}{1728}(b-a)^4 \cdot L.$$

If f''' is L-Lipschitzian on $[a,b]$, then

$$\left| \int_a^b f(t)dt - D(a,b) \right| \leqslant \frac{7}{51\,840}(b-a)^5 \cdot L.$$

Proof. Using (2.4) and (2.5) we get

$$\int_0^1 |F_1(t)|\, dt = \frac{25}{36} \quad \text{and} \quad \int_0^1 |F_2(t)|\, dt = \frac{1}{12}.$$

Therefore, applying (3.8) with $n = 1,2,$ we get the first and the second inequality. Using Corollary 2, we get

$$\int_0^1 |F_3(t)|\, dt = \frac{1}{36} \quad \text{and} \quad \int_0^1 |F_4(t)|\, dt = \frac{7}{270}.$$

Now, the third and the fourth inequalities follow from (3.8) with $n = 3,4$. □

REMARK 2. Let us note that for a function f which is L-Lipschitzian on $[a,b]$ the following inequality holds [6] (see also [9]):

$$\left| \int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leqslant \frac{5}{36}(b-a)^2 L.$$

This inequality is related to the Simpson quadrature formula and gives the error estimate for a function f which is L-Lipschitzian on $[a,b]$. We can compare this with the first inequality in Corollary 3:

$$\left| \int_a^b f(t)dt - \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] \right| \leqslant \frac{25}{288}(b-a)^2 L.$$

We see that, for this class of functions, we will have better error estimate for the Maclaurin quadrature rule than for the Simpson rule.

REMARK 3. Suppose that $f:[a,b]\to\mathbf{R}$ is such that $f^{(n)}$ exists and is bounded on $[a,b]$, for some $n\geqslant 1$ In this case we have for all $t,s\in[a,b]$

$$\left|f^{(n-1)}(t)-f^{(n-1)}(s)\right|\leqslant\|f^{(n)}\|_\infty\cdot|t-s|,$$

which means that $f^{(n-1)}$ is $\|f^{(n)}\|_\infty$-Lipschitzian function on $[a,b]$. Therefore, the inequalities established in Theorem 2 hold with $L=\|f^{(n)}\|_\infty$. Consequently, under appropriate assumptions on f, the inequalities from Corollary 3 hold with $L=\|f'\|_\infty$, $\|f''\|_\infty$, $\|f'''\|_\infty$, $\|f^{(4)}\|_\infty$, respectively.

THEOREM 3. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then*

$$\left|\int_a^b f(t)dt-D(a,b)+\tilde{T}_{n-1}(a,b)\right|\leqslant\frac{(b-a)^n}{8n!}\max_{t\in[0,1]}|F_n(t)|\cdot V_a^b(f^{(n-1)})\tag{3.11}$$

and

$$\left|\int_a^b f(t)dt-D(a,b)+\tilde{T}_n(a,b)\right|\leqslant\frac{(b-a)^n}{8n!}\max_{t\in[0,1]}|G_n(t)|\cdot V_a^b(f^{(n-1)}),\tag{3.12}$$

where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. If $\Phi:[a,b]\to\mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral $\int_{[a,b]}\Phi(t)df^{(n-1)}(t)$ exists, then

$$\left|\int_{[a,b]}\Phi(t)df^{(n-1)}(t)\right|\leqslant\max_{t\in[a,b]}|\Phi(t)|\cdot V_a^b(f^{(n-1)}).\tag{3.13}$$

We apply estimate (3.13) to $\Phi(t)=F_n\left(\frac{t-a}{b-a}\right)$ to obtain

$$\begin{aligned}&\left|\frac{(b-a)^n}{8n!}\int_{[a,b]}F_n\left(\frac{t-a}{b-a}\right)df^{(n-1)}(t)\right|\\ \leqslant\ &\frac{(b-a)^n}{8n!}\max_{t\in[a,b]}\left|F_n\left(\frac{t-a}{b-a}\right)\right|\cdot V_a^b(f^{(n-1)})\\ =\ &\frac{(b-a)^n}{8n!}\max_{t\in[0,1]}|F_n(t)|\cdot V_a^b(f^{(n-1)}).\end{aligned}$$

Now, we use the above inequality and identity (2.3) to obtain (3.11). In the same manner, we apply estimate (3.13) to $\Phi(t)=G_n\left(\frac{t-a}{b-a}\right)$, and then use identity (2.2), to obtain inequality (3.12). □

COROLLARY 4. *Let* $f:[a,b]\to\mathbf{R}$ *be given function.*
If f *is a continuous function of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)dt - D(a,b)\right| \leqslant \frac{5}{24}(b-a)\cdot V_a^b(f).$$

If f' *is a continuous function of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)dt - D(a,b)\right| \leqslant \frac{1}{72}(b-a)^2\cdot V_a^b(f').$$

If f'' *is a continuous function of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)dt - D(a,b)\right| \leqslant \frac{1}{768}(b-a)^3\cdot V_a^b(f'').$$

If f''' *is a continuous function of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b f(t)dt - D(a,b)\right| \leqslant \frac{1}{3456}(b-a)^4\cdot V_a^b(f''').$$

