Mathematical
Inequalities
& Applications
Volume 6, Number 2 (2003), 339–350

AUXILIARY PRINCIPLE TECHNIQUE FOR SOLVING GENERALIZED SET-VALUED NONLINEAR QUASI-VARIATIONAL-LIKE INEQUALITIES

NAN-JING HUANG AND YA-PING FANG

(*communicated by R. P. Agarwal*)

Abstract. In this paper, we introduce and study a new class of generalized set-valued nonlinear quasi-variational-like inequalities in Hilbert spaces and construct some iterative algorithms to compute the approximating solutions of this class of generalized set-valued nonlinear quasi-variational-like inequalities by using the auxiliary principle technique. We also give the convergence analysis of the iterative sequences generated by the algorithms. The results presented in this paper extend and impro ve the corresponding results announced by Ding.

1. Introduction

Variational inequality theory and complementarity problem theory are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide class of problems generated in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. For the past years, many authors have studied various variational inequalities and variational inclusions (see [1, 2, 4–7, 9, 11–14]) by various methods such as the projection method and its variant forms, linear approximation, descent, and Newton's methods. However, these methods are invalid to solve variational-like inequalities. In 1981, Glowinski et al. [4] suggested another technique-auxiliary principle technique for solving a class of variational inequalities. From then on, many authors extended and generalize the auxiliary principle technique to solve variational inequalities. For details, we refer to [3, 4, 7, 13] and the references therein.

Recently, Ding [3] studied a class of generalized mixed implicit quasi-variational inequalities in Hilbert spaces and constructed some new iterative algorithms to compute the approximating solutions by using the auxiliary principle technique. Inspired and motivated by [3, 5, 14], in this paper, we introduce and study a new class of generalized set-valued nonlinear quasi-variational-like inequalities in Hilbert spaces and construct

Mathematics subject classification (2000): 49A29, 49J40, 47H10.

Key words and phrases: Quasi-variational-like inequality; Auxiliary principle technique; Iterative algorithm; Convergence.

This work was supported by the National Science Foundation of China (10171070) and Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

Paper MIA-06-32

some iterative algorithms to compute the approximating solutions of this class of generalized set-valued nonlinear quasi-variational-like inequalities by using the auxiliary principle technique. We also give the convergence analysis of the iterative sequences generated by the algorithms. The results presented in this paper extend and improve the corresponding results of [2, 3, 5, 9, 12–14].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H, where 2^H denotes the family of all the nonempty subset of H. Let $T, A : H \to CB(H)$ be two set-valued mappings, where $CB(H)$ denotes the family of all the nonempty closed bounded subset of H, $N, \eta : H \times H \to H$ be two single-valued mappings, and $b : H \times H \to R$ be a real function. In this paper, we consider the following generalized set-valued nonlinear quasi-variational-like inequalities problem:

Find $x \in H, u \in T(x)$, and $v \in A(x)$ such that

$$x \in K(x), \quad \langle N(u,v), \eta(y,x)\rangle + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in K(x). \tag{2.1}$$

If $\eta(x,y) = g(x) - g(y)$ for all x, y in H, where $g : H \to H$ is a single-valued mapping, then the problem (2.1) is equivalent to finding $x \in H, u \in T(x)$, and $v \in A(x)$ such that

$$x \in K(x), \quad \langle N(u,v), g(y) - g(x)\rangle + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in K(x), \tag{2.2}$$

which is called the generalized set-valued nonlinear implicit quasi-variational inequality.

If $\eta(x,y) = x - y$ for all x, y in H, then the problem (2.1) is equivalent to finding $x \in H, u \in T(x)$, and $v \in A(x)$ such that

$$x \in K(x), \quad \langle N(u,v), y - x\rangle + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in K(x), \tag{2.3}$$

which is called the generalized mixed implicit quasi-variational inequality studied by Ding [3].

If $b(x,y) = 0$ for all x, y in H, then the problem (2.1) is equivalent to finding $x \in H, u \in T(x)$, and $v \in A(x)$ such that

$$x \in K(x), \quad \langle N(u,v), \eta(y,x)\rangle \geqslant 0, \quad \forall y \in K(x), \tag{2.4}$$

which is called the quasi-variational-like inequality.

