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GENERALIZED NONLINEAR QUASI-VARIATIONAL-LIKE INEQUALITIES FOR SET-VALUED MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new class of generalized nonlinear quasi-variational-like inequalities for set-valed mappings in Banach spaces. Using the KKM technique, we prove the existence and uniqueness of solution for this class of generalized nonlinear quasi-variational-like inequalities for set-valued mappings in Banach spaces. Our results extend and improve some main results of Verma.

1. Introduction and preliminaries

Variational inequalities not only have stimulated the new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. In recent years, variational inequalities have been generalized and applied in various directions. For details, we refer to [1]–[9] and the references therein.

Recently, Huang, Fang and Cho [5] introduced and studied a new class of generalized nonlinear mixed quasi-variational inequalities for single-valued mappings, which includes the variational inequalities considered by Verma [7], [8] as special cases.

In this paper, we introduce and study a new class of generalized nonlinear quasi-variational-like inequalities for set-valued mappings in Banach spaces. Using the KKM technique, we prove the existence and uniqueness of solution for this class of generalized nonlinear quasi-variational-like inequalities for set-valued mappings in Banach spaces. Our results extend and improve some main results of Verma [7], [8].

Throughout this paper, let X be a real reflexive Banach space, X^* be its dual space, and K be a nonempty convex closed subset of X. Denote $\langle \omega, x\rangle = \omega(x)$ for all $\omega \in X^*$ and $x \in X$. Let $S, T : K \to 2^{X^*}$ be two set-valued mappings, $\eta : K \times K \to K$ be a single-valued mapping and $f : K \to R \cup \{+\infty\}$ be a proper convex functional.

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Suppose that η is affine with respect to the first argument satisfying $\eta(u,v) = -\eta(v,u)$ for all $u,v \in K$. It is clear that $\eta(u,u) = 0$ for all $u \in K$. We consider the following problem:

For any $\omega \in X^*$, find $u \in K$ such that

$$\sup_{x \in Su, w \in Tu} \langle x - w - \omega, \eta(v,u) \rangle + f(v) - f(u) \geqslant 0 \tag{1.1}$$

for all $v \in K$.

If $S,T : K \to X^*$ is two single-valed mappings, then the problem (1.1) is equivalent to the following problem:

For any $\omega \in X^*$, find $u \in K$ such that

$$\langle Su - Tu - \omega, \eta(v,u) \rangle + f(v) - f(u) \geqslant 0 \tag{1.2}$$

for all $v \in K$.

If $\eta(u,v) = gu - gv$, then the problem (1.1) is equivalent to the following problem:

For any $\omega \in X^*$, find $u \in K$ such that

$$\sup_{x \in Su, w \in Tu} \langle x - w - \omega, gv - gu \rangle + f(v) - f(u) \geqslant 0 \tag{1.3}$$

for all $v \in K$, where $g : K \to K$ is an affine mapping.

If $\eta(u,v) = u - v$, then the problem (1.1) is equivalent to the following problem:

For any $\omega \in X^*$, find $u \in K$ such that

$$\sup_{x \in Su, w \in Tu} \langle x - w - \omega, v - u \rangle + f(v) - f(u) \geqslant 0 \tag{1.4}$$

for all $v \in K$.

REMARK 1.1. For a suitable choice of S, T, η, and f, the problem (1.1) includes many kinds of known variational inequalities as special cases (see [5], [7], [8] and the references therein).

DEFINITION 1.1. A mapping $S : K \to 2^{X^*}$ is said to be *φ-p monotone* with respect to the mapping $\eta : K \times K \to K$ if there exist a mapping $\varphi : [0,+\infty) \to [0,+\infty)$ and a constant $p > 1$ such that

$$\langle x - y, \eta(u,v) \rangle \geqslant \varphi(\|u - v\|)\|u - v\|^p \tag{1.5}$$

for all $u,v \in K, x \in Su$ and $y \in Sv$.

DEFINITION 1.2. A mapping $T : K \to 2^{X^*}$ is said to be *ψ-p Lipschitzian* with respect to the mapping $\eta : K \times K \to K$ if there exist a mapping $\psi : [0,+\infty) \to [0,+\infty)$ and a constant $p > 1$ such that

$$\langle w - z, \eta(u,v) \rangle \leqslant \psi(\|u - v\|)\|u - v\|^p \tag{1.6}$$

for all $u,v \in K, w \in Tu$ and $z \in Tv$.