Proof. From explicit expressions (2.4), (2.5) and (2.6), we get

$$\begin{aligned}\max_{t\in[0,1]}|F_1(t)| &= -F_1\left(\frac{5}{6}\right)=\frac{5}{3},\\ \max_{t\in[0,1]}|F_2(t)| &= F_2\left(\frac{1}{6}\right)=\frac{2}{9}\end{aligned}$$

and

$$\max_{t\in[0,1]}|F_3(t)| = -F_3\left(\frac{1}{4}\right)=\frac{1}{16}.$$

Therefore, applying (3.11) with $n=1,2,3,$ we get the first three inequalities. Further, using Corollary 1, we get

$$\max_{t\in[0,1]}|F_4(t)| = \frac{1}{18}.$$

Now, the fourth inequality follows from (3.11) with $n=4$. □

REMARK 4. In [7] (see also [9]) the following inequality has been proved:

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{1}{3}(b-a)V_a^b(f).$$

This inequality is related to the Simpson quadrature formula and gives the error estimate for a function f of bounded variation on $[a,b]$. We can compare this with the first inequality in Corollary 3:

$$\left|\int_a^b f(t)dt - \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right]\right|$$
$$\leqslant \frac{5}{24}(b-a)V_a^b(f).$$

We see that, for this class of functions, we will have better error estimate for the Maclaurin quadrature rule than for the Simpson rule.

REMARK 5. Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[a,b]$ for some $n \geqslant 1$ In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ and we have

$$V_a^b(f^{(n-1)}) = \int_a^b \left|f^{(n)}(t)\right| dt = \|f^{(n)}\|_1,$$

Therefore, the inequalities established in Theorem 3 hold with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$. However, a similar observation can be made for the results of Corollary 4.

THEOREM 4. *Assume (p,q) is a pair of conjugate exponents, that is $1 < p,q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$ or $p=\infty$, $q=1$. Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 1$. Then we have*

$$\left|\int_a^b f(t)dt - D(a,b) + \tilde{T}_{n-1}(a,b)\right| \leqslant K(n,p)(b-a)^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p, \tag{3.14}$$

and

$$\left|\int_a^b f(t)dt - D(a,b) + \tilde{T}_n(a,b)\right| \leqslant K^*(n,p)(b-a)^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p, \tag{3.15}$$

where

$$K(n,p) = \frac{1}{8n!}\left[\int_0^1 |F_n(t)|^q\, dt\right]^{\frac{1}{q}},$$

and

$$K^*(n,p) = \frac{1}{8n!}\left[\int_0^1 |G_n(t)|^q\, dt\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we have

$$\begin{aligned}
&\left|\frac{(b-a)^n}{3(n!)}\int_a^b F_n\left(\frac{t-a}{b-a}\right)f^{(n)}(t)dt\right| \\
\leqslant\ & \frac{(b-a)^n}{8n!}\left[\int_a^b \left|F_n\left(\frac{t-a}{b-a}\right)\right|^q dt\right]^{\frac{1}{q}}\cdot\left\|f^{(n)}\right\|_p \\
=\ & \frac{(b-a)^{n+\frac{1}{q}}}{8n!}\left[\int_0^1 |F_n(t)|^q\,dt\right]^{\frac{1}{q}}\cdot\left\|f^{(n)}\right\|_p \\
=\ & K(n,p)(b-a)^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p
\end{aligned}$$

Using the above inequality, by Remark 1, from (2.3) we get estimate (3.14). In the same manner, from (2.2) we get estimate (3.15). □

REMARK 6. For $p=\infty$ we have

$$K(n,\infty)=\frac{1}{8n!}\int_0^1 |F_n(t)|\,dt \ \text{ and } \ K^*(n,\infty)=\frac{1}{8n!}\int_0^1 |G_n(t)|\,dt.$$

The results established in Theorem 4 for $p=\infty$ coincide with the results of Theorem 2 with $L=\|f^{(n)}\|_\infty$. Moreover, by Remark 3 and Corollary 3, we have

$$\left|\int_a^b f(t)dt-D(a,b)\right|\leqslant K(n,\infty)\,(b-a)^{n+1}\cdot\|f^{(n)}\|_\infty,\ \ n=1,2,3,4,$$

where

$$K(1,\infty)=\frac{25}{288},\ K(2,\infty)=\frac{1}{192},\ K(3,\infty)=\frac{1}{1728},\ K(4,\infty)=\frac{7}{51840}.$$

REMARK 7. Let us define for $p=1$

$$K(n,1)=\frac{1}{8n!}\max_{t\in[0,1]}|F_n(t)| \qquad\text{and}\qquad K^*(n,1)=\frac{1}{8n!}\max_{t\in[0,1]}|G_n(t)|\,.$$

Then, using Remark 5 and Theorem 3, we can extend the results established in Theorem 4 to the pair $p=1,\ q=\infty$. This means that if we set $\frac{1}{q}=0$, then (3.14) and (3.15) hold for $p=1$. Also, by Remark 5 and Corollary 4, we have

$$\left|\int_a^b f(t)dt-D(a,b)\right|\leqslant K(n,1)\,(b-a)^n\cdot\|f^{(n)}\|_1,\ \ n=1,2,3,4,$$

where

$$K(1,1)=\frac{5}{24},\ K(2,1)=\frac{1}{72},\ K(3,1)=\frac{1}{768},\ K(4,1)=\frac{1}{3456}.$$

REMARK 8. Note that $K^*(1,p) = K(1,p)$, for $1 < p \leqslant \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leqslant \infty$ we can easily calculate $K(1,p)$. We get

$$K(1,p) = \frac{1}{8}\left[\frac{3^{q+1}+4^{q+1}+5^{q+1}}{4(q+1)3^{q+1}}\right]^{\frac{1}{q}},\ 1 < p \leqslant \infty.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{q\to\infty} \frac{1}{8}\left[\frac{3^{q+1}+4^{q+1}+5^{q+1}}{4(q+1)3^{q+1}}\right]^{\frac{1}{q}} = \frac{5}{24} = K(1,1).$$

