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ON MAXIMAL FUNCTION AND FRACTIONAL INTEGRAL, ASSOCIATED WITH THE BESSEL DIFFERENTIAL OPERATOR

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Abstract. In this paper we consider the generalized shift operator, generated by Bessel differential operator B, by means of which maximal functions (B-maximal functions) and fractional integrals (B-fractional integrals) are investigated. The $L_p(B)$-boundedness result for the B-maximal function and $(L_p(B), L_q(B))$-boundedness result for the B-fractional integral are obtained.

1. Introduction

Suppose that $\mathbf{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ are vectors in $\mathbf{R}^n$, $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$, $\mathbf{R}^n_+ = \{x = (x_1, \ldots x_n);\ x_1 > 0, \ldots, x_n > 0\}$, $E_+(x, r) = \{y \in \mathbf{R}^n_+ : |x - y| < r\}$.

The Bessel differential operator $B = (B_1, \ldots, B_n)$ is defined by

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,$$

$\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_1 > 0, \ldots, \gamma_n > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_n$, $x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$, $|E_+(0, r)|_\gamma = \int_{E_+(0,r)} x^\gamma dx = Cr^{n+|\gamma|}$.

For $1 \leqslant p \leqslant \infty$ let $L_p(\mathbf{R}^n_+, B) \equiv L_p(\mathbf{R}^n_+, x^\gamma dx)$ be the space of functions measurable on $\mathbf{R}^n_+$ with the norm

$$\|f\|_{L_p(\mathbf{R}^n_+, B)} = \left(\int_{\mathbf{R}^n_+} |f(x)|^p x^\gamma dx\right)^{1/p},$$

$$\|f\|_{L_\infty(\mathbf{R}^n_+, B)} \equiv \|f\|_{L_\infty(\mathbf{R}^n_+)} = \operatorname*{ess\,sup}_{x \in \mathbf{R}^n_+} |f(x)|.$$

Denote by T^y the generalized shift operator (B-shift operator) defined by

$$T^y f(x) = \pi^{-\frac{n}{2}} \prod_{i=1}^n \Gamma\left(\gamma_i + \frac{1}{2}\right) \Gamma^{-1}(\gamma_i) \int_0^\pi \ldots \int_0^\pi \prod_{i=1}^n \sin^{\gamma_i - 1} \alpha_i \times$$

$$\times f\left(\sqrt{x_1^2 - 2x_1y_1\cos\alpha_1 + y_1^2}, \ldots, \sqrt{x_n^2 - 2x_ny_n\cos\alpha_n + y_n^2}\right) d\alpha_1 \ldots d\alpha_n.$$

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Let f be in $L_p(\mathbf{R}^n_+,B)$, $1 \leqslant p \leqslant \infty$. Then for all $x \in \mathbf{R}^n_+$, the function $T^x f$ belongs to $L_p(\mathbf{R}^n_+,B)$, and

$$\|T^x f\|_{L_p(\mathbf{R}^n_+,B)} \leqslant \|f\|_{L_p(\mathbf{R}^n_+,B)}. \tag{1}$$

We note that T^y is closely connected with the Bessel differential operator $B = (B_1,\dots,B_n)$ (see [1] for details the one-dimensional case). The B-shift T^y generates the corresponding B-convolution

$$(f*g)_B(x) = \int_{\mathbf{R}^n_+} T^y f(x) g(y) y^\gamma dy.$$

We note the following properties of the B-convolution:

$$(f*g)_B = (g*f)_B,$$
$$\|(f*g)_B\|_{L_p(\mathbf{R}^n_+,B)} \leqslant \|f\|_{L_1(\mathbf{R}^n_+,B)} \|g\|_{L_p(\mathbf{R}^n_+,B)}.$$

The Bessel (Hankel) transformations can be defined by

$$\overset{\wedge}{\varphi}(\lambda) = (F_B\varphi)(\lambda) = \int_{\mathbf{R}^n_+} \varphi(x) \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\lambda_i) x^\gamma dx,$$

and its inverse transformations can be given by

$$\overset{\vee}{\varphi}(x) = \left(F_B^{-1}\varphi\right)(x) = C_{n,\gamma} \int_{\mathbf{R}^n_+} \overset{\wedge}{\varphi}(\lambda) \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\lambda_i) \lambda^\gamma d\lambda,$$

where $j_\nu(t) = t^{-\nu} J_\nu(t)$, J_ν being the Bessel function of the first kind.

For $f \in L_p(\mathbf{R}^n_+,B)$, $p = 1$ or 2, we have $F_B(f*g)_B = F_B f F_B g$.

We shall define function spaces, generated by the Bessel differential operator $B = (B_1,\dots,B_n)$.