DEFINITION 1.3. Let E, F be two topological spaces. A mapping $F : E \to 2^F$ is said to be *lower semicontinuous* if, for any net $\{x_\alpha\} \subset E$ with $x_\alpha \to x$ and a point $y \in F(x)$, there exist $\{x_\beta\} \subset \{x_\alpha\}$ and $y_\beta \in F(x_\beta)$ such that $y_\beta \to y$.

DEFINITION 1.4. A mapping $S : K \to X^*$ is said to be *hemicontinuous* if, for all $x, y, z \in K$, the mapping $t \mapsto \langle S(x+ty), z \rangle$ is continuous on $[0,1]$. A mapping $T : K \to 2^{X^*}$ is said to be lower hemicontinuous if, for all $x, y, z \in K$, the set-valued mapping $t \mapsto \langle T(x+ty), z \rangle$ is lower semicontinous on $[0,1]$.

2. Main results

THEOREM 2.1. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty convex closed subset of X. Let $S, T : K \to 2^{X^*}$ be two lower hemicontinuous set-valued mappings satisfying (1.5) and (1.6), respectively, where mappings $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ satisfy $\varphi(t) > \psi(t)$ for all $t \geqslant 0$ and there exists a constant $\delta > 0$ such that $\varphi - \psi$ is bounded on $[0,\delta]$. In addition, suppose that $\eta(u,v) = -\eta(v,u)$ for all $u, v \in K$, $\eta : K \times K \to K$ is affine with respect to the first argument and $f : K \to R \cup \{+\infty\}$ is a proper convex functional. Then, for any $\omega \in X^*$, $u \in K$ is a solution of the problem (1.1) if and only if $u \in K$ is a solution of the following problem:*

Find $u \in K$ such that

$$\begin{aligned} &\langle y - z, \eta(v,u) \rangle + f(v) - f(u) \\ &\qquad \geqslant (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p \end{aligned} \tag{2.1}$$

for all $v \in K, y \in Sv$ and $z \in Tv$.

Proof. Suppose that the problem (1.1) holds. Since the mappings S and T satisfy (1.5) and (1.6), respectively, then, for all $u, v \in K, x \in Su, y \in Sv, w \in Tu$ and $z \in Tv$, we have

$$\begin{aligned} &\langle y - z - \omega, \eta(v,u) \rangle + f(v) - f(u) \\ &\quad = \langle -\omega, \eta(v,u) \rangle + \langle x - y, \eta(u,v) \rangle \\ &\qquad - \langle w - z, \eta(u,v) \rangle + f(v) - f(u) - \langle x - w, \eta(u,v) \rangle \\ &\quad \geqslant \langle x - w - \omega, \eta(v,u) \rangle + f(v) - f(u) \\ &\qquad + (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p. \end{aligned}$$

Since $u \in K$ is a solution of the problem (1.1), this implies that

$$\langle x - w - \omega, \eta(v,u) \rangle + f(v) - f(u) \geqslant 0$$

for all $v \in K$, $x \in Su$ and $w \in Tu$. Thus we have

$$\begin{aligned} &\langle y - z - \omega, \eta(v,u) \rangle + f(v) - f(u) \\ &\qquad \geqslant (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p \end{aligned}$$

for all $v \in K, y \in Sv$ and $z \in Tv$, i.e., (2.1) is true.

Conversely, suppose that (2.1) holds. Without loss of generality, choose a point $v \in K$ such that $f(v) < +\infty$ and so $f(u) < +\infty$. Letting $v_n = (1 - \frac{1}{n})u + \frac{1}{n}v$ for $n = 1, 2, 3 \ldots$, then $v_n \in K$ and $v_n - u = \frac{1}{n}(v - u)$ for all $n \geqslant 1$. Further, since

