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ON THE HYERS–ULAM–RASSIAS STABILITY OF A GENERAL CUBIC FUNCTIONAL EQUATION

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Abstract. In this paper, we solve the generalized Hyers-Ulam-Rassias stability problem for a cubic functional equation $f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)$ in the spirit of Hyers, Ulam, Rassias and Găvruta.

1. Introduction

In 1940, S. M. Ulam [21] raised a question concerning the stability of group homomorphisms:

Let G_1 *be a group and let* G_2 *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta > 0$ *such that if a function* $h : G_1 \to G_2$ *satisfies the inequality* $d(h(xy), h(x)h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$ *?*

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive functions was solved by D. H. Hyers [8] and generalized by Th. M. Rassias [18]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 10, 15]. The terminology generalized Hyers-Ulam-Rassias stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [9, 11, 19].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric biadditive function ([1], [16]). It is natural that the equation (1.1) is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique

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symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [1], [16]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.2}$$

A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [3, 4, 7, 14]. K. W. Jun and Y. H. Lee [13] proved the Hyers-Ulam-Rassias stability of the pexiderized quadratic equation (1.1).

Now, we investigate the following functional equations, which are quite different from (1.1),

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.3}$$

$$\begin{aligned} &f(x+y+z) + f(x+y-z) + 2f(x) + 2f(y) \\ &\quad = 2f(x+y) + f(x+z) + f(x-z) + f(y+z) + f(y-z), \end{aligned} \tag{1.4}$$

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y). \tag{1.5}$$

Since the function $f(x) = cx^3$ on real field is a solution of the each functional equation, each equation is naturally called a cubic functional equation and in particular every solution of the cubic functional equation (1.3) is said to be a cubic function. Let both E_1 and E_2 be real vector spaces. The author [17] proved that a function $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exists a function $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x,x,x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables. The function B is given by

$$B(x,y,z) = \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)] \tag{1.6}$$

for all $x, y, z \in E_1$.

Also he showed that a function $f : E_1 \to E_2$ satisfies the functional equation (1.4) if and only if there exist functions $B : E_1 \times E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that $f(x) = B(x,x,x) + A(x) + c$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables and A is additive.

In this paper, we establish the Hyers-Ulam-Rassias stability problem for the equation (1.5) under the approximately cubic (quadratic or additive) condition.

2. General solution of Eq. (1.5)

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be real vector spaces throughout this paper.

THEOREM 2.1. *A function $f : E_1 \to E_2$ satisfies the functional equation (1.5) if and only if there exist functions $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that* $f(x) = B(x,x,x) + Q(x,x) + A(x) + c$

for all $x \in E_1$, where B is symmetric for each fixed one variable and is additive for fixed two variables, Q is symmetric biadditive and A is additive.

Proof. We first assume that f is a solution of the functional equation (1.5). If we put $g(x) = f(x) - f(0)$, we get g is also a solution of (1.5) and $g(0) = 0$. So we may assume without loss of generality that f is a solution of (1.5) and $f(0) = 0$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$, $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in E_1$. Then $f_e(0) = 0 = f_o(0)$, f_e is even and f_o is odd. Since f is a solution of (1.5), f_e and f_o also satisfy the equation (1.5). Replacing f by f_o, y by $y+z$ in (1.5) and then y by $y-z$ in (1.5), separately, and then adding the resulting two relations, we have

$$\begin{aligned} &f_o(x+2y+2z)+f_o(x+2y-2z)+f_o(x-2y+2z)+f_o(x-2y-2z)+12f_o(x) \\ &\quad = 4f_o(x+y+z)+4f_o(x+y-z)+4f_o(x-y+z)+4f_o(x-y-z). \end{aligned} \tag{2.1}$$

Also, by virtue of (1.5), expanding the left hand side of (2.1), we can rewrite (2.1) in the form

$$\begin{aligned} &f_o(x+y+z)+f_o(x+y-z)+f_o(x-y+z)+f_o(x-y-z)+4f_o(x) \\ &\quad = 2f_o(x+y)+2f_o(x-y)+2f_o(x+z)+2f_o(x-z). \end{aligned} \tag{2.2}$$

Exchanging x with y in (2.2) and then adding the resulting relation to (2.2), we have

$$\begin{aligned} &f_o(x+y+z)+f_o(x+y-z)+2f_o(x)+2f_o(y) \\ &\quad = 2f_o(x+y)+f_o(x+z)+f_o(x-z)+f_o(y+z)+f_o(y-z) \end{aligned} \tag{2.3}$$

for all $x,\ y \in E_1$. Hence, by Theorem 2.3 [17], $f_o(x) = B(x,x,x) + A(x) + c$ for all $x \in E_1$, where $c = f(0)$, B is symmetric for each fixed one variable and is additive for fixed two variables, and A is additive.

