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A NOTE ON HILBERT'S INEQUALITY

HE LEPING, GAO MINGZHE AND WEI SHANGRONG

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Abstract. In this paper, it is shown that some sharp results on Hilbert's inequality for double series can be obtained by means of the refined Cauchy's inequality.

1. Introduction

The inequality of the form

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/(m+n)\right)^2 \leqslant \pi^2\left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right) \tag{1}$$

is called Hilbert's double series theorem (see [1]). And the equality contained in (1) holds if and only if (a_n), or (b_n), is a zero-sequence.

In our previous paper [2], it was shown that the inequality (1) can be improved by introducing a weight function of the form $\pi - 0(n)/\sqrt{n}$ (with $0(n) > 0$). And then Gao and Yang [3] proved that an infimum of $0(n)$ is:

$$0(1) = \pi/2 - 7/24 + \theta/320 \qquad (0 < \theta < 1).$$

However, if we select a proper non-zero real number R_ω, and the right-hand side of (1) can be replaced by

$$\left(\sum_{n=1}^{\infty} \omega(n) a_n^2\right)\left(\sum_{n=1}^{\infty} \omega(n) b_n^2\right) - R_\omega^2,$$

where $\omega(n) = \pi - 0(n)/\sqrt{n}$ and $0(n) > 0$, then the further results of the papers [2] and [3] will be obtained.

The main purpose of this paper is to prove the existence of the non-zero real number R_ω and to find an expression of R_ω.

We first intoduce some notations and function.

The inner product of two elements α and β in an inner product space E is denoted by (α, β), and the norm is denoted by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$.

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We next introduce a binary quadratic form defined by

$$F(x,y) = \|\alpha\|^2 x^2 - 2(\alpha,\beta)xy + \|\beta\|^2 y^2 \tag{2}$$

where $x = (\beta,\gamma)$ and $y = (\alpha,\gamma)$.

We further define

$$R = \|\alpha\|\,|(\beta,\gamma)| - \|\beta\|\,|(\alpha,\gamma)|. \tag{3}$$

The results involve R with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that, if $\|\alpha\|\,|(\beta,\gamma)| = \|\beta\|\,|(\alpha,\gamma)|$, then $R = 0$. Therefore, it is shrewd in every case to choose γ such that $\|\alpha\|\,|(\beta,\gamma)| \neq \|\beta\|\,|(\alpha,\gamma)|$.

2. Lemmas

LEMMA 1. *Let $F(x,y)$ and R be defined by (2) and (3), respectively. If $\|\alpha\|\,|(\beta,\gamma)| \neq \|\beta\|\,|(\alpha,\gamma)|$, then $F(x,y) \geqslant R^2 > 0$, and the equality holds if and only if $(\alpha,\beta)xy = 0$.*

Proof. Applying Cauchy-Schwarz's inequality to (2) we have

$$\begin{aligned} F(x,y) &= \|\alpha\|^2 x^2 - 2(\alpha,\beta)xy + \|\beta\|^2 y^2 \\ &\geqslant \|\alpha\|^2 x^2 - 2\,|(\alpha,\beta)xy| + \|\beta\|^2 y^2 \\ &\geqslant (\|\alpha\|\,|x| - \|\beta\|\,|y|)^2. \end{aligned}$$

Since $\|\alpha\|\,|(\beta,\gamma)| \neq \|\alpha\|\,|(\alpha,\gamma)|$,

$$(\|\alpha\|\,|x| - \|\beta\|\,|y|)^2 = (\|\alpha\|\,|(\beta,\gamma)| - \|\beta\|\,|(\alpha,\gamma)|)^2 > 0.$$

It follows that

$$F(x,y) \geqslant R^2 > 0. \tag{4}$$

Clearly, the equality in (4) holds if and only if $(\alpha,\beta)xy = 0$.