If $K(x) = H$ for all $x \in H$, then the problem (2.1) is equivalent to find $x \in H, u \in T(x)$, and $v \in A(x)$ such that

$$\langle N(u,v), \eta(y,x)\rangle + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in H, \tag{2.5}$$

which is called the generalized set-valued strongly nonlinear mixed variational-like inequality.

In many practical problems, $K(x)$ has the following form:

$$K(x) = m(x) + K \tag{2.6}$$

for all x in H, where $m : H \to H$ is a single-valued mapping and K is a nonempty closed convex set of H.

REMARK 2.1. For a suitable choice of the mappings T, A, K, N, η, and b, one can obtain many known variational inequalities and complementarity problems as special cases from the problem (2.1) (See [2, 3, 5–7, 9, 11–14]).

For our results, we need the following concepts and results.

DEFINITION 2.1. Let $T : H \to CB(H)$ be a set-valued mapping. A mapping $N : H \times H \to H$ is said to be

(i) α-strongly monotone with respect to T in the first argument if there exists a constant α such that

$$\langle N(u,\cdot) - N(v,\cdot), x - y\rangle \geqslant \alpha\|x-y\|^2, \quad \forall x, y \in H, u \in T(x), v \in T(v);$$

(ii) β-Lipschitz continuous in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u,\cdot) - N(v,\cdot)\| \leqslant \beta\|u-v\|, \quad \forall u, v \in H.$$

In a similar way, we can define the monotonicity and Lipschitz continuity of N in the second argument. We note that if $N(\cdot,\cdot)$ is Lipschitz continuous in both arguments, then N is continuous.

DEFINITION 2.2. A mapping $\eta : H \times H \to H$ is said to be

(i) σ-monotone if there exists a constant $\sigma > 0$ such that

$$\langle \eta(u,v), u - v\rangle \geqslant \sigma\|u-v\|^2, \quad \forall u, v \in H;$$

(ii) δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leqslant \delta\|u-v\|, \quad \forall u, v \in H.$$

DEFINITION 2.3. A mapping $m : H \to H$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|m(x) - m(y)\| \leqslant \tau\|x-y\|, \quad \forall x, y \in H.$$

DEFINITION 2.4. A set-valued mapping $T : H \to CB(H)$ is said to be

(i) μ-strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle u - v, x - y\rangle \geqslant \mu\|x-y\|^2, \quad \forall x, y \in H, u \in T(x), v \in T(y);$$

(ii) ξ-H-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$H(T(x), T(y)) \leqslant \xi\|x - y\|, \quad \forall x, y \in H,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

HYPOTHESIS 2.1. Let $b : H \times H \to R$ be a real function satisfying the following conditions:

(i) $b(u, v)$ is linear with respect to u;
(ii) $b(u, v)$ is bounded, i.e., there exists a constant $\gamma > 0$ such that

$$b(u, v) \leqslant \gamma\|u\| \cdot \|v\|, \ \ \forall u, v \in H;$$

(iii) $b(u, v) - b(u, w) \leqslant b(u, v - w), \ \ \forall u, v, w \in H$;
(iv) $b(u, v)$ is convex with respect to v.

REMARK 2.1. From (i)-(iv), we have
(1) $|b(u, v)| \leqslant \gamma\|u\| \cdot \|v\|, \quad b(u, 0) = b(0, v) = 0, \quad \forall u, v \in H$;
(2) $|b(u, v) - b(u, w)| \leqslant \nu\|u\| \cdot \|v - w\|, \quad \forall u, v, w \in H$;
(3) $b(u, v)$ is continuous.

HYPOTHESIS 2.2. Let $\eta : H \times H \to H$ satisfy the following conditions:
(1) $\eta(x, y) + \eta(y, z) = \eta(x, z)$ for all x, y, z in H;
(2) for any x, y, u, v in H, $x - y = u - v$ implies that $\eta(x, y) = \eta(u, v)$;
(3) for given x, u, v in H, the mapping $y \mapsto \langle N(u, v), \eta(y, x)\rangle$ is convex and lower semicontinuous.

HYPOTHESIS 2.3. Let $g : H \to H$ satisfy the following conditions:
(1) for any x, y, u, v in H, $x - y = u - v$ implies that $g(x) - g(y) = g(u) - g(v)$;
(2) for given u, v in H, the mapping $y \mapsto \langle N(u, v), g(y)\rangle$ is convex and lower semicontinuous.

3. Auxiliary Problem

In this section, we give the auxiliary problem for the generalized set-valued nonlinear quasi-variational-like inequalities problem (2.1) and prove the existence and uniqueness of solution of the auxiliary problem.