$\eta : K \times K \to K$ is affine with respect to the first argument and $\eta(u,v) = -\eta(v,u)$ for all $u,v \in K$, we know that $\eta(u,u) = 0$ and $\eta(v_n,u) = \frac{1}{n}\eta(v,u)$. For any $x \in Su$ and $w \in Tu$, since S,T are lower hemicontinuous, there exist subsequence $\{v_{n_j}\} \subset \{v_n\}, y_{n_j} \in Sv_{n_j}$ and $z_{n_j} \in Tv_{n_j}$ such that

$$y_{n_j} \to x, \quad z_{n_j} \to w, \quad \langle y_{n_j} - z_{n_j}, \tau\rangle \to \langle x - w, \tau\rangle \tag{2.2}$$

as $j \to \infty$ for any $\tau \in X$. It follows from (2.1) that

$$\begin{aligned} &\langle y_{n_j} - z_{n_j} - \omega, \eta(v_{n_j},u)\rangle + f(v_{n_j}) - f(u) \\ &= \frac{1}{n_j}\langle y_{n_j} - z_{n_j} - \omega, \eta(v,u)\rangle + f(v_{n_j}) - f(u) \\ &\geqslant (\varphi(\|v_{n_j} - u\|) - \psi(\|v_{n_j} - u\|))\|v_{n_j} - u\|^p \\ &= \left(\frac{1}{n_j}\right)^p \left(\varphi\left(\frac{1}{n_j}\|v-u\|\right) - \psi\left(\frac{1}{n_j}\|v-u\|\right)\right)\|v-u\|^p. \end{aligned} \tag{2.3}$$

Since f is convex and $\eta(v_{n_j},u) = \frac{1}{n_j}\eta(v,u)$, from (2.3), we have

$$\begin{aligned} &\langle y_{n_j} - z_{n_j} - \omega, \eta(v,u)\rangle + f(v) - f(u) \\ &\geqslant \left(\frac{1}{n_j}\right)^{p-1} \left(\varphi\left(\frac{1}{n_j}\|v-u\|\right) - \psi\left(\frac{1}{n_j}\|v-u\|\right)\right)\|v-u\|^p. \end{aligned} \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$\langle x - w - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant 0$$

for all $v \in K, x \in Su$ and $w \in Tu$. This completes the proof.

REMARK 2.1. Theorem 2.1 improves and extends Theorem 2.1 of Verma [7] and [8].

In the sequel, we need the following definition and lemma for our further result.

DEFINITION 2.1. A mapping $F : X \to 2^X$ is said to be a *KKM mapping* if, for any $\{x_1,x_2,\ldots,x_n\} \subset X$, $\mathrm{co}\{x_1,x_2,\ldots,x_n\} \subset \bigcup_{i=1}^{n} F(X_i)$.

LEMMA 2.1. [10] *Let K be a nonempty subset of a topological vector space E and $F : K \to 2^E$ be a KKM mapping. If $F(x)$ is closed in E for every x in K and there exists at least a point $x_0 \in K$ such that $F(x_0)$ is compact, then $\bigcap_{x\in K} F(x) \neq \emptyset$.*

THEOREM 2.2. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty bounded closed convex subset of X. Let S,T,φ,ψ and η be the same as in Theorem 2.1. Suppose that $\varphi - \psi$, η are continuous and $f : K \to R \cup \{+\infty\}$ is proper convex lower semicontinuous. Then the problem (1.1) has a unique solution.*

Proof. We first prove the existence of solution of the problem (1.1). Define the set-valued mappings $F,G : K \to 2^K$ by

$$F(v) = \{u \in K : \langle x - w - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant 0 \quad \text{for some } x \in Su, w \in Tu\}$$

and

$$G(v) = \{u \in K : \langle y - z - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p \quad \text{for all } y \in Sv, z \in Tv\}$$

for all $v \in K$, respectively. We show that F is a KKM mapping. Assume that F is not a KKM mapping. Then there exist $\{v_1, v_2, \dots, v_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \dots, n$, such that

$$\sum_{i=1}^{n} t_i = 1, \quad v = \sum_{i=1}^{n} t_i v_i \notin \bigcup_{i=1}^{n} F(v_i).$$