So, from (3.14) with $n = 1$ we get the following inequality

$$\left|\int_a^b f(t)dt - \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right]\right|$$
$$\leqslant \frac{1}{8}\left[\frac{3^{q+1}+4^{q+1}+5^{q+1}}{4(q+1)3^{q+1}}\right]^{\frac{1}{q}}(b-a)^{1+\frac{1}{q}} \cdot \|f'\|_p.$$

This can be compared with the similar inequality proved in [8] (see also [9]), related to the Simpson rule,

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{1+\frac{1}{q}} \cdot \|f'\|_p.$$

Denote

$$C(1,p) = \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}},\ 1 < p \leqslant \infty.$$

We have

$$\begin{aligned}\frac{C(1,p)}{K(1,p)} &= \left[\frac{4^{q+1}+8^{q+1}}{3^{q+1}+4^{q+1}+5^{q+1}}\right]^{\frac{1}{q}}\\ &= \left[\frac{1+\left(\frac{1}{2}\right)^{q+1}}{\left(\frac{3}{8}\right)^{q+1}+\left(\frac{5}{8}\right)^{q+1}+\left(\frac{1}{2}\right)^{q+1}}\right]^{\frac{1}{q}}\\ &> 1,\end{aligned}$$

since

$$\left(\frac{3}{8}\right)^{q+1} + \left(\frac{5}{8}\right)^{q+1} < \frac{3}{8} + \frac{5}{8} = 1.$$

We see that, for this class of functions, we will have better error estimate for the Maclaurin quadrature rule than for the Simpson rule.

At the end of this section we use formula (2.2) to obtain Grüss type inequality related to Euler-Maclaurin formula. To do this we need the following two technical lemmata proved in the recent paper [5]:

LEMMA 3. *Let* $F, G : [a,b] \to \mathbf{R}$ *be two integrable functions. If*

$$m \leqslant F(t) \leqslant M, \quad a \leqslant t \leqslant b$$

and

$$\int_a^b G(t)dt = 0,$$

then

$$\left|\frac{1}{b-a}\int_a^b F(t)G(t)dt\right| \leqslant \frac{M-m}{2}\sqrt{\frac{1}{b-a}\int_a^b G^2(t)dt}. \tag{3.16}$$

REMARK 9. The result stated in the lemma above is a slight generalization of the following result due to H. Hadwiger and E. Heil (see [12, p. 544]): Let F, G be two integrable functions on $[0,1]$ such that $0 \leqslant F(x) \leqslant 1$, for all $x \in [0,1]$, and $\int_0^1 G(x)dx = 0$. Then

$$\int_0^1 G^2(x)dx \geqslant 4\left(\int_0^1 F(x)G(x)dx\right)^2.$$

LEMMA 4. (i) *Let* $k \geqslant 1$ *and* $\gamma \in \mathbf{R}$. *Then*

$$\int_0^1 B_k^*(\gamma - t)dt = 0.$$

(ii) *Let* $k \geqslant 1$ *and* $\gamma, \delta \in \mathbf{R}$. *Then*

$$\int_0^1 B_k^*(\gamma - t)B_k^*(\delta - t)dt = \frac{(k!)^2}{(2k)!}(-1)^{k-1}B_{2k}^*(\delta - \gamma).$$

(iii) *Let* $k \geqslant 1$ *and* $\gamma \in \mathbf{R}$. *Then*

$$\int_0^1 \left[B_k^*(\gamma - t)\right]^2 dt = \frac{(k!)^2}{(2k)!}\left|B_{2k}\right|.$$

THEOREM 5. *Suppose that* $f : [a,b] \to \mathbf{R}$ *is such that* $f^{(n)}$ *exists and is integrable on* $[a,b]$, *for some* $n \geqslant 1$. *Assume that*

$$m_n \leqslant f^{(n)}(t) \leqslant M_n, \ a \leqslant t \leqslant b,$$

for some constants m_n *and* M_n. *Then*

$$\left|\int_a^b f(t)dt - D(a,b) + \tilde{T}_n(a,b)\right|$$

$$\leqslant \frac{1}{16}(b-a)^{n+1}(M_n - m_n)\sqrt{(1 + 7 \cdot 3^{2-2n})\frac{|B_{2n}|}{(2n)!}}. \tag{3.17}$$

Proof. By Remark 1 we can rewrite $\tilde{R}_n^1(a,b)$ as

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^{n+1}}{8n!} \cdot \frac{1}{b-a} \int_a^b F(t)G(t)dt,$$

where

$$F(t) = f^{(n)}(t), \; G(t) = G_n\left(\frac{t-a}{b-a}\right), \; a \leqslant t \leqslant b.$$

Using Lemma 4 (i) we get

$$\int_a^b G(t)dt = (b-a)\int_0^1 \left[3B_k^*\left(\frac{1}{6}-t\right) + 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right)\right] ds = 0.$$

Also, using Lemma 4 (ii) and (iii) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b G^2(t)dt &= \int_0^1 \left[3B_k^*\left(\frac{1}{6}-t\right) + 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right)\right]^2 ds \\ &= 22\frac{(n!)^2}{(2n)!}\left|B_{2n}\right| + 42\frac{(n!)^2}{(2n)!}(-1)^{n-1}B_{2n}\left(\frac{1}{3}\right) \\ &= (1+7\cdot 3^{2-2n})\frac{(n!)^2}{(2n)!}\left|B_{2n}\right|.\end{aligned}$$