Given $x \in H, u \in T(x)$, and $v \in A(x)$, we consider the following auxiliary problem for the problem (2.1):

Find $w \in K(x)$ such that

$$\langle w, y - w\rangle \geqslant \langle x, y - w\rangle - \rho\langle N(u, v), \eta(y, w)\rangle + \rho b(x, w) - \rho b(x, y), \ \forall y \in K(x), \quad (3.1)$$

where $\rho > 0$ is a constant.

THEOREM 3.1. *Let* $T, A : H \to CB(H)$, *and* $N, \eta : H \times H \to H$ *be four mappings. Let* $K : H \to 2^H$ *be a set-valued mapping such that for each* $x \in H$, $K(x)$ *is a nonempty closed convex subset of* H. *Let* $b : H \times H \to R$ *is a real function such that for any given* $x \in H$, $y \mapsto b(x,y)$ *is convex and lower semi-continuous on* H. *Moreover, suppose that Hypothesis 2.2 holds. Then for any given* $x \in H, u \in T(x)$, *and* $v \in A(x)$, *the following problem:*

$$\min_{y \in K(x)} J(y), \tag{3.2}$$

where

$$\begin{cases} J(y) = \frac{1}{2}\langle y, y\rangle + j(y), \\ j(y) = \rho\langle N(u,v), \eta(y,x)\rangle + \rho b(x,y) - \langle x, y\rangle, \end{cases} \tag{3.3}$$

admits a unique solution and w *is a solution of the problem (3.2) if and only if* w *is a solution of the problem (3.1).*

Proof. Since the function $y \mapsto b(x,y)$ is convex lower semicontinuous, it follows from Hypothesis 2.2 that $j(y)$ is convex lower semicontinuous on $K(x)$ and $J(x)$ is strictly convex and lower semicontinuous on $K(x)$. By Theorem 2.5 of [10, p. 25], j is bounded from below by a hyperplane $f(y) = \langle h, y\rangle + r$, where $h \in H$ and $r \in R$. Hence we have

$$\begin{aligned} J(y) &= \frac{1}{2}\langle y, y\rangle + j(y) \geqslant \frac{1}{2}\|y\|^2 + \langle h, y\rangle + r \\ &= \frac{1}{2}\|y+h\|^2 - \frac{1}{2}\|h\|^2 + r. \end{aligned}$$

This implies that

$$J(y) \to \infty \quad \text{and} \quad \|y\| \to \infty. \tag{3.4}$$

Now let $\{y_n\} \subset K(x)$ be a minimizing sequence of J on $K(x)$, i.e.,

$$\lim_{n\to\infty} J(y_n) = d \quad \text{and} \quad d = \inf_{y \in K(x)} J(y).$$

We claim that $\{y_n\}$ is bounded. If it is false, then there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $\|y_{n_k}\| \geqslant k, k = 1, 2, 3, \ldots$. By (3.4), we have $J(y_{n_k}) \to \infty$ which contradicts the fact $\lim_{k\to\infty} J(y_{n_k}) = d < \infty$. Therefore there exists a constant $r_1 > 0$ such that

$$\{y_n\} \subset K(x) \cap B(0, r_1) = \{y \in K(x) : \|y\| \leqslant r_1\}.$$

By Weierstrass theorem (See, [10, p. 24]), there exists $w \in K(x)$ such that

$$J(w) = \min_{y \in K(x)} J(y).$$

Since J is strictly convex, we know that w is the unique solution of the problem (3.2).

Now suppose that w is a unique solution of the problem (3.2). We show that w is also a solution of the auxiliary problem (3.1). For any $y \in K(x)$ and $t \in [0,1]$, we have

$$\begin{aligned} J(w) &= \frac{1}{2}\langle w, w\rangle + j(w) \leqslant J(w + t(y - w)) \\ &= \frac{1}{2}\langle w + t(y - w), w + t(y - w)\rangle + j(w + t(y - w)) \\ &\leqslant \frac{1}{2}\langle w, w\rangle + \frac{t^2}{2}\langle y - w, y - w\rangle + t\langle w, y - w\rangle + j(w) + t(j(y) - j(w)). \end{aligned}$$

This implies that

$$\frac{t}{2}\langle y - w, y - w\rangle + \langle w, y - w\rangle + j(y) - j(w) \geqslant 0.$$