DEFINITION 1. [2] Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n+|\gamma|$, $[t]_1 = \min\{1,t\}$. We denote by $L_{p,\lambda}(\mathbf{R}^n_+,B)$ B-Morrey spaces and by $\widetilde{L}_{p,\lambda}(\mathbf{R}^n_+,B)$ modified B-Morrey spaces which are the sets of functions f locally integrable on $\mathbf{R}^n_+$, with finite norms

$$\|f\|_{L_{p,\lambda}(\mathbf{R}^n_+,B)} = \sup_{\mathbf{R}^n_+\times(0,\infty)} \left(t^{-\lambda} \int_{E_+(0,t)} T^y |f(x)|^p y^\gamma dy \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbf{R}^n_+,B)} = \sup_{\mathbf{R}^n_+\times(0,\infty)} \left([t]_1^{-\lambda} \int_{E_+(0,t)} T^y |f(x)|^p y^\gamma dy \right)^{1/p}.$$

DEFINITION 2. Let us now introduce, as in [2], the B-BMO space $BMO(\mathbf{R}^n_+,B)$ as the set of functions locally integrable on $\mathbf{R}^n_+$, with finite norm

$$\|f\|_{*,B} = \sup_{x,r} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} |T^y f(x) - f_{E_+(0,r)}(x)| y^\gamma dy < \infty,$$

where

$$f_{E_+(0,r)}(x) = |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y f(x) y^\gamma dy.$$

Note that

$$\widetilde{L}_{p,0}(\mathbf{R}_+^n, B) = L_{p,0}(\mathbf{R}_+^n, B) = L_p(\mathbf{R}_+^n, B),$$
$$L_{p,n+|\gamma|}(\mathbf{R}_+^n, B) = L_\infty(\mathbf{R}_+^n, B),$$

$$\widetilde{L}_{p,\lambda}(\mathbf{R}_+^n, B) \subset\!\succ L_p(\mathbf{R}_+^n, B) \quad \text{and} \quad \|f\|_{L_p(\mathbf{R}_+^n,B)} \leqslant \|f\|_{\widetilde{L}_{p,\lambda}(\mathbf{R}_+^n,B)}.$$

LEMMA 1. *Let* $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n+|\gamma|$. *Then for* $\alpha p = n+|\gamma|-\lambda$

$$L_{p,\lambda}(\mathbf{R}_+^n, B) \subset L_{1,n+|\gamma|-\alpha}(\mathbf{R}_+^n, B) \quad \textit{and} \quad \|f\|_{L_{1,n+|\gamma|-\alpha}(\mathbf{R}_+^n,B)} \leqslant C\,\|f\|_{L_{p,\lambda}(\mathbf{R}_+^n,B)}.$$

Proof. Let $f \in L_{p,\lambda}(\mathbf{R}_+^n, B)$, $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n+|\gamma|$, $1/p+1/p' = 1$ and $\alpha p = n+|\gamma|-\lambda$. By applying Hölder's inequality we have

$$\int_{E_+(0,t)} T^y|f(x)|y^\gamma dy \leqslant \left(\int_{E_+(0,t)} (T^y|f(x)|)^p y^\gamma dy\right)^{1/p} \left(\int_{E_+(0,t)} y^\gamma dy\right)^{1/p'}$$
$$\leqslant Ct^{(n+|\gamma|)/p'} \left(\int_{E_+(0,t)} T^y|f(x)|^p y^\gamma dy\right)^{1/p}.$$

Moreover

$$t^{\alpha-(n+|\gamma|)} \int_{E_+(0,t)} T^y|f(x)|y^\gamma dy \leqslant Ct^{\alpha-(n+|\gamma|)/p} \left(\int_{E_+(0,t)} T^y|f(x)|^p y^\gamma dy\right)^{1/p}$$
$$\leqslant C\left(t^{-\lambda} \int_{E_+(0,t)} T^y|f(x)|^p y^\gamma dy\right)^{1/p}$$
$$\leqslant C\,\|f\|_{L_{p,\lambda}(\mathbf{R}_+^n,B)}.$$

Therefore $f \in L_{1,n+|\gamma|-\alpha}(\mathbf{R}_+^n, B)$ and

$$\|f\|_{L_{1,n+|\gamma|-\alpha}(\mathbf{R}_+^n,B)} \leqslant C\,\|f\|_{L_{p,\lambda}(\mathbf{R}_+^n,B)}.$$

2. $L_p(B)$-boundedness of B-maximal function

We now consider the B-maximal operator (see [2], [3])

$$M_B f(x) = \sup_{r>0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y|f(x)|y^\gamma dy.$$

THEOREM 1. *1. If $f \in L_1(\mathbf{R}^n_+, B)$, then for every $\alpha > 0$*

$$|\{x : M_B f(x) > \alpha\}|_\gamma \leqslant \frac{C}{\alpha}\int_{\mathbf{R}^n_+} |f(x)| x^\gamma dx,$$

where $C > 0$ is independent of f.

2. If $f \in L_p(\mathbf{R}^n_+, B)$, $\quad 1 < p \leqslant \infty$, then $M_B f \in L_p(\mathbf{R}^n_+, B)$ and

$$\|M_B f\|_{L_p(\mathbf{R}^n_+, B)} \leqslant C_p \|f\|_{L_p(\mathbf{R}^n_+, B)},$$

where $C_p > 0$ is independent of f.