Now, we apply the inequality (3.16) to obtain the estimate

$$\begin{aligned}\left|\tilde{R}_n^1(a,b)\right| &\leqslant \frac{(b-a)^{n+1}}{8n!} \cdot \frac{M_n - m_n}{2}\sqrt{\frac{1}{b-a}\int_a^b G^2(t)dt} \\ &= \frac{1}{16}(b-a)^{n+1}(M_n - m_n)\sqrt{(1+7\cdot 3^{2-2n})\frac{|B_{2n}|}{(2n)!}},\end{aligned}$$

which proves our assertion. $\square$

REMARK 10. In [5] the following inequality, related to the Euler-Simpson formula, has been proved:

$$\begin{aligned}&\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - S_n^S(a,b)\right| \\ &\leqslant \Gamma_n := \frac{1}{6}(b-a)^{n+1}(M_n - m_n)\sqrt{(1+2^{3-2n})\frac{|B_{2n}|}{(2n)!}},\end{aligned}$$

where $S_1^S(a,b) = S_2^S(a,b) = S_3^S(a,b) = 0$ and

$$S_n^S(a,b) = \frac{1}{3}\sum_{j=2}^{\left[\frac{n}{2}\right]} \frac{(b-a)^{2j}}{(2j)!}(1-2^{2-2j})B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geqslant 4$. It should be noted that the above inequality for $n = 1,2,3$ has been proved earlier in [13]. On the other side, from the discussion given at the beginning of this section we know that $\tilde{T}_1(a,b) = \tilde{T}_2(a,b) = \tilde{T}_3(a,b) = 0$ and

$$\tilde{T}_n(a,b) = \frac{1}{8}\sum_{j=2}^{\left[\frac{n}{2}\right]} \frac{(b-a)^{2j}}{(2j)!}\left(1-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geqslant 4$. The inequality (3.17) can be rewritten as

$$\left|\int_a^b f(t)dt - \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] - \tilde{T}_n(a,b)\right|$$
$$\leqslant \ \tilde{\Gamma}_n := \frac{1}{16}(b-a)^{n+1}(M_n - m_n)\sqrt{(1+7\cdot 3^{2-2n})\frac{|B_{2n}|}{(2n)!}}.$$

Note that

$$\frac{\Gamma_n}{\tilde{\Gamma}_n} = \frac{8}{3}\sqrt{\frac{1+2^{3-2n}}{1+7\cdot 3^{2-2n}}} \geqslant \sqrt{\frac{8}{3}}, \quad \text{for all } n \geqslant 1.$$

This shows that the quadrature rule based on the Euler-Maclaurin formula will have a better Grüss type error estimate than the quadrature rule based on the Euler-Simpson formula.

4. Error estimates for Euler-Maclaurin quadrature formulae

Let us divide the interval $[a,b]$ into ν subintervals of equal length $h = \frac{1}{\nu}(b-a)$. Assume that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geqslant 1$. We consider the repeated Euler-Maclaurin formula

$$\int_a^b f(t)dt = D_\nu(f) - \sigma_{n-1}(f) + \rho_n(f) \tag{4.1}$$

and the repeated modified Euler-Maclaurin formula

$$\int_a^b f(t)dt = D_\nu(f) - \sigma_n(f) + \tilde{\rho}_n(f), \tag{4.2}$$

where $\rho_n(f)$ and $\tilde{\rho}_n(f)$ are the remainders, $D_\nu(f)$ is given by

$$D_\nu(f) = \sum_{i=1}^{\nu} D\left(a+(i-1)h, a+ih\right),$$

where $D\left(a+(i-1)h, a+ih\right)$ is equal to

$$\frac{h}{8}\left[3f\left(a+\left(i-\frac{5}{6}\right)h\right) + 2f\left(a+\left(i-\frac{1}{2}\right)h\right) + 3f\left(a+\left(i-\frac{1}{6}\right)h\right)\right]$$

Further, $\sigma_m(f)$ is given by

$$\sigma_m(f) = \sum_{i=1}^{\nu} \tilde{T}_m\left(a+(i-1)h, a+ih\right),\ m \geqslant 0.$$

We see that

$$\sigma_0(f) = \sigma_1(f) = \sigma_2(f) = \sigma_3(f) = 0, \tag{4.3}$$

while for $m \geqslant 4,$ we have

$$\begin{aligned} \sigma_m(f) &= \sum_{i=1}^{\nu} \frac{1}{8} \sum_{j=2}^{\left[\frac{m}{2}\right]} \frac{h^{2j}}{(2j)!} \tilde{B}_{2j} \left[f^{(2j-1)}(a+ih) - f^{(2j-1)}(a+(i-1)h)\right] \\ &= \frac{1}{8} \sum_{j=2}^{\left[\frac{m}{2}\right]} \frac{h^{2j}}{(2j)!} \tilde{B}_{2j} \sum_{i=1}^{\nu} \left[f^{(2j-1)}(a+ih) - f^{(2j-1)}(a+(i-1)h)\right] \\ &= \frac{1}{8} \sum_{j=2}^{\left[\frac{m}{2}\right]} \frac{h^{2j}}{(2j)!} \left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right) B_{2j} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]. \end{aligned} \tag{4.4}$$

The remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$ can be written as

$$\rho_n(f) = \sum_{i=1}^{\nu} \rho_n(f;i), \quad \text{and} \quad \tilde{\rho}_n(f) = \sum_{i=1}^{\nu} \tilde{\rho}_n(f;i), \tag{4.5}$$

where, for $i = 1, \ldots, \nu,$

$$\rho_n(f;i) = \int_{a+(i-1)h}^{a+ih} f(t)dt - D\left(a+(i-1)h, a+ih\right) + \tilde{T}_{n-1}\left(a+(i-1)h, a+ih\right)$$

and

$$\tilde{\rho}_n(f;i) = \int_{a+(i-1)h}^{a+ih} f(t)dt - D\left(a+(i-1)h, a+ih\right) + \tilde{T}_n\left(a+(i-1)h, a+ih\right).$$

We shall apply the results from the preceding section to obtain some estimates for the remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$. Before doing this, note that for $n = 2k-1,\ k \geqslant 3,$ we have

$$\begin{aligned} \sigma_{2k-2}(f) &= \sigma_{2k-1}(f) \\ &= \frac{1}{8} \sum_{j=2}^{k-1} \frac{h^{2j}}{(2j)!} \left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right) B_{2j} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]. \end{aligned}$$

Thus

$$\rho_{2k-1}(f) = \tilde{\rho}_{2k-1}(f),$$

so that (4.1) and (4.2) coincide in this case. This shows that (4.2) can be interesting only when $n = 2k,\ k \geqslant 2$. In this case we have

$$\begin{aligned}\tilde{\rho}_{2k}(f) &= \rho_{2k}(f) + \sigma_{2k}(f) - \sigma_{2k-1}(f) \\ &= \rho_{2k}(f) + \frac{h^{2k}}{8(2k)!}\left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right)B_{2j}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right].\end{aligned}$$

In fact we have

$$\tilde{\rho}_{2k-2}(f) = \rho_{2k}(f),\ k \geqslant 3.$$

Therefore, for $k \geqslant 3$ we can approximate $\int_a^b f(t)dt$ by

$$I_{2k}(f;\nu) = D_\nu(f) - \frac{1}{8}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

using either (4.1) with $n = 2k-1$ or (4.2) with $n = 2k-2$. If we apply (4.1), we must assume that $f^{(2k-2)}$ is a continuous function of bounded variation on $[a,b]$. For (4.2), it is enough to assume that $f^{(2k-3)}$ is a continuous function of bounded variation on $[a,b]$.

THEOREM 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then we have*

$$|\rho_n(f)| \leqslant \frac{\nu h^{n+1}}{8n!}\int_0^1 |F_n(t)|\,dt \cdot L$$

and

$$|\tilde{\rho}_n(f)| \leqslant \frac{\nu h^{n+1}}{8n!}\int_0^1 |G_n(t)|\,dt \cdot L.$$

Specially

$$|\rho_1(f)| \leqslant \frac{25}{288}\nu h^2 L, \quad |\rho_2(f)| \leqslant \frac{1}{192}\nu h^3 L,$$

and

$$|\rho_3(f)| \leqslant \frac{1}{1728}\nu h^4 L, \quad |\rho_4(f)| \leqslant \frac{7}{51840}\nu h^5 L.$$

Proof. Applying (3.8) and (3.9) we get for $i = 1,\dots,\nu$,

$$|\rho_n(f;i)| \leqslant \frac{h^{n+1}}{8n!}\int_0^1 |F_n(t)|\,dt \cdot L$$

and

$$|\tilde{\rho}_n(f;i)| \leqslant \frac{h^{n+1}}{8n!}\int_0^1 |G_n(t)|\,dt \cdot L.$$

Using the above estimates and the triangle inequality, we get from (4.5)

$$|\rho_n(f)| \leqslant \sum_{i=1}^{\nu}|\rho_n(f;i)| \leqslant \frac{\nu h^{n+1}}{8n!}\int_0^1 |F_n(t)|\,dt \cdot L$$

and

$$|\tilde{\rho}_n(f)| \leqslant \sum_{i=1}^{\nu} |\tilde{\rho}_n(f;i)| \leqslant \frac{\nu h^{n+1}}{8n!} \int_0^1 |G_n(t)|\, dt \cdot L.$$

The last assertion follows from Corollary 3. □

REMARK 11. Instead of the assumption that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$, we can use the stronger assumption that $f^{(n)}$ exists and is bounded on $[a,b]$, for some $n \geqslant 1$. In this case Theorem 6 applies with L replaced by $\left\|f^{(n)}\right\|_\infty$ (see Remark 3).

THEOREM 7. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then we have*

$$|\rho_n(f)| \leqslant \frac{h^n}{8n!} \max_{t\in[0,1]} |F_n(t)| \cdot V_a^b(f^{(n-1)})$$

and

$$|\tilde{\rho}_n(f)| \leqslant \frac{h^n}{8n!} \max_{t\in[0,1]} |G_n(t)| \cdot V_a^b(f^{(n-1)}).$$

Specially

$$|\rho_1(f)| \leqslant \frac{5}{24} h V_a^b(f), \quad |\rho_2(f)| \leqslant \frac{1}{72} h^2 V_a^b(f')$$

and

$$|\rho_3(f)| \leqslant \frac{1}{768} h^3 V_a^b(f''), \quad |\rho_4(f)| \leqslant \frac{1}{3456} h^4 V_a^b(f''').$$

