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A NOTE ON WIRTINGER–BEESACK'S INTEGRAL INEQUALITIES

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Abstract. In this note we show that Wirtinger's integral inequalities and Beesack's integral inequality follow from a unified integral inequality. We also consider Banach space valued versions of these inequalities.

Wirtinger's integral inequalities (cf. [1]) are as follows:

$$\int_0^1 f(t)^2 dt \leqslant \frac{1}{\pi^2}\int_0^1 f'(t)^2 dt,$$
$$(f \in C^1([0,1],\mathbb{R}) \text{ with } f(0) = f(1) = 0),$$

$$\int_0^1 f(t)^2 dt \leqslant \frac{4}{\pi^2}\int_0^1 f'(t)^2 dt,$$
$$(f \in C^1([0,1],\mathbb{R}) \text{ with } f(0) = 0).$$

Also Beesack's integral inequality (cf. [3, p. 145]) is as follows:

$$\int_0^1 f(t)^2 t^4 dt \leqslant \frac{4}{25}\int_0^1 f'(t)^2 t^6 dt,$$
$$(f \in C^1([0,1],\mathbb{R}) \text{ with } f(0) = f(1) = 0).$$

In this note we show that these inequalities follow from a unified integral inequality and we also consider Banach space valued versions of these inequalities. Recently, Wirtinger-Beesack's integral inequalities have studied by B. Florkiewicz and K. Wojteczek [2], but our unified integral inequality is different from their one.

Throughout this note, let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$, $\varphi \in C^1((0,1],\mathbb{R})$ such that $\lim_{t\searrow 0} t\varphi(t)$ and $\lim_{t\searrow 0} t^2\varphi'(t)$ exist and $\psi \in C^1([0,1],\mathbb{R})$. Let $x, y \in C^1([0,1],H)$ with $x(0) = y(0) = 0$ and set $\alpha = \lim_{t\searrow 0} t\varphi(t)$ and $\beta = \lim_{t\searrow 0} t^2\varphi'(t)$. Then

$$\begin{aligned}\lim_{t\searrow 0}(\varphi\psi)'(t)\langle x(t),y(t)\rangle &= \beta\psi(0)\langle x'(0),y'(0)\rangle + \alpha\psi'(0)\langle x'(0),y(0)\rangle \\ &= \beta\psi(0)\langle x'(0),y'(0)\rangle\end{aligned}$$

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and hence the integral $\int_0^1 (\varphi\psi)'(t)\langle x(t), y(t)\rangle\, dt$ exists. Also since

$$\lim_{t\searrow 0} \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) = \|\alpha x'(0)\|^2 + \|\alpha y'(0)\|^2,$$

it follows that the integral $\int_0^1 \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) dt$ exists. In this case, we have the following

THEOREM 1. *Let* $x, y \in C^1([0,1], H)$ *with* $\varphi(1)\psi(1)\langle x(1), y(1)\rangle = 0$ *and* $x(0) = y(0) = 0$. *Then*

$$2\left|\int_0^1 (\varphi\psi)'(t)\langle x(t), y(t)\rangle\, dt\right| \leqslant \int_0^1 \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) dt + \int_0^1 \psi^2(t)\left(\|x'(t)\|^2 + \|y'(t)\|^2\right) dt$$

holds. The equality is attained if and only if either

$$\varphi(t)x(t) + \psi(t)y'(t) = \psi(t)x'(t) + \varphi(t)y(t) = 0, \quad (0 < t \leqslant 1)$$

or

$$-\varphi(t)x(t) + \psi(t)y'(t) = \psi(t)x'(t) - \varphi(t)y(t) = 0, \quad (0 < t \leqslant 1).$$