In turn, since f_e satisfies the equation (1.5), we obtain

$$f_e(2x+y)+f_e(2x-y)+6f_e(y) = 4f_e(x+y)+4f_e(x-y). \tag{2.4}$$

Replacing x and y by $x+y$ and $x-y$ in (2.4), respectively, we have

$$f_e(3x+y)+f_e(x+3y)+6f_e(x-y) = 16f_e(x)+16f_e(y). \tag{2.5}$$

Putting $x+y$ instead of y in (2.4), one obtains

$$f_e(3x+y)+f_e(x-y)+6f_e(x+y) = 4f_e(2x+y)+4f_e(y). \tag{2.6}$$

Interchange x and y in (2.6) to get the relation

$$f_e(x+3y)+f_e(x-y)+6f_e(x+y) = 4f_e(2y+x)+4f_e(x). \tag{2.7}$$

Adding (2.7) to (2.6) and using (2.5), we lead to

$$12f_e(x)+12f_e(y)-4f_e(x-y)+12f_e(x+y) = 4f_e(2x+y)+4f_e(2y+x). \tag{2.8}$$

Replacing y by $-y$ in (2.8) and then adding the resulting relation to (2.8) together with (2.4), we have

$$f_e(x+y)+f_e(x-y) = 2f_e(x)+2f_e(y) \tag{2.9}$$

for all $x,\ y \in E_1$. Therefore, $f_e(x) = Q(x,x)$, where Q is a symmetric biadditive function. As a result, $f(x) = f_e(x) + f_o(x) = B(x,x,x) + Q(x,x) + A(x) + c$ for all $x \in E_1$.

Conversely, if there exist functions $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c such that $f(x) = B(x,x,x) + Q(x,x) + A(x) + c$ for all $x \in E_1$, where A is additive, Q is symmetric biadditive, and B is symmetric for fixed one variable and is additive for fixed two variables, then it is obvious that f satisfies the equation (1.5).

3. Stability of Eq. (1.5)

We now investigate the Hyers-Ulam-Rassias stability problem for the equation (1.5). Thus we find the condition that there exists a true cubic function near a approximately cubic function. From now on, let X be a real vector space and let Y be a real Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and $\mathbb{N}$ the set of all positive integers. In the following theorem, the Hyers-Ulam-Rassias stability of (1.5) is proved under the approximately cubic condition.

THEOREM 3.1. *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x)}{27^i}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{27^n} = 0$$

for all $x, y \in X \setminus \{0\}$. Suppose that a function $f : X \to Y$ satisfies

$$\|f(x+2y) + f(x-2y) + 6f(x) - 4f(x+y) - 4f(x-y)\| \leqslant \phi(x,y), \tag{3.1}$$

$$\|f(2x) + 8f(-x)\| \leqslant \delta \tag{3.2}$$

for all $x, y \in X \setminus \{0\}$ and for some $\delta \geqslant 0$. Then there exists a unique cubic function $T : X \to Y$ which satisfies the equation (1.5) and the inequality

$$\begin{aligned}\|f(x) - T(x)\| \leqslant \sum_{i=1}^{\infty} \Big[&\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\phi(3^{i-1}x, 3^{i-1}x) \\ &+ \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\phi(-3^{i-1}x, -3^{i-1}x)\Big] + \frac{2\delta + 2\|f(0)\|}{13}\end{aligned} \tag{3.3}$$

for all $x \in X$. The function T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n} \tag{3.4}$$

for all $x \in X$.

Proof. If we replace x, y by $2x, x$ in (3.1), we have

$$\|f(4x) + 6f(2x) - 4f(3x) - 4f(x)\| \leqslant \phi(2x, x) + \|f(0)\| \tag{3.5}$$

for all $x \in X \setminus \{0\}$. Replacing x, y and z by x in (3.1), we get

$$\|f(3x) + f(-x) + 6f(x) - 4f(2x)\| \leqslant \phi(x, x) + 4\|f(0)\| \tag{3.6}$$

for all $x \in X \setminus \{0\}$. Applying (3.2) to (3.5) and using (3.6), we have

$$\|f(x) + f(-x)\| \leqslant \frac{1}{84}[\phi(2x, x) + 4\phi(x, x) + 19\delta + 17\|f(0)\|] \tag{3.7}$$

for all $x \in X \setminus \{0\}$. Utilizing (3.2) and (3.6), we obtain

$$\|f(3x) + 6f(x) + 33f(-x)\| \leqslant \phi(x, x) + 4\delta + 4\|f(0)\|. \tag{3.8}$$