Proof. Applying (3.11) and (3.12) we get for $i = 1,\dots,\nu$, respectively,

$$|\rho_n(f;i)| \leqslant \frac{h^n}{8n!} \max_{t\in[0,1]} |F_n(t)| \cdot V_{a+(i-1)h}^{a+ih}(f^{(n-1)})$$

and

$$|\tilde{\rho}_n(f;i)| \leqslant \frac{h^n}{8n!} \max_{t\in[0,1]} |G_n(t)| \cdot V_{a+(i-1)h}^{a+ih}(f^{(n-1)}).$$

Using the above estimates and the triangle inequality, we get from (4.5)

$$\begin{aligned} |\rho_n(f)| &\leqslant \sum_{i=1}^{\nu} |\rho_n(f;i)| \\ &\leqslant \frac{h^n}{8n!} \max_{t\in[0,1]} |F_n(t)| \cdot \sum_{i=1}^{\nu} V_{a+(i-1)h}^{a+ih}(f^{(n-1)}) \\ &= \frac{h^n}{8n!} \max_{t\in[0,1]} |F_n(t)| \cdot V_a^b(f^{(n-1)}), \end{aligned}$$

which proves the first inequality. The second inequality is proved similarily. The last assertion follows from Corollary 4. □

REMARK 12. If $f:[a,b]\to\mathbf{R}$ is such that $f^{(n)}\in L_1[a,b]$ for some $n\geqslant 1$, then $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ and $V_a^b(f^{(n-1)})=\|f^{(n)}\|_1$. Therefore, Theorem 7 applies with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$ (see Remark 5).

THEOREM 8. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ *or* $p=\infty$, $q=1$. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n)}\in L_p[a,b]$ *for some* $n\geqslant 1$. *Then we have*

$$|\rho_n(f)|\leqslant \nu K(n,p)h^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p$$

and

$$|\tilde{\rho}_n(f)|\leqslant \nu K^*(n,p)h^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p,$$

where $K(n,p)$ *and* $K^*(n,p)$ *are defined as in Theorem 4.*

Proof. For $i=1,\ldots,\nu$ consider the function $g_i(t)=f^{(n)}(t)$, $t\in[a+(i-1)h\,,\,a+ih]$. Obviously we have

$$\|g_i\|_p\leqslant\|f^{(n)}\|_p,$$

where the norm $\|g_i\|_p$ is taken over the interval $[a+(i-1)h,a+ih]$, while the norm $\|f^{(n)}\|_p$ is taken over the interval $[a,b]$. Applying (3.14) and (3.15) and using the above inequality, we get for $i=1,\ldots,\nu$

$$|\rho_n(f;i)|\leqslant K(n,p)h^{n+\frac{1}{q}}\cdot\|g_i\|_p\leqslant K(n,p)h^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p$$

and

$$|\tilde{\rho}_n(f;i)|\leqslant K^*(n,p)h^{n+\frac{1}{q}}\cdot\|g_i\|_p\leqslant K^*(n,p)h^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p.$$

The result follows from (4.5) by the triangle inequality. □

THEOREM 9. *Suppose that* $f:[a,b]\to\mathbf{R}$ *is such that* $f^{(n)}$ *exists and is integrable on* $[a,b]$, *for some* $n\geqslant 1$. *Assume that*

$$m_n\leqslant f^{(n)}(t)\leqslant M_n,\ a\leqslant t\leqslant b,$$

for some constants m_n *and* M_n. *Then*

$$|\tilde{\rho}_n(f)|\leqslant\frac{\nu}{16}(b-a)^{n+1}(M_n-m_n)\sqrt{(1+7\cdot 3^{2-2n})\frac{|B_{2n}|}{(2n)!}}.$$

Proof. Applying (3.17) we get

$$|\tilde{\rho}_n(f;i)|\leqslant\frac{1}{16}(b-a)^{n+1}(M_n-m_n)\sqrt{(1+7\cdot 3^{2-2n})\frac{|B_{2n}|}{(2n)!}},$$

for all $i=1,\ldots,\nu$. The result follows from (4.5) using the triangle inequality. □

In the following discussion we assume that $f : [a,b] \to \mathbf{R}$ has a continuous derivative of order n, for some $n \geqslant 1$. In this case we can use (2.3) and the second formula from Remark 1 to obtain, for $i = 1, \ldots, \nu$,

$$\begin{aligned}\rho_n(f\,;i) &= \frac{h^n}{8n!}\int_{a+(i-1)h}^{a+ih} F_n\left(\frac{t-a-(i-1)h}{h}\right) f^{(n)}(t)dt \\ &= \frac{h^{n+1}}{8n!}\int_0^1 F_n(s) f^{(n)}(a+(i-1)h+hs)ds.\end{aligned}$$

Therefore, by (4.5) we get

$$\rho_n(f) = \frac{h^{n+1}}{8n!}\int_0^1 F_n(s)\Phi_n(s)ds, \tag{4.6}$$

where

$$\Phi_n(s) = \sum_{i=1}^{\nu} f^{(n)}(a+(i-1)h+hs),\ \ 0 \leqslant s \leqslant 1. \tag{4.7}$$

Similarly, we get

$$\tilde{\rho}_n(f) = \frac{h^{n+1}}{8n!}\int_0^1 G_n(s)\Phi_n(s)ds.$$

Obviously, $\Phi_n(s)$ is a continuous function on $[0,1]$ and

$$\begin{aligned}\int_0^1 \Phi_n(s)ds &= h^{-1}\sum_{i=1}^{\nu}\left[f^{(n-1)}(a+ih) - f^{(n-1)}(a+(i-1)h)\right] \\ &= h^{-1}\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right].\end{aligned} \tag{4.8}$$

From the discussion given at the beginning of this section it follows that it is the most interesting to consider the repeated Euler-Maclaurin formula (4.1) for $n = 2k$, $k \geqslant 2$, which can be rewritten as

$$\int_a^b f(t)dt = I_{2k}(f\,;\nu) + \rho_{2k}(f), \tag{4.9}$$

where

$$I_{2k}(f\,;\nu) = D_\nu(f) - \frac{1}{8}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right].$$

We assume the sum on the right hand side in the above equality to be zero when $k = 2$.