Proof. Set $f(t) = \varphi(t)\psi(t)\langle x(t), y(t)\rangle$, $(0 < t \leqslant 1)$ and $f(0) = 0$. Since $\lim_{t\searrow 0} f(t) = \alpha\psi(0)\langle x'(0), y(0)\rangle = 0$, it follows that $f \in C([0,1], \mathbb{R})$. Also since

$$\begin{aligned} f'(t) &= \varphi'(t)\psi(t)\langle x(t), y(t)\rangle + \varphi(t)\psi'(t)\langle x(t), y(t)\rangle \\ &\quad + \varphi(t)\psi(t)\langle x'(t), y(t)\rangle + \varphi(t)\psi(t)\langle x(t), y'(t)\rangle, \qquad (0 < t \leqslant 1), \end{aligned}$$

$$\begin{aligned} \lim_{t\searrow 0} f'(t) &= \beta\psi(0)\langle x'(0), y'(0)\rangle + \alpha\psi'(0)\langle x'(0), y(0)\rangle \\ &\quad + \alpha\psi(0)\langle x'(0), y'(0)\rangle + \alpha\psi(0)\langle x'(0), y'(0)\rangle \\ &= (2\alpha + \beta)\psi(0)\langle x'(0), y'(0)\rangle \end{aligned}$$

namely $\lim_{t\searrow 0} f'(t)$ exists. It follows that $f \in C^1([0,1], \mathbb{R})$ and

$$\int_0^1 f'(t)dt = f(1) - f(0) = \varphi(1)\psi(1)\langle x(1), y(1)\rangle = 0. \tag{1}$$

Now put

$$p(t) = \|\varphi(t)x(t) + \psi(t)y'(t)\|^2 + \|\psi(t)x'(t) + \varphi(t)y(t)\|^2, \ (0 < t \leqslant 1).$$

Since $\lim_{t\searrow 0} \varphi(t)x(t) = \alpha x'(0)$ and $\lim_{t\searrow 0} \varphi(t)y(t) = \alpha y'(0)$, it follows that p can be extended to a continuous function on $[0,1]$ and the integral $\int_0^1 p(t)dt$ exists. Note that

$$\begin{aligned} &2\varphi(t)\psi(t)\langle x(t), y(t)\rangle' \\ &\quad = p(t) - \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) \\ &\qquad - \psi^2(t)\left(\|x'(t)\|^2 + \|y'(t)\|^2\right), \qquad (0 < t \leqslant 1) \end{aligned}$$

and hence, the function $t \mapsto \varphi(t)\psi(t)\langle x(t), y(t)\rangle'$ can be extended to a continuous function on $[0,1]$. It follows that the integral $\int_0^1 \varphi(t)\psi(t)\langle x(t), y(t)\rangle' dt$ exists and

$$2\int_0^1 \varphi(t)\psi(t)\langle x(t), y(t)\rangle' dt \tag{2}$$
$$= \int_0^1 p(t)dt - \int_0^1 \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) dt - \int_0^1 \psi^2(t)\left(\|x'(t)\|^2 + \|y'(t)\|^2\right) dt.$$

Also, by (1) we have

$$\int_0^1 \varphi(t)\psi(t)\langle x(t), y(t)\rangle' dt \tag{3}$$
$$= \int_0^1 f'(t)dt - \int_0^1 (\varphi\psi)'(t)\langle x(t), y(t)\rangle\, dt$$
$$= -\int_0^1 (\varphi\psi)'(t)\langle x(t), y(t)\rangle\, dt.$$

By (2) and (3), we obtain the desired inequality

$$2\int_0^1 (\varphi\psi)'(t)\langle x(t), y(t)\rangle\, dt \leqslant \int_0^1 \varphi^2(t)\left(\|x(t)\|^2 + \|y(t)\|^2\right) dt + \int_0^1 \psi^2(t)\left(\|x'(t)\|^2 + \|y'(t)\|^2\right) dt.$$

The equality is attained if and only if $\int_0^1 p(t)dt = 0$, that is

$$\varphi(t)x(t) + \psi(t)y'(t) = \psi(t)x'(t) + \varphi(t)y(t) = 0$$

for all $t \in (0,1]$. Replacing φ by $-\varphi$ in the above argument, we can obtain the desired result. □