By substituting $-x$ for x in (3.8), we have

$$\|f(-3x) + 6f(-x) + 33f(x)\| \leqslant \phi(-x, -x) + 4\delta + 4\|f(0)\|. \tag{3.9}$$

We use induction on $n \in \mathbb{N}$ to prove our next relation:

$$\begin{aligned} &\left\| f(x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{39^n} - \frac{1}{27^n}\Big) f(3^n x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{39^n} + \frac{1}{27^n}\Big) f(-3^n x) \right\| \\ &\leqslant \sum_{i=1}^{n} \Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{39^i}\Big)\phi(3^{i-1}x, 3^{i-1}x) + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{39^i}\Big)\phi(-3^{i-1}x, -3^{i-1}x)\Big] \\ &+ \sum_{i=1}^{n} \frac{4\delta + 4\|f(0)\|}{27^i} \end{aligned} \tag{3.10}$$

for all $x \in X \setminus \{0\}$. By (3.8) and (3.9), we get

$$\begin{aligned} &\|f(x) - \frac{2}{351} f(3x) + \frac{11}{351} f(-3x)\| \\ &\leqslant \frac{2}{351}\|-f(3x) - 6f(x) - 33f(-x)\| + \frac{11}{351}\|f(-3x) + 6f(-x) + 33f(x)\| \\ &\leqslant \frac{2}{351}\phi(x, x) + \frac{11}{351}\phi(-x, -x) + \frac{4\delta + 4\|f(0)\|}{27}, \end{aligned} \tag{3.11}$$

which proves the validity of the inequality (3.10) for $n = 1$. By using (3.8), (3.9), and the following relation:

$$\begin{aligned} &f(x) + \frac{1}{2}\Big(\frac{(-1)^n}{39^{n+1}} - \frac{1}{27^{n+1}}\Big) f(3^{n+1}x) + \frac{1}{2}\Big(\frac{(-1)^n}{39^{n+1}} + \frac{1}{27^{n+1}}\Big) f(-3^{n+1}x) \\ &= f(x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{39^n} - \frac{1}{27^n}\Big) f(3^n x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{39^n} + \frac{1}{27^n}\Big) f(-3^n x) \\ &+ \frac{1}{2}\Big(\frac{(-1)^n}{39^{n+1}} - \frac{1}{27^{n+1}}\Big)[f(3^{n+1}x) + 6f(3^n x) + 33f(-3^n x)] \\ &+ \frac{1}{2}\Big(\frac{(-1)^n}{39^{n+1}} + \frac{1}{27^{n+1}}\Big)[f(-3^{n+1}x) + 6f(-3^n x) + 33f(3^n x)], \end{aligned} \tag{3.12}$$

we can easily verify the relation (3.10) for $n+1$.

It follows from (3.10) and (3.7) that

$$\begin{aligned}&\left\|\frac{f(3^n x)}{27^n}-f(x)\right\| \qquad (3.13)\\ &\leqslant\sum_{i=1}^{n}\Big[\frac{1}{2}\Big(\frac{1}{27^i}-\frac{(-1)^{i-1}}{39^i}\Big)\phi(3^{i-1}x,3^{i-1}x)+\frac{1}{2}\Big(\frac{1}{27^i}+\frac{(-1)^{i-1}}{39^i}\Big)\phi(-3^{i-1}x,-3^{i-1}x)\Big]\\ &+\sum_{i=1}^{n}\frac{4\delta+4\|f(0)\|}{27^i}+\frac{1}{168}\Big(\frac{1}{27^n}+\frac{(-1)^{n-1}}{39^n}\Big)\\ &\quad\Big[\phi(2\cdot 3^n x,3^n x)+4\phi(3^n x,3^n x)+19\delta+17\|f(0)\|\Big]\end{aligned}$$

for all $x\in X\setminus\{0\}$.

In order to prove convergence of the sequence $\{\frac{f(3^n x)}{27^n}\}$, we show that the sequence is a Cauchy sequence in Y. By (3.13), we obtain that for $n>m>0$,