Proof. We need to introduce a maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudo-metric ρ and a positive measure μ satisfying the doubling condition

$$\mu(E(x, 2r)) \leqslant C\mu(E(x, r)), \tag{2}$$

where C is independent of x and $r > 0$. Here $E(x, r) = \{y \in X : \rho(x, y) < r\}$. Let (X, ρ, μ) be a space of homogeneous type. Define

$$M_\mu f(x) = \sup_{r>0} \mu(E(x, r))^{-1} \int_{E(x,r)} |f(y)| d\mu(y).$$

It is well known that the maximal operator M_μ is of weak type $(1, 1)$ and is bounded on $L_p(X, d\mu)$ for $1 < p < \infty$ (see [4]). We shall use this result in the case in which $X = \mathbf{R}^n_+$, $\rho(x, y) = |x - y|$, $d\mu(x) = x^\gamma dx$. It is clear that this measure satisfies the doubling condition (2).

Also

$$\begin{aligned}
\mu E(x, r) = |E_+(x, r)|_\gamma &= \int_{\{y \in \mathbf{R}^n_+ : |x-y| < r\}} y^\gamma dy \\
&\leqslant \prod_{i=1}^n \int_{\{y_i > 0;\ |x_i - y_i| < r\}} y_i^{\gamma} dy_i = \prod_{i=1}^n \int_{\max\{0, x_i - r\}}^{x_i + r} y_i^{\gamma} dy_i \\
&\leqslant C \prod_{i=1}^n \begin{cases} r^{1+\gamma_i}, & r > x_i \\ r x_i^{\gamma_i}, & r \leqslant x_i \end{cases} = C r^{n+|\gamma|} \prod_{i=1}^n \max\{1, (x_i/r)^{\gamma_i}\}.
\end{aligned}$$

We shall show that

$$M_B f(x) \leqslant C M_\mu f(x).$$

By the definition of the B-shift operator it follows that $T^y \chi_{E_+(0,r)}(x)$ is supported in $E_+(x, r)$.

Next we estimate $T^y \chi_{E_+(0,r)}(x)$.

$$T^y \chi_{E_+(0,r)}(x) \leqslant C \int\limits_{\{(\alpha_1, \ldots, \alpha_n) \in (0,\pi)^n :\ \sum_{i=1}^n (x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2) < r^2\}} \!\!\!\!\!\!\int \sin^{\gamma_i - 1} \alpha_i \, d\alpha_1 \ldots d\alpha_n$$

$$\leqslant C\prod_{i=1}^{n}\int\limits_{\{\alpha_i\in(0,\pi):x_i^2-2x_iy_i\cos\alpha_i+y_i^2<r^2\}}\sin^{\gamma_i-1}\alpha_i\,d\alpha_i$$

$$=C\prod_{i=1}^{n}\int\limits_{\left\{\alpha_i\in(0,\pi):\frac{x_i^2+y_i^2-r^2}{2x_iy_i}<\cos\alpha_i\right\}}\sin^{\gamma_i-2}\alpha_i\,d\cos\alpha_i$$

$$=C\prod_{i=1}^{n}\int\limits_{\max\left\{-1,\frac{x_i^2+y_i^2-r^2}{2x_iy_i}\right\}}^{1}\left(1-t_i^2\right)^{\frac{\gamma_i}{2}-1}dt_i$$

$$\leqslant C\prod_{i=1}^{n}\min\left\{1,\frac{r^{\gamma_i/2}(r-|x_i-y_i|)^{\gamma_i/2}}{(x_iy_i)^{\gamma_i/2}}\right\}\leqslant C\prod_{i=1}^{n}\min\{1,(r/x_i)^{-\gamma_i}\}.$$

Consequently there exists $C>0$ such that for all $x\in\mathbf{R}_+^n,\ r>0$ and $y\in E_+(x,r)$

$$T^y\chi_{E_+(0,r)}(x)\leqslant C\prod_{i=1}^{n}\min\{1,(r/x_i)^{\gamma_i}\}.$$

Thus

$$M_Bf(x)\leqslant\sum_{k=0}^{n}M_{B,k}f(x)$$

$$=\sum_{k=0}^{n}\sup_{\substack{r>x_{i_j},\,j=\overline{1,k}\\ r\leqslant x_{i_j},\,j=\overline{k+1,n}\\ i_j\neq i_p,\,j\neq p}}|E_+(0,r)|_\gamma^{-1}\int_{E_+(x,r)}|f(y)|T^y\chi_{E_+(0,r)}(x)y^\gamma\,dy,$$

where

$$M_{B,0}f(x)=\sup_{r\leqslant x_j,\,j=\overline{1,n}}|E_+(0,r)|_\gamma^{-1}\int_{E_+(x,r)}|f(y)|T^y\chi_{E_+(0,r)}(x)y^\gamma\,dy,$$

$$M_{B,n}f(x)=\sup_{r>x_j,\,j=\overline{1,n}}|E_+(0,r)|_\gamma^{-1}\int_{E_+(x,r)}|f(y)|T^y\chi_{E_+(0,r)}(x)y^\gamma\,dy.$$

Without loss of generality we assume that $i_j\equiv j,\ j=1,\dots,n.$ Then

$$M_{B,k}f(x)=\sup_{\substack{r>x_j,\,j=\overline{1,k}\\ r\leqslant x_j,\,j=\overline{k+1,n}}}|E_+(0,r)|_\gamma^{-1}\int_{E_+(x,r)}|f(y)|T^y\chi_{E_+(0,r)}(x)y^\gamma\,dy.$$