In particular, putting $x = y$ in the above arguments, we have

COROLLARY 2. *Let $x \in C^1([0,1], H)$ be such that $\varphi(1)\psi(1)x(1) = 0$ and $x(0) = 0$. Then $\int_0^1 \left((\varphi\psi)'(t) - \varphi^2(t)\right)\|x(t)\|^2 dt \leqslant \int_0^1 \psi^2(t)\|x'(t)\|^2 dt$ holds. The equality is attained if and only if $\varphi(t)x(t) + \psi(t)x'(t) = 0$ for all $0 < t \leqslant 1$.*

REMARK 1. Let $0 < \theta < \pi$. Set $\varphi_\theta(t) = \cot\theta t$, $(0 < t \leqslant 1)$ and $\psi_\theta(t) = -1/\theta$, $(0 \leqslant t \leqslant 1)$. Then φ_θ is a function in $C^1((0,1], \mathbb{R})$ such that $\lim_{t\searrow 0} t\varphi_\theta(t)$ and $\lim_{t\searrow 0} t^2\varphi_\theta{}'(t)$ exist. Also ψ_θ is clearly a function in $C^1([0,1], \mathbb{R})$. Note that $(\varphi_\theta\psi_\theta)'(t) - \varphi_\theta{}^2(t) = 1$, $(0 < t \leqslant 1)$. Hence, Corollary 2 implies the following Hilbert space valued version of Wirtinger's inequality:

$$\int_0^1 \|x(t)\|^2 dt \leqslant \frac{1}{\pi^2}\int_0^1 \|x'(t)\|^2 dt$$

for all $x \in C^1([0,1],H)$ with $x(0)=x(1)=0$. Note also that $\varphi_\theta(1)=0$ if $\theta=\pi/2$. Then Corollary 2 also implies the following Hilbert space valued version of Wirtinger's inequality:

$$\int_0^1 \|x(t)\|^2 dt \leqslant \frac{4}{\pi^2}\int_0^1 \|x'(t)\|^2 dt$$

for all $x \in C^1([0,1],H)$ with $x(0)=0$. However, we see later that this inequality holds for an arbitrary Banach space (see Remark 3).

REMARK 2. Let $\gamma \in \mathbb{R}$ and $\lambda \geqslant -1$ with $\lambda \neq -1/2$. Set $\varphi_{\gamma,\lambda}(t)=\gamma t^\lambda$, $(0<t\leqslant 1)$ and $\psi_\lambda(t)=t^{\lambda+1}$, $(0\leqslant t\leqslant 1)$. Then $\varphi_{\gamma,\lambda}$ is a function in $C^1((0,1],\mathbb{R})$ such that $\lim_{t\searrow 0} t\varphi_{\gamma,\lambda}(t)$ and $\lim_{t\searrow 0} t^2\varphi_{\gamma,\lambda}{}'(t)$ exist. Also ψ_λ is a function in $C^1([0,1],\mathbb{R})$. Note that

$$(\varphi_{\gamma,\lambda}\,\psi_\lambda)'(t)-\varphi_{\gamma,\lambda}{}^2(t)=\left(\gamma(2\lambda+1)-\gamma^2\right)t^{2\lambda}$$

for all $0<t\leqslant 1$. Note also that $\max_{\gamma\in\mathbb{R}}(\gamma(2\lambda+1)-\gamma^2)=(2\lambda+1)^2/4$. Then Corollary 2 implies that

$$\int_0^1 \|x(t)\|^2 t^{2\lambda}dt \leqslant \frac{4}{(2\lambda+1)^2}\int_0^1 \|x'(t)\|^2 t^{2(\lambda+1)}dt$$