$$\begin{aligned}\left\|\frac{f(3^n x)}{27^n}-\frac{f(3^m x)}{27^m}\right\| &= \frac{1}{27^m}\left\|\frac{f(3^{n-m}3^m x)}{27^{n-m}}-f(3^m x)\right\| \qquad (3.14)\\ &\leqslant \frac{1}{27^m}\sum_{i=1}^{n-m}\Big[\frac{1}{2}\Big(\frac{1}{27^i}-\frac{(-1)^{i-1}}{39^i}\Big)\phi(3^{m+i-1}x,3^{m+i-1}x)\\ &+\frac{1}{2}\Big(\frac{1}{27^i}+\frac{(-1)^{i-1}}{39^i}\Big)\phi(-3^{m+i-1}x,-3^{m+i-1}x)\Big]\\ &+\sum_{i=1}^{n-m}\frac{4\delta+2\|f(0)\|}{27^{m+i}}+\frac{1}{27^m}\frac{1}{168}\Big(\frac{1}{27^{n-m}}+\frac{(-1)^{n-m-1}}{39^{n-m}}\Big)\\ &\Big[\phi(2\cdot 3^n x,3^n x)+4\phi(3^n x,3^n x)+19\delta+17\|f(0)\|\Big].\end{aligned}$$

Since the right hand side of the inequality (3.14) tends to 0 as m tends to infinity, the sequence $\{\frac{f(3^n x)}{27^n}\}$ is a Cauchy sequence. Therefore, we may define

$$T(x)=\lim_{n\to\infty}3^{-3n}f(3^n x)$$

for all $x\in X$. By letting $n\to\infty$ in (3.13), we arrive at the formula (3.3).

To show that T satisfies the equation (1.5), replace x,y by $3^n x, 3^n y$, respectively, in (3.1) and divide by 27^n, then it follows that

$$\begin{aligned}27^{-n}\|f(3^n(x+2y))+f(3^n(x-2y))+6f(3^n x)\\ -4f(3^n(x+y))-4f(3^n(x-y))\| \leqslant 27^{-n}\phi(3^n x,3^n y).\end{aligned}$$

Taking the limit as $n\to\infty$, we find that T satisfies (1.5) for all $x,y\in X$. Obviously, it follows from (3.2) and (3.7) that $T(x)+T(-x)=0$, and $T(2x)+8T(-x)=0$. These facts and Theorem 2.1 imply that T is a cubic function.

To prove the uniqueness of the cubic function T subject to (3.3), let us assume that there exists a cubic function $S:X\to Y$ which satisfies (1.5) and the inequality

(3.3). Obviously, we have $S(3^n x) = 27^n S(x)$ and $T(3^n x) = 27^n T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.3) that

$$\begin{aligned}
\|S(x)-T(x)\| &= 27^{-n}\|S(3^n x)-T(3^n x)\| \\
&\leqslant 27^{-n}(\|S(3^n x)+f(0)-f(3^n x)\|+\|f(3^n x)-f(0)-T(3^n x)\|) \\
&\leqslant \frac{2}{27^n}\sum_{i=1}^{\infty}\Big[\frac{1}{2}\Big(\frac{1}{27^i}-\frac{(-1)^{i-1}}{39^i}\Big)\phi(3^{n+i-1}x, 3^{n+i-1}x) \\
&+ \frac{1}{2}\Big(\frac{1}{27^i}+\frac{(-1)^{i-1}}{39^i}\Big)\phi(-3^{n+i-1}x, -3^{n+i-1}x)\Big]+\frac{4\delta+4\|f(0)\|}{27^n 13}
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T. This completes the proof of the theorem.

From the main Theorem 3.1, we obtain the following corollary concerning the stability of the equation (1.5).

COROLLARY 3.2. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$, $p < 3$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned}
\|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| &\leqslant \varepsilon(\|x\|^p+\|y\|^p), \qquad (3.15) \\
\|f(2x)+8f(-x)\| &\leqslant \delta
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and for some $\delta \geqslant 0$. Then there exists a unique cubic function $T : X \to Y$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - T(x)\| \leqslant \frac{2\varepsilon\|x\|^p}{27 - 3^p} + \frac{2\delta + 2\|f(0)\|}{13} \qquad (3.16)$$

for all $x \in X$. The function T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$$

for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

Proof. The conclusion follows from Theorem 3.1. The proof of the last assertion in the above corollary goes through in the same way as that of [4].

The following corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned}
\|f(x + 2y) + f(x - 2y) + 6f(x) - 4f(x + y) - 4f(x - y)\| &\leqslant \varepsilon, \qquad (3.17) \\
\|f(2x) + 8f(-x)\| &\leqslant \delta
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$ *and for some* $\delta \geqslant 0$. *Then there exists a unique cubic function* $T : X \to Y$ *defined by* $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$ *which satisfies the equation (1.5) and the inequality*

$$\|f(x) - T(x)\| \leqslant \frac{\varepsilon + 4\delta + 4\|f(0)\|}{26} \tag{3.18}$$

for all $x \in X$. *Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* Y *is continuous, then* $T(rx) = r^3 T(x)$ *for all* $r \in \mathbb{R}$.