In the case $k=0,$ by taking into account that $\mu E_+(x,r)\leqslant Cr^{n+|\gamma|}$, $|E_+(0,r)|_\gamma=r^{n+|\gamma|}$ and $T^y\chi_{E_+(0,r)}\leqslant 1$, we have

$$M_{B,0}f(x)\leqslant\sup_{r\leqslant x_j,\,j=\overline{1,n}}\frac{1}{\mu E_+(x,r)}\int_{E_+(x,r)}|f(y)|y^\gamma\,dy\leqslant CM_\mu f(x).$$

In the case $1 \leqslant k \leqslant n$, by taking into account

$$\mu E_+(x,r) \leqslant C r^{n+|\gamma|} \prod_{i=1}^{n} \max\{1, (x_i/r)^{\gamma_i}\} = C r^{n+|\gamma|} \prod_{i=1}^{k} (x_i/r)^{\gamma_i},$$

$$T^y \chi_{E_+(0,r)} \leqslant C \prod_{i=1}^{n} \min\{1, (r/x_i)^{\gamma_i}\} = \prod_{i=1}^{k} (r/x_i)^{\gamma_i},$$

we have

$$M_{B,k} f(x) \leqslant C \sup_{\substack{r > x_j,\, j=\overline{1,k} \\ r \leqslant x_j,\, j=\overline{k+1,n}}} |E_+(0,r)|_\gamma^{-1} r^{n+|\gamma|} \prod_{i=1}^{k} (x_i/r)^{\gamma_i} \prod_{i=1}^{k} (r/x_i)^{\gamma_i} \times$$
$$\times \frac{1}{\mu E_+(x,r)} \int_{E_+(x,r)} |f(y)| y^\gamma dy \leqslant C M_\mu f(x).$$

Finally we get

$$M_B f(x) \leqslant C M_\mu f(x).$$

This completes the proof.

COROLLARY 1. *If* $f \in L_p(\mathbf{R}_+^n, B)$, $1 \leqslant p \leqslant \infty$, *then*

$$\lim_{r \to 0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y f(x) y^\gamma dy = f(x)$$

for a. e. $x \in \mathbf{R}_+^n$.

REMARK 1. In the one-dimensional case Theorem 1 was proved earlier by another method by K. Stempak [5].

3. Sobolev – type theorem for the B-fractional integral

Consider the B-Riesz potentials

$$I_B^\alpha f(x) = \int_{\mathbf{R}_+^n} T^y |x|^{\alpha-n-|\gamma|} f(y) y^\gamma dy, \quad 0 < \alpha < n + |\gamma|,$$

and the modified of B-Riesz potentials

$$\widetilde{I}_B^\alpha f(x) = \int_{\mathbf{R}_+^n} \left(T^y |x|^{\alpha-n-|\gamma|} - |y|^{\alpha-n-|\gamma|} \chi_{E_+^*(0,1)}(y) \right) f(y) y^\gamma dy,$$

where $E_+^*(0,r) = \mathbf{R}_+^n \setminus E_+(0,r)$, $r > 0$.

The examples considered below show that if $p \geqslant \frac{n+|\gamma|}{\alpha}$, then the B-potentials I_B^α are not defined for all functions $f \in L_p(\mathbf{R}_+^n, B)$.

EXAMPLE 1. Let $x \in \mathbf{R}_+^n$, $0 < \alpha < n + |\gamma|$, $f(x) = \frac{1}{|x|^\alpha \ln |x|} \chi_{E_+^*(0,2)}(x)$. For $p = \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$ and $I_B^\alpha f(x) = +\infty$.

EXAMPLE 2. Let $x \in \mathbf{R}_+^n$, $0 < \alpha < n + |\gamma|$, $f(x) = |x|^{-\alpha} \chi_{E_+^*(0,2)}(x)$. For $p > \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$ and $I_B^\alpha f(x) = +\infty$.

For these B-Riesz potentials the following analogue of Hardy–Littlewood–Sobolev theorem is valid.

THEOREM 2. *Let* $0 < \alpha < n + |\gamma|$, $1 \leqslant p < \frac{n+|\gamma|}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$.

a) If $f \in L_p(\mathbf{R}_+^n, B)$, *then the integral* $I_B^\alpha f$ *is absolutely convergent for almost every* $x \in \mathbf{R}_+^n$.

b) If $1 < p < \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$, *then* $I_{B_n}^\alpha f \in L_q(\mathbf{R}_+^n, B)$ *and*

$$\|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n, B)} \leqslant C_p \|f\|_{L_p(\mathbf{R}_+^n, B)}, \tag{3}$$

where $C_p > 0$ *is independent of* f.

c) If $f \in L_1(\mathbf{R}_+^n, B)$, $\frac{1}{q} = 1 - \frac{\alpha}{n+|\gamma|}$, *then*

$$|\{x \in \mathbf{R}_+^n : I_B^\alpha f(x) > \beta\}|_\gamma \leqslant \left(\frac{C}{\beta} \|f\|_{L_1(\mathbf{R}_+^n, B)} \right)^q, \qquad \beta > 0, \tag{4}$$

where $C > 0$ *is independent of* f.