THEOREM 10. *If $f : [a,b] \to \mathbf{R}$ is such that $f^{(2k)}$ is continuous on $[a,b]$, for some $k \geqslant 2$, then there exists a point $\eta \in [a,b]$ such that*

$$\rho_{2k}(f) = -\nu\frac{h^{2k+1}}{8(2k)!}\left(1-2^{1-2j}\right)\left(1-3^{2-2j}\right)B_{2j}f^{(2k)}(\eta). \tag{4.10}$$

Proof. Using (4.6) we can rewrite $\rho_{2k}(f)$ as

$$\rho_{2k}(f) = (-1)^k \frac{h^{2k+1}}{8(2k)!} J_k, \tag{4.11}$$

where

$$J_k = \int_0^1 (-1)^k F_{2k}(s)\Phi_{2k}(s)ds. \tag{4.12}$$

If

$$m = \min_{t\in[a,b]} f^{(2k)}(t), \quad M = \max_{t\in[a,b]} f^{(2k)}(t),$$

then from (4.7) we get

$$\nu m \leqslant \Phi_{2k}(s) \leqslant \nu M, \quad 0 \leqslant s \leqslant 1.$$

On the other side, from Corollary 1 it follows that

$$(-1)^k F_{2k}(s) \geqslant 0, \quad 0 \leqslant s \leqslant 1,$$

which implies

$$\nu m \int_0^1 (-1)^k F_{2k}(s)ds \leqslant J_k \leqslant \nu M \int_0^1 (-1)^k F_{2k}(s)ds.$$

We have already calculated in the proof of Corollary 2 that $\int_0^1 F_{2k}(s)ds = -\tilde{B}_{2k}$, so that we have

$$\nu m(-1)^{k-1}\tilde{B}_{2k} \leqslant J_k \leqslant \nu M(-1)^{k-1}\tilde{B}_{2k}.$$

By the continuity of $f^{(2k)}(s)$ on $[a,b]$, it follows that there must exist a point $\eta \in [a,b]$ such that

$$J_k = \nu(-1)^{k-1}\tilde{B}_{2k} f^{(2k)}(\eta).$$

Combining this with (4.11) we get (4.10). □

REMARK 13. The repeated Euler-Maclaurin formula (4.9) is a generalization of Maclaurin formula (1.2). Namely, for $k=2$ and $\nu=1$ formula (4.10) reduces to

$$\rho_4(f) = \frac{7(b-a)^5}{51840} f^{(4)}(\eta)$$

i.e. to (1.2).

REMARK 14. In [11, p. 222] the following repeated Euler-Simpson formula has been considered:

$$\int_a^b f(t)dt = I_{S,2k}(f;\nu) + \rho_{S,2k}(f),$$

where

$$I_{S,2k}(f,\nu) = \frac{h}{6}\sum_{i=1}^{\nu}\left[f(a+(i-1)h)+4f(a+(i-1/2)h)+f(a+ih)\right]$$
$$+\frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}(1-2^{2-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right].$$

It has been proved that, under the assumptions of Theorem 10, there exists a point $\xi \in [a,b]$ such that [11, p. 225]

$$\rho_{S,2k}(f) = \nu\frac{h^{2k+1}}{3(2k)!}(1-2^{2-2k})B_{2k}f^{(2k)}(\xi).$$

We can compare the remainders $\rho_{S,2k}(f)$ and $\rho_{2k}(f)$. From the above expression and (4.10) we get

$$\frac{\rho_{2k}(f)}{\rho_{S,2k}(f)} = -\frac{3\left(1-2^{1-2k}\right)\left(1-3^{2-2k}\right)}{8(1-2^{2-2k})}\cdot\frac{f^{(2k)}(\eta)}{f^{(2k)}(\xi)}.$$

Therefore, if $f^{(2k)}$ does not change its sign on $[a,b]$, then $\rho_{2k}(f)$ and $\rho_{S,2k}(f)$ have opposite signs. Also note that for the numerical coefficients

$$K = \nu\frac{h^{2k+1}}{8(2k)!}\left(1-2^{1-2k}\right)(1-3^{2-2k})B_{2k}$$

and

$$K_S = \nu\frac{h^{2k+1}}{3(2k)!}(1-2^{2-2k})B_{2k}$$

we have

$$\frac{3}{8} < \frac{K}{K_S} = \frac{3\left(1-2^{1-2k}\right)(1-3^{2-2k})}{8(1-2^{2-2k})} \leqslant \frac{7}{18},\ k \geqslant 2.$$

Therefore, if $f^{(2k)}$ changes very slowly, then the approximate equality $\int_a^b f(t)dt \approx I_{2k}(f;\nu)$ will be more accurate than the approximate equality $\int_a^b f(t)dt \approx I_{S,2k}(f;\nu)$.