In the next part, we investigate the Hyers-Ulam-Rassias stability problem for the equation (1.5) under the approximately quadratic condition.

THEOREM 3.4. *Let* $\phi : X^2 \to \mathbb{R}^+$ *be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x)}{9^i}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{9^n} = 0$$

for all $x, y \in X \setminus \{0\}$. *Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| \leqslant \phi(x,y), \tag{3.19}$$

$$\|f(2x)-4f(-x)\| \leqslant \delta \tag{3.20}$$

for all $x, y \in X \setminus \{0\}$ *and for some* $\delta \geqslant 0$. *Then there exists a unique quadratic function* $Q : X \to Y$ *which satisfies the equation (1.5) and the inequality*

$$\begin{aligned} \|f(x) - Q(x)\| \leqslant & \sum_{i=1}^{\infty} \Big[\frac{1}{2}\Big(\frac{1}{9^i} - \frac{(-1)^{i-1}}{21^i}\Big)\phi(3^{i-1}x, 3^{i-1}x) \\ & + \frac{1}{2}\Big(\frac{1}{9^i} + \frac{(-1)^{i-1}}{21^i}\Big)\phi(-3^{i-1}x, -3^{i-1}x)\Big] + \frac{\delta + \|f(0)\|}{2} \end{aligned} \tag{3.21}$$

for all $x \in X$. *The function* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n} \tag{3.22}$$

for all $x \in X$.

Proof. Similarly, repeating the process from (3.5) to (3.7), we have

$$\|f(x) - f(-x)\| \leqslant \frac{1}{36}[\phi(2x,x) + 4\phi(x,x) + 15\delta + 17\|f(0)\|] \tag{3.23}$$

for all $x \in X \setminus \{0\}$. Utilizing (3.20) and (3.6), we obtain

$$\|f(3x) + 6f(x) - 15f(-x)\| \leqslant \phi(x,x) + 4\delta + 4\|f(0)\|. \tag{3.24}$$

By substituting $-x$ for x in (3.24), we have

$$\|f(-3x)+6f(-x)-15f(x)\| \leqslant \phi(-x,-x)+4\delta+4\|f(0)\|. \tag{3.25}$$

We use induction on n to obtain our next relation:

$$\begin{aligned} &\left\|f(x)+\frac{1}{2}\Big(-\frac{1}{9^n}+\frac{(-1)^{n-1}}{21^n}\Big)f(3^n x)+\frac{1}{2}\Big(-\frac{1}{9^n}-\frac{(-1)^{n-1}}{21^n}\Big)f(-3^n x)\right\| \\ &\quad\leqslant \sum_{i=1}^{n}\Big[\frac{1}{2}\Big(\frac{1}{9^i}-\frac{(-1)^{i-1}}{21^i}\Big)\phi(3^{i-1}x,3^{i-1}x) \\ &\quad+\frac{1}{2}\Big(\frac{1}{9^i}+\frac{(-1)^{i-1}}{21^i}\Big)\phi(-3^{i-1}x,-3^{i-1}x)\Big]+\sum_{i=1}^{n}\frac{4\delta+4\|f(0)\|}{9^i} \end{aligned} \tag{3.26}$$

for all $x \in X\setminus\{0\}$.

It follows from (3.26) and (3.23) that

$$\begin{aligned} \left\|f(x)-\frac{f(3^n x)}{9^n}\right\| \leqslant\ & \sum_{i=1}^{n}\Big[\frac{1}{2}\Big(\frac{1}{9^i}-\frac{(-1)^{i-1}}{21^i}\Big)\phi(3^{i-1}x,3^{i-1}x) \\ &+\ \frac{1}{2}\Big(\frac{1}{9^i}+\frac{(-1)^{i-1}}{21^i}\Big)\phi(-3^{i-1}x,-3^{i-1}x)\Big] \\ &+\ \sum_{i=1}^{n}\frac{4\delta+4\|f(0)\|}{9^i}+\frac{1}{72}\Big(\frac{1}{9^n}+\frac{(-1)^{n-1}}{21^n}\Big) \\ &\Big[\phi(2\cdot 3^n x,3^n x)+4\phi(3^n x,3^n x)+15\delta+17\|f(0)\|\Big] \end{aligned} \tag{3.27}$$

for all $x \in X\setminus\{0\}$. Now, using the same argument as that of Theorem 3.1, we obtain that there exists a unique function $Q : X \to Y$, defined by $Q(x) = \lim_{n\to\infty}\frac{f(3^n x)}{9^n}$, which satisfies the equation (1.5) and the inequality (3.21). It is clear from (3.20) and (3.23) that $Q(x)-Q(-x)=0$, and $Q(2x)-4Q(-x)=0$. These facts and Theorem 2.1 imply that the function Q is additive. This completes the proof.