Proof. a) Let $f \in L_p(\mathbf{R}_+^n, B)$, $1 \leqslant p < \frac{n+|\gamma|}{\alpha}$. We denote

$$f_1(x) = f(x) \chi_{E_+(0,1)}(x), \quad f_2(x) = f(x) - f_1(x).$$

Then

$$I_B^\alpha f(x) = I_B^\alpha f_1(x) + I_B^\alpha f_2(x) = J_1(x) + J_2(x).$$

Let us estimate $J_1(x)$.

$$\begin{aligned} |J_1(x)| &\leqslant \int_{E_+(0,1)} |y|^{\alpha - n - |\gamma|} T^y |f(x)| y^\gamma dy \\ &= \int_{\mathbf{R}_+^n} |y|^{\alpha - n - |\gamma|} \chi_{E_+(0,1)}(y) T^y |f(x)| y^\gamma dy. \end{aligned}$$

By Young's inequality

$$\|J_1(\cdot)\|_{L_p(\mathbf{R}_+^n, B)} \leqslant C \left\| |\cdot|^{\alpha - n - |\gamma|} \chi_{E_+(0,1)} \right\|_{L_1(\mathbf{R}_+^n, B)} \|f\|_{L_p(\mathbf{R}_+^n, B)}.$$

Here

$$\begin{aligned} \left\| |\cdot|^{\alpha - n - |\gamma|} \chi_{E_+(0,1)} \right\|_{L_1(\mathbf{R}_+^n, B)} &= \int_{E_+(0,1)} |y|^{\alpha - n - |\gamma|} y^\gamma dy \\ &= C \int_0^1 r^{\alpha - 1} dr = C_1 < \infty. \end{aligned}$$

Then for $f \in L_p(\mathbf{R}_+^n, B)$, $1 \leqslant p \leqslant \infty$

$$\|J_1(\cdot)\|_{L_p(\mathbf{R}_+^n,B)} \leqslant C_1 \|f\|_{L_p(\mathbf{R}_+^n,B)},$$

e.g. $J_1(x)$ is absolutely convergent almost every $x \in \mathbf{R}_+^n$.

By Hölder's inequality we have

$$|J_2(x)| \leqslant \int\limits_{\mathbf{R}_+^n \setminus E_+(0,1)} |y|^{\alpha-n-|\gamma|} T^y |f(x)| y^\gamma dy$$

$$\leqslant \|T^x|f(\cdot)|\|_{L_p(\mathbf{R}_+^n,B)} \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,1)} |y|^{(\alpha-n-|\gamma|)p'} \right)^{\frac{1}{p'}}.$$

By using inequality (1)

$$|J_2(x)| \leqslant C\|f\|_{L_p(\mathbf{R}_+^n,B)} \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,1)} |y|^{(\alpha-n-|\gamma|)p'} \right)^{\frac{1}{p'}}.$$

Hence for $1 \leqslant p < \frac{n+|\gamma|}{\alpha}$ for some $C > 0$

$$|J_2(x)| \leqslant C\|f\|_{L_p(\mathbf{R}_+^n,B)}, \qquad x \in \mathbf{R}_+^n.$$

Thus for all functions $f \in L_p(\mathbf{R}_+^n, B)$, $1 \leqslant p < \frac{n+|\gamma|}{\alpha}$ B-Riesz potentials $I_B^\alpha f(x)$ are absolutely convergent for almost every $x \in \mathbf{R}_+^n$.

b) We have

$$I_B^\alpha f\,(x) = \left(\int\limits_{E_+(0,t)} + \int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} \right) T^y f(x) |y|^{\alpha-n-|\gamma|} y^\gamma dy = A(x,t) + C(x,t).$$

By taking sum with respect to all integer $k < 0$, we get

$$|A(x,t)| \leqslant \int\limits_{E_+(0,t)} |T^y f(x)|\, |y|^{\alpha-n-|\gamma|} y^\gamma dy$$

$$= \sum_{k=-\infty}^{-1} \int\limits_{2^k t \leqslant |y| < 2^{k+1} t} |T^y f(x)|\, |y|^{\alpha-n-|\gamma|} y^\gamma dy$$

$$\leqslant C \sum_{k=-\infty}^{-1} \left(2^k t\right)^{\alpha-n-|\gamma|} \int\limits_{2^k t \leqslant |y| < 2^{k+1} t} |T^y f(x)|\, y^\gamma dy$$

$$\leqslant C t^\alpha \left(M_B f\right)(x).$$

Therefore it follows that

$$|A(x,t)| \leqslant Ct^{\alpha} M_B f(x), \tag{5}$$

where $C>0$ does not depend f, x and t.