For any $y \in Sv$ and $z \in Tv$, by the definition of F, we have

$$\langle y - z - \omega, \eta(v_i, v)\rangle + f(v_i) - f(v) < 0$$

for $i = 1, 2, \dots, n$. It follows that

$$\begin{aligned} 0 = \langle y - z - \omega, \eta(v,v)\rangle &= \left\langle y - z - \omega, \eta\left(\sum_{i=1}^{n} t_i v_i, v\right)\right\rangle \\ &= \sum_{i=1}^{n} t_i \langle y - z - \omega, \eta(v_i, v)\rangle < \sum_{i=1}^{n} t_i (f(v) - f(v_i)) \\ &= f(v) - \sum_{i=1}^{n} t_i f(v_i) \leqslant f(v) - f(v) = 0, \end{aligned}$$

which is a contradiction. This implies that F is a KKM mapping. Now we prove that $F(v) \subset G(v)$ for all $v \in K$. Letting $u \in F(v)$, then there exist $x \in Su, w \in Tu$ such that

$$\langle x - w - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant 0.$$

Since the mappings S and T satisfy (1.5) and (1.8), respectively, we have

$$\begin{aligned} &\langle y - z - \omega, \eta(v,u)\rangle + f(v) - f(u) \\ &\quad = \langle x - y, \eta(u,v)\rangle - \langle w - z, \eta(u,v)\rangle \\ &\qquad + \langle -\omega, \eta(v,u)\rangle - \langle x - w, \eta(u,v)\rangle + f(v) - f(u) \\ &\quad \geqslant (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p \\ &\qquad + \langle x - w - \omega, \eta(v,u)\rangle + f(v) - f(u) \\ &\quad \geqslant (\varphi(\|v-u\|) - \psi(\|v-u\|))\|v-u\|^p \end{aligned}$$

for all $v \in K, y \in Sv$ and $z \in Tv$. This implies that $u \in G(v)$, and so G is also a KKM mapping. From the assumptions, we know that $G(v)$ is weakly closed for all v in K. Since K is bounded closed convex, we know that K is weakly compact and so $G(v)$ is weakly compact in K for all $v \in K$. It follows from Lemma 2.1 that $\bigcap_{v \in K} G(v) \neq \emptyset$. Hence, there exists a point $u_0 \in K$ such that

$$\begin{aligned} &\langle y - z, \eta(v, u_0)\rangle + f(v) - f(u_0) \\ &\qquad \geqslant (\varphi(\|v-u_0\|) - \psi(\|v-u_0\|))\|v-u_0\|^p \end{aligned}$$

for all $v \in K$, $y \in Sv$ and $z \in Tv$. By Theorem 2.1, we know that

$$\sup_{x_0 \in Su_0, w_0 \in Tu_0} \langle x_0 - w_0 - \omega, \eta(v, u_0) \rangle + f(v) - f(u_0) \geqslant 0$$

for all $v \in K$.

To show the uniqueness of the solution, let $u_1, u_2 \in K$ be two solutions of the problem (1.1). For any $x_1 \in Su_1, w_1 \in Tu_1, x_2 \in Su_2$ and $w_2 \in Tu_2$, we have

$$\langle x_1 - w_1 - \omega, \eta(v, u_1) \rangle + f(v) - f(u_1) \geqslant 0 \tag{2.5}$$

and

$$\langle x_2 - w_2 - \omega, \eta(v, u_2) \rangle + f(v) - f(u_2) \geqslant 0 \tag{2.6}$$

for all $v \in K$. Setting $v = u_2$ in (2.5) and $v = u_1$ in (2.6) and adding them, then we have

$$\langle x_1 - w_1 - (x_2 - w_2), \eta(u_2, u_1) \rangle \geqslant 0. \tag{2.7}$$

By (1.5) and (1.6), we obtain

$$\begin{aligned} &\langle x_1 - w_1 - (x_2 - w_2), \eta(u_2, u_1) \rangle \\ &\quad = \langle x_1 - x_2, \eta(u_2, u_1) \rangle - \langle w_1 - w_2, \eta(u_2, u_1) \rangle \\ &\quad \leqslant -(\varphi(\|u_1 - u_2\|) - \psi(\|u_1 - u_2\|))\|u_1 - u_2\|^p. \end{aligned} \tag{2.8}$$

Since $\varphi(t) > \psi(t)$ for all $t > 0$, it follows from (2.7) and (2.8) that $\|u_1 - u_2\|^p = 0$. This implies that $u_1 = u_2$. This completes the proof.

REMARK 2.2. Theorem 2.2 improves and extends Theorem 2.4 of Verma [8].

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