LEMMA 2. *Suppose α,β and γ are three arbitrary vectors of the space E. If $\|\gamma\| = 1$ and $\|\alpha\|\,|(\beta,\gamma)| \neq \|\beta\|\,|(\alpha,\gamma)|$, then*

$$(\alpha,\beta)^2 \leqslant \|\alpha\|^2\|\beta\|^2 - R^2 \tag{5}$$

where R is defined by (3). And the equality contained in (5) holds if and only if the vector γ is a linear combination of α and β, and $(\alpha,\beta)xy = 0$.

Proof. Consider the Gramm determinant constructed by the vectors α,β and γ. According to [4] we have

$$\begin{vmatrix} (\alpha,\alpha) & (\alpha,\beta) & (\alpha,\gamma) \\ (\beta,\alpha) & (\beta,\beta) & (\beta,\gamma) \\ (\gamma,\alpha) & (\gamma,\beta) & (\gamma,\gamma) \end{vmatrix} \geqslant 0. \tag{6}$$

The equality contained in (6) holds if and only if the vectors α, β and γ are linearly dependent. Since $\|\alpha\|\,|(\beta,\gamma)| \neq \|\beta\|\,|(\alpha,\gamma)|$, the vectors α and β are linearly independent. Hence the equality in (6) holds if and only if the vector γ is a linear combination of α and β.

Expanding this determinant and using the condition $\|\gamma\| = 1$, we obtain from (6) that

$$\|\alpha\|^2\|\beta\|^2 - (\alpha,\beta)^2 - F(x,y) \geqslant 0,$$

where $F(x,y)$ is defined by (2). And then using the inequality (4), the result follows at once.

Actually, the inequality (5) is a refinement on Cauchy's inequality.

3. Main results

THEOREM. *If* $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^2 < +\infty$, *then*

$$\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/(m+n)\Big)^2 < \Big(\sum_{n=1}^{\infty}\omega(n)a_n^2\Big)\Big(\sum_{n=1}^{\infty}\omega(n)b_n^2\Big) - R_\omega^2 \tag{7}$$

where $\omega(n) = \pi - 0(n)/\sqrt{n},\ 0(n) > 0$ *and* $R_\omega^2 > 0$.

Proof. Define two functions by

$$\alpha = \frac{\alpha_m}{(m+n)^{1/2}}\Big(\frac{m}{n}\Big)^{1/4} \quad \text{and} \quad \beta = \frac{b_n}{(m+n)^{1/2}}\Big(\frac{n}{m}\Big)^{1/4}. \tag{8}$$

An inner product of the vectors α and β is defined by

$$(\alpha,\beta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha\beta. \tag{9}$$

Using the inequality (5) we have

$$\begin{aligned}\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/(m+n)\Big)^2 &= \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha\beta\Big)^2 \\ &\leqslant \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^2\Big)\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^2\Big) - R^2,\end{aligned} \tag{10}$$

where R is defined by (3).

Notice that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^2 = \sum_{n=1}^{\infty}\omega(n)a_n^2 \quad \text{and} \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^2 = \sum_{n=1}^{\infty}\omega(n)b_n^2,$$

where $\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/2}$.

According to the paper [2] we have

$$\omega(n) = \pi - 0(n)/\sqrt{n},$$

where $0(n) > 0$ for all $n \in N$.

The norm with weight $\omega(n)$ is denoted by

$$\|x\|_\omega = \Big(\sum_{n=1}^{\infty}(\pi - 0(n)/\sqrt{n})x_n^2\Big). \tag{11}$$

We may write R in the form

$$R_\omega = \|a\|_\omega \,|(\beta,\gamma)| - \|b\|_\omega \,|(\alpha,\gamma)| \tag{12}$$

where α and β are given by (8), and γ is choosen by

$$\gamma = 6/mn\pi^2. \tag{13}$$

It is easy to deduce that

$$\begin{aligned}\|\gamma\| &= \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{6}{mn\pi^2}\Big)^2\Big)^{1/2} \\ &= \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{36}{m^2n^2\pi^2}\Big)^{1/2} \\ &= \Big(\frac{36}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big)^{1/2} = 1.\end{aligned}$$

Hence the vector γ satisfies the condition of Lemma 2 and it is obvious that $\|\alpha\|_\omega \,|(\beta,\gamma)| \neq \|b\|_\omega \,|(\alpha,\gamma)|$. Whence the inequality (10) can be written as

$$\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/(m+n)\Big)^2 \leqslant \|a\|_\omega^2 \|b\|_\omega^2 - R_\omega^2 \tag{14}$$

where R_ω is given by (12).