for all $x \in C^1([0,1],H)$ with $x(0)=x(1)=0$. However, we see later that this inequality holds for an arbitrary Banach space whenever $-1\leqslant\lambda<-1/2$ (see Remark 4). In case of $\lambda=2$ and $H=\mathbb{R}$, the above inequality is just Beesack's inequality. We note that the above Beesack's type inequality for $\lambda=-1/2$ does not hold. In fact, we can easily see that there is no constant $M>0$ such that

$$\int_0^1 \|x(t)\|^2 t^{-1}dt \leqslant M\int_0^1 \|x'(t)\|^2 dt$$

for all $x \in C^1([0,1],H)$ with $x(0)=x(1)=0$.

COROLLARY 3. *Let E be a real Banach space and $x \in C^1([0,1],E)$ with $x(0)=0$. Suppose that $\varphi(1)\psi(1)=0$ and $\varphi^2(t)<(\varphi\psi)'(t)$, $(0<t<1)$. Then*

$$\int_0^1 \left((\varphi\psi)'(t)-\varphi^2(t)\right)\|x(t)\|^2 dt \leqslant \int_0^1 \psi^2(t)\|x'(t)\|^2 dt$$

holds. The equality is attained if and only if $\varphi(t)\|x(t)\|+\psi(t)\|x'(t)\|=0$, $(0<t<1)$ and $d\|x(t)\|/dt=\|x'(t)\|$, $(0<t<1)$.

Proof. Set $f(t)=\int_0^t \|x'(\tau)\|d\tau$, $(0\leqslant t\leqslant 1)$. Then $f'(t)=\|x'(t)\|$, $(0\leqslant t\leqslant 1)$ and $f(0)=0$. Then Corollary 2 implies that

$$\int_0^1 \left((\varphi\psi)'(t)-\varphi^2(t)\right)f(t)^2 dt \leqslant \int_0^1 \psi^2(t)f'(t)^2 dt = \int_0^1 \psi^2(t)\|x'(t)\|^2 dt.$$

Note that $x(t)=\int_0^t x'(\tau)d\tau$, $(0\leqslant t\leqslant 1)$ and then $\|x(t)\|\leqslant\int_0^t \|x'(\tau)\|d\tau=f(t)$, $(0\leqslant t\leqslant 1)$. Therefore

$$\left((\varphi\psi)'(t)-\varphi^2(t)\right)\|x(t)\|^2 \leqslant \left((\varphi\psi)'(t)-\varphi^2(t)\right)f(t)^2,\ (0<t<1)$$

by hypothesis and hence we obtain the desired inequality. The equality is attained if and only if

$$\int_0^1 \left((\varphi\psi)'(t) - \varphi^2(t)\right) f(t)^2 dt = \int_0^1 \psi^2(t) f'(t)^2 dt \tag{4}$$

and

$$\int_0^1 \left((\varphi\psi)'(t) - \varphi^2(t)\right) \|x(t)\|^2 dt = \int_0^1 \left((\varphi\psi)'(t) - \varphi^2(t)\right) f(t)^2 dt. \tag{5}$$

Corollary 2 implies that (4) holds if and only if $\varphi(t)f(t) + \psi(t)f'(t) = 0$ for all $0 < t < 1$. Note also that (5) holds if and only if $\|x(t)\| = f(t)$ for all $0 < t < 1$. Therefore, we obtain the desired equality condition. □

REMARK 3. Remark 1 and Corollary 3 imply that if E is a real Banach space then

$$\int_0^1 \|x(t)\|^2 dt \leqslant \frac{4}{\pi^2} \int_0^1 \|x'(t)\|^2 dt$$

holds for all $x \in C^1([0,1], E)$ with $x(0) = 0$.

PROBLEM 1. Given a Banach space E, does

$$\int_0^1 \|x(t)\|^2 dt \leqslant \frac{1}{\pi^2} \int_0^1 \|x'(t)\|^2 dt$$

hold for all $x \in C^1([0,1], E)$ with $x(0) = x(1) = 0$?