Letting $t \to 0$ in the above inequality, we obtain

$$\begin{aligned} \langle w, y &- w\rangle + \rho\langle N(u, v), \eta(y, x)\rangle + \rho b(x, y) - \langle x, y\rangle \\ &- \rho\langle N(u, v), \eta(w, x)\rangle - \rho b(x, w) + \langle x, w\rangle \geqslant 0. \end{aligned}$$

It follows from (1) of Hypothesis 2.2 that

$$\langle w, y - w\rangle \geqslant \langle x, y - w\rangle - \rho\langle N(u, v), \eta(y, w)\rangle + \rho b(x, w) - \rho b(x, y), \quad \forall y \in K(x).$$

This prove that w is a solution of the auxiliary problem (3.1).

Conversely, suppose that w is a solution of the auxiliary problem (3.1). It follows from (3.1) that

$$\begin{aligned} \frac{1}{2}[\langle y, y\rangle &- \langle w, w\rangle] \\ &= \langle w, y - w\rangle + \frac{1}{2}\langle y - w, y - w\rangle \geqslant \langle w, y - w\rangle \\ &\geqslant \langle x, y - w\rangle - \rho\langle N(u, v), \eta(y, w)\rangle + \rho b(x, w) - \rho b(x, y) \\ &= \langle x, y\rangle - \langle x, w\rangle - \rho\langle N(u, v), \eta(y, x)\rangle \\ &\quad + \rho\langle N(u, v), \eta(w, x)\rangle + \rho b(x, w) - \rho b(x, y), \quad \forall y \in K(x). \end{aligned}$$

This implies that $J(y) \geqslant J(w)$ for all $y \in K(x)$ and so w is a solution of the problem (3.2). The proof is complete.

4. Iterative Algorithms and Convergence

In this section, by using the auxiliary technique, we construct some new iterative algorithms for solving the problems (2.1)–(2.4) and give the convergence analysis of the iterative sequences generated by the algorithms.

Based on Theorem 3.1, we give one algorithm for the problem (2.1) as follows:

ALGORITHM 4.1. Suppose that T, A, K, N, η, and b are the same as in Theorem 3.1 and that Hypothesis 2.2 holds. For any given $x_0 \in H, u_0 \in T(x_0)$, and $v_0 \in A(x_0)$, based on Theorem 3.1 and Nadler [8], there exist $\{x_n\},\{u_n\}$, and $\{v_n\}$ such that

$$\begin{cases} x_{n+1} \in K(x_n), \\ u_{n+1} \in T(x_{n+1}), \quad \|u_{n+1}-u_n\| \leqslant (1+(1+n)^{-1})H(T(x_{n+1}),T(x_n)), \\ v_{n+1} \in A(x_{n+1}), \quad \|v_{n+1}-v_n\| \leqslant (1+(1+n)^{-1})H(A(x_{n+1}),A(x_n)), \\ \langle x_{n+1}, y-x_{n+1}\rangle \geqslant \langle x_n, y-x_{n+1}\rangle - \rho\langle N(u_n,v_n),\eta(y,x_{n+1})\rangle \\ \qquad +\rho b(x_n,x_{n+1}) - \rho b(x_n,y), \quad \forall y \in K(x_n). \quad n=0,1,2\ldots \end{cases} \tag{4.1}$$

If $\eta(x,y) = g(x)-g(y)$ for all x,y in H, then Algorithm 4.1 reduces to the following algorithm for the problem (2.2).

ALGORITHM 4.2. Let T,A,K,N, and b be the same as in Theorem 3.1. Suppose that Hypothesis 2.3 holds. For any given $x_0 \in H, u_0 \in T(x_0)$, and $v_0 \in A(x_0)$, based on Theorem 3.1 and Nadler [8], there exist $\{x_n\},\{u_n\}$, and $\{v_n\}$ such that

$$\begin{cases} x_{n+1} \in K(x_n), \\ u_{n+1} \in T(x_{n+1}), \quad \|u_{n+1}-u_n\| \leqslant (1+(1+n)^{-1})H(T(x_{n+1}),T(x_n)), \\ v_{n+1} \in A(x_{n+1}), \quad \|v_{n+1}-v_n\| \leqslant (1+(1+n)^{-1})H(A(x_{n+1}),A(x_n)), \\ \langle x_{n+1}, y-x_{n+1}\rangle \geqslant \langle x_n, y-x_{n+1}\rangle - \rho\langle N(u_n,v_n),g(y)-g(x_{n+1})\rangle \\ \qquad +\rho b(x_n,x_{n+1}) - \rho b(x_n,y), \quad \forall y \in K(x_n). \quad n=0,1,2\ldots \end{cases}$$

If $\eta(x,y) = x-y$ for all x,y in H, then Algorithm 4.1 reduces to the following algorithm for the problem (2.3).