Evidently, the vectors α, β and γ defined respectively by (8) and (13) are linearly independent, it is impossible to tako equality in (14). And owing to the fact that the conditions of Lemma 1 are satisfied, we have $R_\omega^2 > 0$.

The proof of the theorem is completed.

Note: According to the paper [2], the expression of $O(n)$ contained in (7) is given as follows:

$$O(n) = A_2(n) + C_2(n) - n/2(n+1)$$

where $A_2(n) = 2\sqrt{n}\Big(\operatorname{arc\,tg}(1/\sqrt{n})\Big)$ and

$$C_2(n) = -\sum_{k=1}^{s-1}\frac{\sqrt{n}B_{2k}}{(2k)!}\Big(\frac{1}{n}\Big)^{2k} f^{(2k-1)}\Big(\frac{1}{n}\Big) + \rho_s,$$

$f(t) = \frac{1}{1+t}\left(\frac{1}{t}\right)^{1/2}$, $t \in (0,1]$, $B_{j's}$, are Bernoulli numbers, viz. $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, etc. ρ_s is the remaider having the same sign as that of the first to be dropped, and having smaller absolute value in comparison.

COROLLARY 1. *If* $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$, *then*

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n/(m+n)\right)^2 < \left(\sum_{n=1}^{\infty}(\pi - O(n)/\sqrt{n})a_n^2\right)^2 - \tilde{R}_\omega^2$$

where $O(n) > 0$ *and* $\tilde{R}_\omega^2 > 0$.

Proof. We need only show that $\tilde{R}_\omega^2 > 0$. We obtain easily from (12) that

$$\tilde{R}_\omega^2 = \|a\|_\omega^2\Big(|(\beta,\gamma)| - |(\alpha,\gamma)|\Big)^2 \tag{15}$$

where α and γ are given respectively by (8) and (13), and β is defined by

$$\beta = \frac{a_n}{(m+n)^{1/2}}\left(\frac{n}{m}\right)^{1/2}.$$

Notice that $\beta/\alpha = \sqrt{n}a_n/\sqrt{m}a_m \neq k$, for all $m, n \in N$, where k is a constant.
Hence $\tilde{R}_\omega^2$ in (15) is not equal to zero. In other words, we have $\tilde{R}_\omega^2 > 0$.

COROLLARY 2. *With the assumptions as in the theorem, then*

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/(m+n)\right)^2 < \left(\sum_{n=1}^{\infty}(\pi - \alpha/\sqrt{n})a_n^2\right)\left(\sum_{n=1}^{\infty}(\pi - \alpha/\sqrt{n})b_n^2\right) - R_\omega^2 \tag{16}$$

where $\alpha = \pi/2 - 7/24 + \theta/320$ $(O < \theta < 1)$ *and* R_ω *is given by (12).*

Proof. In the paper [3], it was shown that

$$\omega(n) \leqslant \pi - \alpha/\sqrt{n}$$

where $\omega(n)$ is a weight function of the form

$$\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/2},$$

and $\alpha = \pi/2 - 7/24 + \theta/320 \quad (O < \theta < 1)$.
Thus the corollary follows.

If R_ω^2 contained in (16) is replaced by zero, then the result of the paper [3] is yielded. Clearly the inequality (16) is an improvement of the corresponding result of the paper [3].

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He Leping
Department of Mathematics and Computer Science
Normal College, Jishou University
Jishou Hunan 416000
People's Republic of China
e-mail: lianheping@163.com

Gao Mingzhe
Department of Mathematics and Computer Science
Normal College, Jishou University
Jishou Hunan 416000
People's Republic of China
e-mail: mingzhegao@163.com

Wei Shangrong
Department of Mathematics
Zhaoqing College
Zhaoqing Guangdong 526062
People's Republic of China