REMARK 4. Let $-1 \leqslant \lambda < -1/2$. Set $\varphi_\lambda(t) = (2\lambda+1)t^\lambda/2$, $(0 < t \leqslant 1)$ and $\psi_\lambda(t) = (1-t)t^{\lambda+1}$, $(0 \leqslant t \leqslant 1)$. Then φ_λ is a function in $C^1((0,1], \mathbb{R})$ such that $\lim_{t\searrow 0} t\varphi_\lambda(t)$ and $\lim_{t\searrow 0} t^2\varphi_\lambda{}'(t)$ exist. Also ψ_λ is a function in $C^1([0,1], \mathbb{R})$ with $\psi_\lambda(1) = 0$. Note that

$$\begin{aligned} (\varphi_\lambda\psi_\lambda)'(t) - \varphi_\lambda{}^2(t) &= \left(-(2\lambda+1)(\lambda+1)t + \frac{(2\lambda+1)^2}{4}\right) t^{2\lambda} \\ &\geqslant \frac{(2\lambda+1)^2}{4} t^{2\lambda} > 0 \end{aligned}$$

for all $0 < t < 1$. Then Corollary 3 implies that

$$\begin{aligned} \int_0^1 \frac{(2\lambda+1)^2}{4} \|x(t)\|^2 t^{2\lambda} dt &\leqslant \int_0^1 \left((\varphi_\lambda\psi_\lambda)'(t) - \varphi_\lambda{}^2(t)\right) \|x(t)\|^2 dt \\ &\leqslant \int_0^1 \psi_\lambda{}^2(t) \|x'(t)\|^2 dt \leqslant \int_0^1 (1-t)^2 \|x'(t)\|^2 t^{2(\lambda+1)} dt \\ &\leqslant \int_0^1 \|x'(t)\|^2 t^{2(\lambda+1)} dt \end{aligned}$$

and hence

$$\int_0^1 \|x(t)\|^2 t^{2\lambda} dt \leqslant \frac{4}{(2\lambda+1)^2} \int_0^1 \|x'(t)\|^2 t^{2(\lambda+1)} dt$$

for all $x \in C^1([0,1], E)$ with $x(0) = 0$.

PROBLEM 2. Given a Banach space E, does

$$\int_0^1 \|x(t)\|^2 t^{2\lambda} dt \leqslant \frac{4}{(2\lambda+1)^2} \int_0^1 \|x'(t)\|^2 t^{2(\lambda+1)} dt$$

hold for all $x \in C^1([0,1],E)$ with $x(0) = x(1) = 0$?

COROLLARY 4. *Let* $\varphi \in C^1((0,1],\mathbb{R})$ *such that* $\lim_{t\searrow 0} t\varphi(t)$ *and* $\lim_{t\searrow 0} t^2\varphi'(t)$ *exist. Let* $\lambda \in C^1([0,1],\mathbb{R})$ *such that* $\lambda'(t) < 0, (0 < t < 1), \lambda(0) = 1$ *and* $\lambda(1) = 0$. *Let* E *be a real Banach space and* $x \in C^1([0,1],E)$ *with* $x(1) = 0$. *Let* $\psi \in C^1([0,1],\mathbb{R})$ *and suppose that* $\varphi(1)\psi(1) = 0$ *and* $\varphi^2(t) < (\varphi\psi)'(t)$, $(0 < t < 1)$. *Then*

$$\int_0^1 \frac{(\varphi\psi)'(\lambda^{-1}(t)) - \varphi^2(\lambda^{-1}(t))}{-\lambda'(\lambda^{-1}(t))} \|x(t)\|^2 dt \leqslant \int_0^1 -\lambda'(\lambda^{-1}(t))\psi^2(\lambda^{-1}(t))\|x'(t)\|^2 dt.$$

Proof. Consider the function $t \mapsto x(\lambda(t))$ and apply Corollary 3. □

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