By Hölder's inequality and the inequality (1) we have

$$|C(x,t)| \leqslant \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |T^y f(x)|^p \, y^{\gamma} dy \right)^{\frac{1}{p}} \times \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |y|^{(\alpha-n-|\gamma|)p'} y^{\gamma} dy \right)^{\frac{1}{p'}}$$

$$\leqslant \|T^y f\|_{L_p(\mathbf{R}_+^n,B)} \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |y|^{(\alpha-n-|\gamma|)p'} y^{\gamma} dy \right)^{\frac{1}{p'}}$$

$$\leqslant \|f\|_{L_p(\mathbf{R}_+^n,B)} \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |y|^{(\alpha-n-|\gamma|)p'} y^{\gamma} dy \right)^{\frac{1}{p'}} \leqslant Ct^{-(n+|\gamma|)/q} \|f\|_{L_p(\mathbf{R}_+^n,B)}.$$

Consequently

$$|C(x,t)| \leqslant Ct^{-(n+|\gamma|)/q} \|f\|_{L_p(\mathbf{R}_+^n,B)}. \tag{6}$$

Thus, from (5) and (6), we have

$$|I_B^{\alpha} f(x)| \leqslant C \left(t^{\alpha} M_B f(x) + t^{-(n+|\gamma|)/q} \|f\|_{L_p(\mathbf{R}_+^n,B)} \right).$$

Minimum of the right-hand side is attained at $t = \left[(M_B f(x))^{-1} \|f\|_{L_p(\mathbf{R}_+^n,B)} \right]^{p/(n+|\gamma|)}$, so

$$|I_B^{\alpha} f(x)| \leqslant C \left(M_B f(x) \right)^{p/q} \|f\|_{L_p(\mathbf{R}_+^n,B)}^{1-p/q}.$$

Hence, by the Theorem 1, we have

$$\int_{E_+(0,t)} |I_B^{\alpha} f(y)|^q \, y^{\gamma} dy \leqslant C \|f\|_{L_p(\mathbf{R}_+^n,B)}^{q-p} \int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} (M_B f(y))^p \, y^{\gamma} dy \leqslant C \|f\|_{L_p(\mathbf{R}_+^n,B)}^q,$$

and (3) follows.

c) Let $f \in L_1(\mathbf{R}_+^n, B)$. Note that

$$|\{x : |I_B^{\alpha} f(x)| > 2\beta\}|_{\gamma} \leqslant |\{x : |A(x,t)| > \beta\}|_{\gamma} + |\{x : |C(x,t)| > \beta\}|_{\gamma}.$$

By taking into account the inequality (6) and applying Theorem 1 we have

$$\beta \, |\{x \in \mathbf{R}_+^n : |A(x,t)| > \beta\}|_{\gamma} = \beta \int\limits_{\{x \in \mathbf{R}_+^n :\ |A(x,t)| > \beta\}} x^{\gamma} dx \leqslant \beta \int\limits_{\{x \in \mathbf{R}_+^n :\ Ct^{\alpha} M_B f(x) > \beta\}} x^{\gamma} dx$$

$$= \beta \left| \left\{ x \in \mathbf{R}_+^n :\ M_B f(x) > \frac{\beta}{Ct^{\alpha}} \right\} \right|_{\gamma}$$

$$\leqslant \beta \cdot \frac{C_1 t^{\alpha}}{\beta} \int_{\mathbf{R}_+^n} |f(x)| x^{\gamma} dx = C_1 t^{\alpha} \|f\|_{L_1(\mathbf{R}_+^n,B)}$$

and also

$$
\begin{aligned}
|C(x,t)| &\leqslant \int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |T^y f(x)|\, |y|^{\alpha-n-|\gamma|} y^\gamma dy \\
&\leqslant t^{\alpha-n-|\gamma|} \int\limits_{\mathbf{R}_+^n \setminus E_+(0,t)} |T^y f(x)|\, y^\gamma dy \\
&= t^{-\frac{n+|\gamma|}{q}} \int\limits_{\mathbf{R}_+^n} |f(x)| x^\gamma dx = t^{-\frac{n+|\gamma|}{q}} \|f\|_{L_1(\mathbf{R}_+^n,B)} .
\end{aligned}
$$

Thus, if $t^{-\frac{n+|\gamma|}{q}} \|f\|_{L_1(\mathbf{R}_+^n,B)} = \beta$, then $|C(x,t)| \leqslant \beta$ and, consequently,

$$
|\{x \in \mathbf{R}_+^n :\ |C(x,t)| > \beta\}|_\gamma = 0.
$$

Finally

$$
\begin{aligned}
|\{x \in \mathbf{R}_+^n :\ |I_B^\alpha f(x)| > 2\beta\}|_\gamma &\leqslant \frac{C_1}{\beta} t^\alpha \|f\|_{L_1(\mathbf{R}_+^n,B)} = C_1 t^{\alpha+\frac{n+|\gamma|}{q}} \\
&= C_1 t^{n+|\gamma|} = C_1 \beta^{-q} \|f\|^q_{L_1(\mathbf{R}_+^n,B)} = \frac{C}{\beta} \|f\|^q_{L_1(\mathbf{R}_+^n)} .
\end{aligned}
$$

Therefore the mapping $f \to R_B^\alpha f$ is of weak type $(1,q)$.
Theorem has been proved.

THEOREM 3. *Let* $0 < \alpha < n+|\gamma|$, $1 < p < \frac{n+|\gamma|}{\alpha}$, *then the condition*

$$
\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}
$$

is necessary for inequality (3) to be valid.