ALGORITHM 4.3. Let T,A,K,N, and b be the same as in Theorem 3.1. For any given $x_0 \in H, u_0 \in T(x_0)$, and $v_0 \in A(x_0)$, based on Theorem 3.1 and Nadler [8], there exist $\{x_n\},\{u_n\}$, and $\{v_n\}$ such that

$$\begin{cases} x_{n+1} \in K(x_n), \\ u_{n+1} \in T(x_{n+1}), \quad \|u_{n+1}-u_n\| \leqslant (1+(1+n)^{-1})H(T(x_{n+1}),T(x_n)), \\ v_{n+1} \in A(x_{n+1}), \quad \|v_{n+1}-v_n\| \leqslant (1+(1+n)^{-1})H(A(x_{n+1}),A(x_n)), \\ \langle x_{n+1}, y-x_{n+1}\rangle \geqslant \langle x_n, y-x_{n+1}\rangle - \rho\langle N(u_n,v_n),y-x_{n+1}\rangle \\ \qquad +\rho b(x_n,x_{n+1}) - \rho b(x_n,y), \quad \forall y \in K(x_n), \quad n=0,1,2\ldots \end{cases}$$

If $b(x,y) = 0$ for all x,y in H, then Algorithm 4.1 reduces to the following algorithm for the problem (2.4).

ALGORITHM 4.4. Suppose that T,A,K,N, and η are the same as in Theorem 3.1 and that Hypothesis 2.2 holds. For any given $x_0 \in H, u_0 \in T(x_0)$, and $v_0 \in A(x_0)$, based on Theorem 3.1 and Nadler [8], there exist $\{x_n\},\{u_n\}$, and $\{v_n\}$ such that

$$\begin{cases} x_{n+1} \in K(x_n), \\ u_{n+1} \in T(x_{n+1}), \quad \|u_{n+1}-u_n\| \leqslant (1+(1+n)^{-1})H(T(x_{n+1}),T(x_n)), \\ v_{n+1} \in A(x_{n+1}), \quad \|v_{n+1}-v_n\| \leqslant (1+(1+n)^{-1})H(A(x_{n+1}),A(x_n)), \\ \langle x_{n+1}, y-x_{n+1}\rangle \geqslant \langle x_n, y-x_{n+1}\rangle - \rho\langle N(u_n,v_n),\eta(y,x_{n+1})\rangle, \quad \forall y \in K(x), \\ \qquad n=0,1,2\ldots \end{cases}$$

THEOREM 4.1. *Let $K : H \to 2^H$ be a set-valued mapping such that $K(x)$ has the form of (2.6). Let $T, A : H \to CB(H)$ be γ-H-Lipschitz continuous and μ-H-Lipschitz continuous, respectively. Let $N : H \times H \to H$ be α-strongly monotone with respect to T and β-Lipschitz continuous in the first argument, and ξ-Lipschitz conouous in the second argument. Let $m : H \to H$ be σ-Lipschitz continuous and $\eta : H \times H \to H$ be s-strongly monotone and τ-Lipschitz continuous. Suppose that Hypotheses 2.1 and 2.2 hold and that the following inequalities hold:*

$$\begin{cases} |\rho - \frac{\alpha-k}{\beta^2\gamma^2-k^2}| < \frac{\sqrt{(\alpha-k)^2-(\beta^2\gamma^2-k^2)4\sigma(1-\sigma)}}{\beta^2\gamma^2-k^2}, \\ k = \beta\gamma\sqrt{1-2s+\tau^2} + \tau\xi\mu + \nu, \quad k < \beta\gamma, \\ \alpha > k + \sqrt{(\beta^2\gamma^2-k^2)4\sigma(1-\sigma)}, \quad \rho k + 2\sigma < 1. \end{cases} \tag{4.2}$$

Then the iterative sequences $\{x_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm 4.1 converge strongly to x, u, and v respectively and (x, u, v) is a solution of the problem (2.1).