THEOREM 11. *If $f : [a,b] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[a,b]$, for some $k \geqslant 2$, and does not change its sign on $[a,b]$, then there exists a point $\theta \in [0,1]$ such that*

$$\rho_{2k}(f) = -\theta\frac{h^{2k}}{8(2k)!}\left(2-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]. \quad (4.13)$$

Proof. Suppose that $f^{(2k)}(t) \geqslant 0,\ a \leqslant t \leqslant b$. Then from (4.7) we get

$$\Phi_{2k}(s) \geqslant 0,\ \ 0 \leqslant s \leqslant 1.$$

From Corollary 1 it follows that

$$0 \leqslant (-1)^k F_{2k}(s) \leqslant (-1)^k F_{2k}\left(\frac{1}{2}\right),\ \ 0 \leqslant s \leqslant 1.$$

Therefore, if J_k is given by (4.12), then

$$0 \leqslant J_k \leqslant (-1)^k F_{2k}\left(\frac{1}{2}\right)\int_0^1 \Phi_{2k}(s)ds.$$

Using (3.6) and (4.8), we get

$$0 \leqslant J_k \leqslant (-1)^{k-1}\left(2-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}h^{-1}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right],$$

which means that there must exist a point $\theta \in [0,1]$ such that

$$J_k = \theta(-1)^{k-1}\left(2-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}h^{-1}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].$$

Combining this with (4.11) we get (4.13). The argument is the same when $f^{(2k)}(t) \leqslant 0$, $a \leqslant t \leqslant b$, since in that case we get

$$(-1)^{k-1}\left(2-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}h^{-1}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right] \leqslant J_k \leqslant 0. \qquad \square$$

REMARK 15. If we approximate $\int_a^b f(t)dt$ by $I_{2k}(f;\nu)$, then the next approximation will be $I_{2k+2}(f;\nu)$. The difference

$$\Delta_{2k}(f;\nu) = I_{2k+2}(f;\nu) - I_{2k}(f;\nu)$$

is equal to the last term in $I_{2k+2}(f;\nu)$, that is

$$\begin{aligned}\Delta_{2k}(f;\nu) &= -\frac{h^{2k}}{8(2k)!}\tilde{B}_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\\ &= -\frac{h^{2k}}{8(2k)!}\left(1-2^{1-2k}\right)(1-3^{2-2k})B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].\end{aligned}$$

We see that, under the assumptions of Theorem 11, $\rho_{2k}(f)$ and $\Delta_{2k}(f;\nu)$ are of the same sign. Moreover, we have

$$\rho_{2k}(f) = \frac{\theta\left(2-2^{1-2k}\right)}{1-2^{1-2k}}\Delta_{2k}(f;\nu)$$

which gives the following estimate for the remainder $\rho_{2k}(f)$:

$$|\rho_{2k}(f)| \leqslant 3\,|\Delta_{2k}(f;\nu)|.$$

THEOREM 12. *Suppose that $f:[a,b]\to\mathbf{R}$ is such that $f^{(2k+2)}$ is continuous on $[a,b]$, for some $k \geqslant 2$. If for each $x\in[a,b]$ either*

$$f^{(2k)}(x) \geqslant 0 \textit{ and } f^{(2k+2)}(x) \geqslant 0$$

or

$$f^{(2k)}(x) \leqslant 0 \textit{ and } f^{(2k+2)}(x) \leqslant 0,$$

then the remainder $\rho_{2k}(f)$ has the same sign as the first neglected term $\Delta_{2k}(f;\nu)$ and

$$|\rho_{2k}(f)| \leqslant |\Delta_{2k}(f;\nu)|.$$

Proof. We have

$$\Delta_{2k}(f;\nu) + \rho_{2k+2}(f) = \rho_{2k}(f),$$

that is

$$\Delta_{2k}(f;\nu) = \rho_{2k}(f) - \rho_{2k+2}(f). \tag{4.14}$$

By (4.6) we have

$$\rho_{2k}(f) = \frac{h^{2k+1}}{8(2k)!}\int_0^1 F_{2k}(s)\Phi_{2k}(s)ds$$

and

$$-\rho_{2k+2}(f) = \frac{h^{2k+3}}{8(2k+2)!}\int_0^1 [-F_{2k+2}(s)]\Phi_{2k+2}(s)ds.$$

Under the assumptions made on f , we see that for all $s \in [a,b]$ either

$$\Phi_{2k}(s) \geqslant 0 \text{ and } \Phi_{2k+2}(s) \geqslant 0$$

or

$$\Phi_{2k}(s) \leqslant 0 \text{ and } \Phi_{2k+2}(s) \leqslant 0.$$

Also, from Corollary 1 it follows that for all $s \in [a,b]$

$$(-1)^k F_{2k}(s) \geqslant 0 \text{ and } (-1)^k[-F_{2k+2}(s)] \geqslant 0.$$

We conclude that $\rho_{2k}(f)$ has the same sign as $-\rho_{2k+2}(f)$. Therefore, because of (4.14), $\Delta_{2k}(f;\nu)$ must have the same sign as $\rho_{2k}(f)$ and $-\rho_{2k+2}(f)$. Moreover, it follows that

$$|\rho_{2k}(f)| \leqslant |\Delta_{2k}(f;\nu)| \text{ and } |-\rho_{2k+2}(f)| \leqslant |\Delta_{2k}(f;\nu)|. \qquad \square$$

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