From the main theorem 3.4, we obtain the following corollary concerning the stability of the equation (1.5).

COROLLARY 3.5. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$, $p<2$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} \|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| &\leqslant \varepsilon(\|x\|^p+\|y\|^p), \\ \|f(2x)-4f(-x)\| &\leqslant \delta \end{aligned} \tag{3.28}$$

for all $x,y \in X\setminus\{0\}$ and for some $\delta \geqslant 0$. Then there exists a unique quadratic function $Q : X \to Y$, given by $Q(x)=\lim_{n\to\infty}\frac{f(3^n x)}{9^n}$ which satisfies the equation (1.5) and the inequality

$$\|f(x)-Q(x)\| \leqslant \frac{2\varepsilon\|x\|^p}{9-3^p}+\frac{\delta+\|f(0)\|}{2} \tag{3.29}$$

for all $x \in X$*. Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* Y *is continuous, then* $Q(rx) = r^2Q(x)$ *for all* $r \in \mathbb{R}$*.*

The proof of the last assertion in the above corollary goes through in the same way as that of [8, 18].

The following corollary is an immediate consequence of Theorem 3.4.

COROLLARY 3.6. *Let* X *and* Y *be a real normed space and a Banach space, respectively, and let* $\varepsilon \geqslant 0$ *be a real number. Suppose that a function* $f : X \to Y$ *satisfies*

$$\begin{aligned} \|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| &\leqslant \varepsilon, \qquad (3.30)\\ \|f(2x)-4f(-x)\| &\leqslant \delta \end{aligned}$$

for all $x, y \in X\setminus\{0\}$ *and for some* $\delta \geqslant 0$*. Then there exists a unique quadratic function* $Q : X \to Y$ *defined by* $Q(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ *which satisfies the equation (1.5) and the inequality*

$$\|f(x) - Q(x)\| \leqslant \frac{\varepsilon + 4\delta + 4\|f(0)\|}{8} \tag{3.31}$$

for all $x \in X$*. Further, if for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* Y *is continuous, then* $Q(rx) = r^2Q(x)$ *for all* $r \in \mathbb{R}$*.*

In the next part, we investigate the Hyers-Ulam-Rassias stability problem for the equation (1.5) under the approximately odd condition.

THEOREM 3.7. *Let* $\phi : X^2 \to \mathbb{R}^+$ *be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x)}{3^i}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{3^n} = 0$$

for all $x, y \in X\setminus\{0\}$*. Suppose that a function* $f : X \to Y$ *satisfies*

$$\|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| \leqslant \phi(x,y), \tag{3.32}$$

$$\|f(2x)+2f(-x)\| \leqslant \delta \tag{3.33}$$

for all $x, y \in X\setminus\{0\}$ *and for some* $\delta \geqslant 0$*. Then there exists a unique additive function* $A : X \to Y$ *which satisfies the equation (1.5) and the inequality*

$$\begin{aligned} &\|f(x)-A(x)\| \qquad (3.34)\\ &\leqslant \sum_{i=1}^{\infty}\left[\frac{1}{2}\left(\frac{1}{3^i}-\frac{(-1)^{i-1}}{15^i}\right)\phi(3^{i-1}x, 3^{i-1}x)+\frac{1}{2}\left(\frac{1}{3^i}+\frac{(-1)^{i-1}}{15^i}\right)\phi(-3^{i-1}x, -3^{i-1}x)\right]\\ &\quad +2\delta+2\|f(0)\| \end{aligned}$$

for all $x \in X$. *The function* A *is given by*

$$A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{3.35}$$

for all $x \in X$.

Proof. Similarly, repeating the process from (3.5) to (3.7), we have

$$\|f(x) + f(-x)\| \leqslant \frac{1}{24}[\phi(2x,x) + 4\phi(x,x) + 13\delta + 17\|f(0)\|] \tag{3.36}$$

for all $x \in X \setminus \{0\}$. Utilizing (3.33) and (3.6), we obtain

$$\|f(3x) + 6f(x) + 9f(-x)\| \leqslant \phi(x,x) + 4\delta + 4\|f(0)\|. \tag{3.37}$$

By substituting $-x$ for x in (3.37), we have

$$\|f(-3x) + 6f(-x) + 9f(x)\| \leqslant \phi(-x,-x) + 4\delta + 4\|f(0)\|. \tag{3.38}$$