Proof. It follows from (4.1) that

$$\begin{aligned} \langle x_{n+1}, y - x_{n+1}\rangle \;&\geqslant\; \langle x_n, y - x_{n+1}\rangle - \rho\langle N(u_n, v_n), \eta(y, x_{n+1})\rangle \\ &\quad + \rho b(x_n, x_{n+1}) - \rho b(x_n, y), \quad \forall y \in K(x_n) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \langle x_{n+2}, y - x_{n+2}\rangle \;&\geqslant\; \langle x_{n+1}, y - x_{n+2}\rangle - \rho\langle N(u_{n+1}, v_{n+1}), \eta(y, x_{n+2})\rangle \\ &\quad + \rho b(x_{n+1}, x_{n+2}) - \rho b(x_{n+1}, y), \quad \forall y \in K(x_{n+1}). \end{aligned} \tag{4.4}$$

Adding $\langle -m(x_n), y - x_{n+1}\rangle$ to two sides of the inequality (4.3) and then, taking $y = x_{n+2} - m(x_{n+1}) + m(x_n) \in K(x_n)$, we obtain

$$\begin{aligned} \langle x_{n+1} \;-\; & m(x_n), x_{n+2} - x_{n+1} - m(x_{n+1}) + m(x_n)\rangle \\ \geqslant\; & \langle x_n - m(x_n), x_{n+2} - x_{n+1} - m(x_{n+1}) + m(x_n)\rangle \\ & - \rho\langle N(u_n, v_n), \eta(x_{n+2} - m(x_{n+1}) + m(x_n), x_{n+1})\rangle \\ & + \rho b(x_n, x_{n+1}) - \rho b(x_n, x_{n+2} - m(x_{n+1}) + m(x_n)). \end{aligned} \tag{4.5}$$

Adding $\langle -m(x_{n+1}), y - x_{n+2}\rangle$ to two sides of the inequality (4.4) and then, taking $y = x_{n+1} - m(x_n) + m(x_{n+1}) \in K(x_{n+1})$, we obtain

$$\begin{aligned} \langle x_{n+2} \;-\; & m(x_{n+1}), x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1})\rangle \\ \geqslant\; & \langle x_{n+1} - m(x_{n+1}), x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1})\rangle \\ & - \rho\langle N(u_{n+1}, v_{n+1}), \eta(x_{n+1} - m(x_n) + m(x_{n+1}), x_{n+2})\rangle \\ & + \rho b(x_{n+1}, x_{n+2}) - \rho b(x_{n+1}, x_{n+1} - m(x_n) + m(x_{n+1})). \end{aligned} \tag{4.6}$$

From Hypothesis 2.2, we know that

$$\begin{aligned} \eta(x_{n+1} - m(x_n) + m(x_{n+1}), x_{n+2}) \;&=\; -\eta(x_{n+2} - m(x_{n+1}) + m(x_n), x_{n+1}) \\ &=\; \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1})). \end{aligned} \tag{4.7}$$

It follows from (4.5)–(4.7) that

$$\begin{aligned}
\| x_{n+1} & - x_{n+2} - (m(x_n) - m(x_{n+1}))\|^2 \\
& \leqslant \langle x_n - x_{n+1} - (m(x_n) - m(x_{n+1})), x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\rangle \\
& \quad -\rho\langle N(u_n, v_n), \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\rangle \\
& \quad +\rho\langle N(u_{n+1}, v_{n+1}), \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\rangle \\
& \quad +\rho[b(-x_n, x_{n+1}) - b(-x_n, x_{n+2} - m(x_{n+1}) + m(x_n)) \\
& \quad +b(x_{n+1}, x_{n+1} - m(x_n) + m(x_{n+1})) - b(x_{n+1}, x_{n+2})] \\
& \leqslant \langle x_n - x_{n+1} - (m(x_n) - m(x_{n+1})), x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\rangle \\
& \quad -\rho\langle N(u_n, v_n) - N(u_{n+1}, v_{n+1}), \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\rangle \\
& \quad +\rho b(x_{n+1} - x_n, x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))) \\
& \leqslant \langle x_n - x_{n+1} - \rho(N(u_n, v_n) - N(u_{n+1}, v_n)), x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\rangle \\
& \quad -\langle m(x_n) - m(x_{n+1}), x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\rangle \\
& \quad +\rho\langle N(u_n, v_n) - N(u_{n+1}, v_n), x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1})) \\
& \quad -\eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\rangle \\
& \quad -\rho\langle N(u_{n+1}, v_n) - N(u_{n+1}, v_{n+1}), \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\rangle \\
& \quad +\rho\nu\|x_{n+1} - x_n\| \cdot \|x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\| \\
& \leqslant \{\|m(x_n) - m(x_{n+1})\| + \|x_n - x_{n+1} - \rho(N(u_n, v_n) - N(u_{n+1}, v_n))\| \\
& \quad +\rho\nu\|x_{n+1} - x_n\|\} \cdot \|x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\| \\
& \quad +\rho\|N(u_n, v_n) - N(u_{n+1}, v_n)\| \cdot \|x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1})) \\
& \quad -\eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\| \\
& \quad +\rho\|N(u_{n+1}, v_n) - N(u_{n+1}, v_{n+1})\| \cdot \|\eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\|. \qquad (4.8)
\end{aligned}$$