Proof. Let $1 < p < \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$ and assume that the inequality

$$
\|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n,B)} \leqslant C_p \|f\|_{L_p(\mathbf{R}_+^n,B)}
$$

holds.

Define $f_t(x) := f(t^a x)$, then

$$
\begin{aligned}
\|f_t\|_{L_p(\mathbf{R}_+^n,B)} &= \left(\int_{\mathbf{R}_+^n} |f(tx)|^p x^\gamma dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbf{R}_+^n} |f(y)|^p t^{-n-|\gamma|} y^\gamma dy \right)^{\frac{1}{p}} = t^{-\frac{n+|\gamma|}{p}} \|f\|_{L_p(\mathbf{R}_+^n,B)}
\end{aligned}
$$

and

$$
\|I_B^\alpha f_t\|_{L_q(\mathbf{R}_+^n,B)} = \left\| \int_{\mathbf{R}_+^n} f_t(y) T^y |x|^{\alpha-n-|\gamma|} y^\gamma dy \right\|_{L_q(\mathbf{R}_+^n,B)}
$$

$$= \left\| \int_{\mathbf{R}_+^n} T^y |x|^{\alpha-n-|\gamma|} f(ty) y^\gamma dy \right\|_{L_q(\mathbf{R}_+^n, B)}$$

$$= \left(\int_{\mathbf{R}_+^n} \left| \int_{\mathbf{R}_+^n} T^{ty} |x|^{\alpha-n-|\gamma|} f(y) t^{-n-|\gamma|} y^\gamma dy \right|^q x^\gamma dx \right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbf{R}_+^n} \left| \int \mathbf{R}_+^n t^{n+|\gamma|-\alpha} T^y |tx|^{\alpha-n-|\gamma|} f(y) t^{-n-|\gamma|} y^\gamma dy \right|^q x^\gamma dx \right)^{\frac{1}{q}}$$

$$= t^{-\alpha-\frac{n+|\gamma|}{q}} \|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n, B)}.$$

By inequality (3)

$$\|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n)} \leqslant C_{p,q} t^{\alpha+\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}} \|f\|_{L_p(\mathbf{R}_+^n, B)}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then in the case $t \to 0$ we have $\|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n, B)} = 0$, for all $f \in L_p(\mathbf{R}_+^n, B)$, which is impossible. Similarly, if $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then at $t \to \infty$ we obtain $\|I_B^\alpha f\|_{L_q(\mathbf{R}_+^n, B)} = 0$, for all $f \in L_p(\mathbf{R}_+^n, B)$, which is also impossible.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$.

THEOREM 4. *Let* $0 < \alpha < n+|\gamma|$, $p = \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$. *Then* $\widetilde{I}_B^\alpha f \in BMO(\mathbf{R}_+^n, B)$ *and*

$$\left\| \widetilde{I}_B^\alpha f \right\|_{BMO(\mathbf{R}_+^n, B)} \leqslant C_p \|f\|_{L_p(\mathbf{R}_+^n, B)}.$$

Proof. Let $f \in L_p(\mathbf{R}_+^n, B)$. Given $t > 0$ we denote

$$f_1(x) = f(x) \chi_{E_+(0,2t)}(x), \quad f_2(x) = f(x) - f_1(x),$$

where $\chi_{E_+(0,2t)}$ is the characteristic function of the set $E_+(0,2t)$. Then

$$\widetilde{I}_B^\alpha f(x) = \widetilde{I}_B^\alpha f_1(x) + \widetilde{I}_B^\alpha f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int_{E_+(0,2t)} \left(T^y |x|^{\alpha-n-|\gamma|} - |y|^{\alpha-n-|\gamma|} \chi_{E_+^*(0,1)}(y) \right) f(y) y^\gamma dy,$$

$$F_2(x) = \int_{\mathbf{R}_+^n \setminus E_+(0,2t)} \left(T^y |x|^{\alpha-n-|\gamma|} - |y|^{\alpha-n-|\gamma|} \chi_{E_+^*(0,1)}(y) \right) f(y) y^\gamma dy.$$

Note that the function f_1 has compact support and thus

$$a_1 = - \int_{E_+(0,2t) \setminus E_+(0,\min\{1,2t\})} |y|^{\alpha-n-|\gamma|} f(y) y^\gamma dy$$

is finite.

Note also that

$$F_1(x)-a_1=\int_{E_+(0,2t)} T^y|x|^{\alpha-n-|\gamma|}f(y)y^\gamma dy$$
$$-\int_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})} |y|^{\alpha-n-|\gamma|}f(y)y^\gamma dy$$
$$+\int_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})} |y|^{\alpha-n-|\gamma|}f(y)y^\gamma dy$$
$$=\int_{\mathbf{R}^n_+} T^y|x|^{\alpha-n-|\gamma|}f_1(y)y^\gamma dy=I^\alpha_B f_1(x).$$

Therefore

$$|F_1(x)-a_1|\leqslant\int_{\mathbf{R}^n_+}|y|^{\alpha-n-|\gamma|}\,|T^y f_1(x)|\,y_n^\gamma dy$$
$$=\int_{\{y\in\mathbf{R}^n_+:\ T^y|x|<2t\}}|y|^{\alpha-n-|\gamma|}\,|T^y f(x)|\,y^\gamma dy.$$

Furthermore, for $|x|<t$ and $T^y|x|<2t$ we have

$$|y|\leqslant|x|+|x-y|\leqslant|x|+T^y|x|<3t.$$

Consequently

$$|F_1(x)-a_1|\leqslant\int_{E_+(0,3t)}|y|^{\alpha-n-|\gamma|}\,|T^y f(x)|\,y^\gamma dy, \qquad (7)$$

if $x\in E_+(0,t)$.