We use induction on n to obtain our next relation:

$$\begin{aligned} &\left\| f(x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{15^n} - \frac{1}{3^n}\Big) f(3^n x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{15^n} + \frac{1}{3^n}\Big) f(-3^n x) \right\| \\ &\leqslant \sum_{i=1}^{n} \Big[\frac{1}{2}\Big(\frac{1}{3^i} - \frac{(-1)^{i-1}}{15^i}\Big)\phi(3^{i-1}x, 3^{i-1}x) \\ &+ \frac{1}{2}\Big(\frac{1}{3^i} + \frac{(-1)^{i-1}}{15^i}\Big)\phi(-3^{i-1}x, -3^{i-1}x)\Big] + \sum_{i=1}^{n} \frac{4\delta + 4\|f(0)\|}{3^i} \end{aligned} \tag{3.39}$$

for all $x \in X \setminus \{0\}$.

It follows from (3.39) and (3.36) that

$$\begin{aligned} &\left\| f(x) - \frac{f(3^n x)}{3^n} \right\| \\ &\leqslant \sum_{i=1}^{n} \Big[\frac{1}{2}\Big(\frac{1}{3^i} - \frac{(-1)^{i-1}}{15^i}\Big)\phi(3^{i-1}x, 3^{i-1}x) + \frac{1}{2}\Big(\frac{1}{3^i} + \frac{(-1)^{i-1}}{15^i}\Big)\phi(-3^{i-1}x, -3^{i-1}x)\Big] \\ &+ \sum_{i=1}^{n} \frac{4\delta + 4\|f(0)\|}{3^i} + \frac{1}{48}\Big(\frac{1}{3^n} + \frac{(-1)^{n-1}}{15^n}\Big) \\ &\quad \Big[\phi(2\cdot 3^n x, 3^n x) + 4\phi(3^n x, 3^n x) + 13\delta + 17\|f(0)\|\Big] \end{aligned} \tag{3.40}$$

for all $x \in X \setminus \{0\}$. Now, using the same argument as that of Theorem 3.1, we obtain that there exists a unique function $A : X \to Y$, defined by $A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$, which satisfies the equation (1.5) and the inequality (3.34). It is clear from (3.33) and (3.36) that $A(x) + A(-x) = 0$, and $A(2x) + 2A(-x) = 0$. These facts and Theorem 2.1 imply that the function A is additive. This completes the proof.

From the main Theorem 3.7, we obtain the following corollary concerning the stability of the equation (1.5).

COROLLARY 3.8. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$, $p < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} \|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| &\leqslant \varepsilon(\|x\|^p+\|y\|^p), \quad (3.41)\\ \|f(2x)+2f(-x)\| &\leqslant \delta \end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and for some $\delta \geqslant 0$. Then there exists a unique additive function $A : X \to Y$, given by $A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{2\varepsilon\|x\|^p}{3-3^p} + 2\delta + 2\|f(0)\| \tag{3.42}$$

for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

The proof of the last assertion in the above corollary goes through in the same way as that of [8, 18].

The following corollary is an immediate consequence of Theorem 3.7.

COROLLARY 3.9. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} \|f(x+2y)+f(x-2y)+6f(x)-4f(x+y)-4f(x-y)\| &\leqslant \varepsilon, \quad (3.43)\\ \|f(2x)+2f(-x)\| &\leqslant \delta \end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and for some $\delta \geqslant 0$. Then there exists a unique additive function $A : X \to Y$ defined by $A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - A(x)\| \leqslant \frac{\varepsilon}{2} + 2\delta + 2\|f(0)\| \tag{3.44}$$

for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $\|\cdot\|$ and $||\cdot||$, respectively. A cubic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-cubic if

$$T(ax) = a^3 T(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

COROLLARY 3.10. *Let $\varepsilon \geqslant 0$, $p < 3$ be real numbers. Suppose that a function $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$\begin{aligned} ||f(\alpha x+2\alpha y)+f(\alpha x-2\alpha y)+6f(\alpha x)-4\alpha^3 f(x+y)-4\alpha^3 f(x-y)|| &\leqslant \varepsilon(\|x\|^p+\|y\|^p),\\ ||f(2\alpha x)+8\alpha^3 f(-x)|| &\leqslant \delta \end{aligned}$$

for all $\alpha \in B$ $(|\alpha| = 1)$, *for some* $\delta \geqslant 0$ *and for all* $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$, *and* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in {}_B\mathbb{B}_1$. *Then there exists a unique* B*-cubic function* $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, *defined by (3.4), which satisfies the equation (1.5) and the inequality (3.16).*

Proof. By Corollary 3.2, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique cubic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$, which satisfies the equation (1.5) and the inequality (3.16). Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, by the same reasoning as the proof of [4], the cubic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$T(tx) = t^3 T(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, T is $\mathbb{R}$-cubic. For each fixed $\alpha \in B$ $(|\alpha| = 1)$, we have $T(\alpha x) = \alpha^3 T(x)$ for all $x \in {}_B\mathbb{B}_1$. The last relation is also true for $\alpha = 0$. Since T is $\mathbb{R}$-cubic and $T(\alpha x) = \alpha^3 T(x)$ for each element $\alpha \in B(|\alpha| = 1)$, for each element $a \in B$ $(a \neq 0)$ $a = |a| \cdot \frac{a}{|a|}$ and

$$\begin{aligned} T(ax) = T\left(|a| \cdot \frac{a}{|a|} x\right) &= |a|^3 \cdot T\left(\frac{a}{|a|} x\right) = |a|^3 \cdot \frac{a^3}{|a|^3} \cdot T(x) \\ &= a^3 T(x), \qquad \forall a \in B(a \neq 0), \forall x \in {}_B\mathbb{B}_1. \end{aligned}$$

So the unique $\mathbb{R}$-cubic function $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also B-cubic, as desired. This completes the proof of the corollary.

Since $\mathbb{C}$ is a Banach algebra, the complex Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

COROLLARY 3.11. *Let* E_1 *and* E_2 *be Banach spaces over the complex field* $\mathbb{C}$, *and let* $\varepsilon \geqslant 0$ *be a real number. Suppose that a function* $f : E_1 \to E_2$ *satisfies*

$$\begin{aligned} \|f(\alpha x+2\alpha y)+f(\alpha x-2\alpha y)+6f(\alpha x)-4\alpha^3 f(x+y)-4\alpha^3 f(x-y)\| &\leqslant \varepsilon, \\ \|f(2\alpha x)+8\alpha^3 f(-x)\| &\leqslant \delta \end{aligned}$$

for all $\alpha \in \mathbb{C}$ *with* $|\alpha| = 1$, *for some* $\delta \geqslant 0$ *and for all* $x, y \in E_1 \setminus \{0\}$, *and* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E_1$. *Then there exists a unique* $\mathbb{C}$*-cubic function* $T : E_1 \to E_2$ *which satisfies the equation (1.5) and the inequality*

$$\|f(x) - T(x)\| \leqslant \frac{\varepsilon + 4\delta + 4\|f(0)\|}{26}$$

for all $x \in E_1$.

Similarly, we obtain the alternative results of Corollary 3.10 and Corollary 3.11 for the approximately quadratic (or additive) case.

REFERENCES

[1] J. ACZÉL AND J. DHOMBRES, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.
[2] J. BAKER, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80**(1980), 411–416.
[3] P. W. CHOLEWA, *Remarks on the stability of functional equations*, Aequationes Math. **27**(1984), 76–86.
[4] S. CZERWIK, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62**(1992), 59–64.
[5] DAN AMIR, *Characterizations of inner product spaces*, Dirkhäuser-Verlag, Basel, 1986.
[6] P. GĂVRUTA, *A generalization of the Hyers-Ulam-Rassias Stability of approximately additive mappings*, J. Math. Anal. Appl. **184**(1994), 431–436.
[7] A. GRABIEC, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48**(1996), 217–235.
[8] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27**(1941), 222–224.
[9] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *"Stability of Functional Equations in Several Variables"*, Birkhäuser, Basel, 1998.
[10] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *On the asymptoticity aspect of Hyers-Ulam stability of mappings*, Proc. Amer. Math. Soc. **126**(1998), 425–430.
[11] D. H. HYERS AND TH. M. RASSIAS, *Approximate homomorphisms*, Aequationes Math. **44**(1992), 125–153.
[12] P. JORDAN AND J. VON NEUMANN, *On inner products in linear metric spaces*, Ann. of Math. **36**(1935), 719–723.
[13] K. W. JUN AND Y. H. LEE, *On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality*, Math. Ineq. Appl. **4(1)**(2001), 93–118.
[14] S. - M. JUNG, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222**(1998), 126–137.
[15] S. - M. JUNG, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. **232**(1999), 384–393.
[16] PL. KANNAPPAN, *Quadratic functional equation and inner product spaces*, Results Math. **27**(1995), 368–372.
[17] H. M. KIM, *On a Cubic Functional Equation*, Kyungpook Math. J. (preprint).
[18] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(1978), 297–300.
[19] TH. M. RASSIAS, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251**(2000), 264–284.
[20] F. SKOF, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano, **53**(1983), 113–129.
[21] S. M. ULAM, *Problems in Modern Mathematics*, Chap. VI, Science ed. Wiley, New York, 1964.

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