By Theorem 1, (1) and (7) for $\alpha p=n+|\gamma|$

$$|E_+(0,t)|_\gamma^{-1}\int_{E_+(0,t)}|T^z F_1(x)-a_1|\,z^\gamma dz$$
$$\leqslant|E_+(0,t)|_\gamma^{-1}\int_{E_+(0,t)}T^z\,|F_1(x)-a_1|\,z^\gamma dz$$
$$\leqslant|E_+(0,t)|_\gamma^{-1}\int_{E_+(0,t)}\left(\int_{E_+(0,3t)}|y|^{\alpha-n-|\gamma|}T^yT^z|f(x)|y^\gamma dy\right)z^\gamma dz$$
$$\leqslant Ct^{-n-|\gamma|}\cdot t^\alpha\cdot t^{(n+|\gamma|)/p'}\left(\int_{E_+(0,t)}(M_B(T^z f(x)))^p\,z^\gamma dz\right)^{1/p}$$
$$\leqslant C_p\,\|T^z f\|_{L_p(\mathbf{R}^n_+,B)}\leqslant C_p\|f\|_{L_p(\mathbf{R}^n_+,B)}. \qquad (8)$$

Denote

$$a_2=\int_{E_+(0,\max\{1,2t\})\setminus E_+(0,2t)}|y|^{\alpha-n-|\gamma|}f(y)y^\gamma dy.$$

Let us estimate $|F_2(x) - a_2|$. By applying Hölder's inequality we have

$$\begin{aligned} |F_2(x) - a_2| &\leqslant \int\limits_{\mathbf{R}_+^n \setminus E_+(0,2t)} |f(y)| \left| T^y |x|^{\alpha-n-|\gamma|} - |y|^{\alpha-n-|\gamma|} \right| y^\gamma dy \\ &\leqslant C|x| \int\limits_{\mathbf{R}_+^n \setminus E_+(0,2t)} |f(y)| |y|^{\alpha-n-|\gamma|-1} y^\gamma dy \\ &\leqslant C|x| \|f\|_{L_p(\mathbf{R}_+^n, B)} \left(\int\limits_{\mathbf{R}_+^n \setminus E_+(0,2t)} |y|^{(\alpha-n-|\gamma|-1)p'} y^\gamma dy \right)^{1/p'} \\ &\leqslant C|x| t^{\alpha-1-\frac{n+|\gamma|}{p}} \|f\|_{L_p(\mathbf{R}_+^n, B)} \leqslant C|x| t^{-1} \|f\|_{L_p(\mathbf{R}_+^n, B)}. \end{aligned}$$

Note that if $|x| \leqslant t$, $|z| \leqslant 2t$, then $T^z|x| \leqslant |x| + |z| \leqslant 3t$. Thus for $\alpha p = n + |\gamma|$ we obtain

$$|T^z F_2(x) - a_2| \leqslant T^z |F_2(x) - a_2| \leqslant C T^z |x| t^{-1} \|f\|_{L_p(\mathbf{R}_+^n, B)} \leqslant C \|f\|_{L_p(\mathbf{R}_+^n, B)}. \qquad (9)$$

Denote

$$a_f = a_1 + a_2 = \int\limits_{E_+(0, \max\{1, 2t\})} |y|^{\alpha-n-|\gamma|} f(y) y^\gamma dy.$$

Finally, by (8) and (9) we have

$$\sup_{x,t} |E_+(0,t)|_\gamma^{-1} \int_{E_+(0,t)} \left| T^y \widetilde{I}_B^\alpha f(x) - a_f \right| y^\gamma dy \leqslant C \|f\|_{L_p(\mathbf{R}_+^n, B)}.$$

Finally

$$\begin{aligned} \left\| \widetilde{I}_B^\alpha f \right\|_{BMO(\mathbf{R}_+^n, B)} &\leqslant 2 \sup_{x,t} |E_+(0,t)|_\gamma^{-1} \int_{E_+(0,t)} \left| T^y \widetilde{I}_B^\alpha f(x) - a_f \right| y^\gamma dy \\ &\leqslant C \|f\|_{L_p(\mathbf{R}_+^n, B)}. \end{aligned}$$

and the statement of the theorem follows.

COROLLARY 2. *Let* $p = \frac{n+|\gamma|}{\alpha}$, $f \in L_p(\mathbf{R}_+^n, B)$.
If integral $I_B^\alpha f$ *exists everywhere, then* $I_B^\alpha f \in BMO(\mathbf{R}_+^n, B)$ *and the inequality*

$$\|I_B^\alpha f\|_{BMO(\mathbf{R}_+^n, B)} \leqslant C_p \|f\|_{L_p(\mathbf{R}_+^n, B)}$$

is valid.

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