Since η is s-strongly monotone and τ-Lipschitz continuous, we have

$$\|\eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\| \leqslant \tau\|x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\| \qquad (4.9)$$

and

$$\begin{aligned}
\|x_{n+1} & - x_{n+2} - (m(x_n) - m(x_{n+1})) - \eta(x_{n+1} - x_{n+2}, m(x_n) - m(x_{n+1}))\| \\
& \leqslant \sqrt{1 - 2s + \tau^2}\|x_{n+1} - x_{n+2} - (m(x_n) - m(x_{n+1}))\|. \qquad (4.10)
\end{aligned}$$

It follows from (4.8)–(4.10) that

$$\begin{aligned}
\|x_{n+1} - x_{n+2}\| \leqslant\ & 2\|m(x_n) - m(x_{n+1})\| + \|x_n - x_{n+1} - \rho(N(u_n, v_n) - N(u_{n+1}, v_n))\| \\
& +\rho\nu\|x_n - x_{n+1}\| + \rho\sqrt{1 - 2s + \tau^2}\|N(u_n, v_n) - N(u_{n+1}, v_n)\| \\
& +\rho\tau\|N(u_{n+1}, v_n) - N(u_{n+1}, v_{n+1})\|. \qquad (4.11)
\end{aligned}$$

Since $N(\cdot,\cdot)$ is α-strongly monotone with respect to T and β-Lipschitz continuous in the first argument and T is γ-H-Lipschitz continuous, we have

$$\begin{aligned}\|N(u_n, v_n) - N(u_{n+1}, v_n)\| &\leqslant \beta\|u_n - u_{n+1}\| \\ &\leqslant \beta H(T(x_n), T(x_{n+1})) \\ &\leqslant \beta\gamma\left(1 + \frac{1}{1+n}\right)\|x_n - x_{n+1}\| \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\|x_n &- x_{n+1} - \rho(N(u_n, v_n) - N(u_{n+1}, v_n))\|^2 \\ &\leqslant \|x_n - x_{n+1}\|^2 - 2\rho\langle x_n - x_{n+1}, N(u_n, v_n) - N(u_{n+1}, v_n)\rangle \\ &+ \rho^2\|N(u_n, v_n) - N(u_{n+1}, v_n)\|^2 \\ &\leqslant \left(1 - 2\rho\alpha + \rho^2\beta^2\gamma^2\left(1 + \frac{1}{1+n}\right)\right)\|x_n - x_{n+1}\|^2. \end{aligned} \tag{4.13}$$

By using ξ-Lipschitz continuity of $N(\cdot,\cdot)$ in the second argument and μ-H-Lipschitz continuity of A, we get

$$\begin{aligned}\|N(u_{n+1}, v_n) - N(u_{n+1}, v_{n+1})\| &\leqslant \xi\|v_n - v_{n+1}\| \\ &\leqslant \xi\left(1 + \frac{1}{1+n}\right)H(A(x_n), A(x_{n+1})) \\ &\leqslant \xi\mu\left(1 + \frac{1}{1+n}\right)\|x_n - x_{n+1}\|. \end{aligned} \tag{4.14}$$

Since m is σ-Lipschitz continuous, we obtain

$$\|m(x_n) - m(x_{n+1})\| \leqslant \sigma\|x_n - x_{n+1}\|. \tag{4.15}$$

From (4.11)–(4.15), we have

$$\|x_{n+1} - x_{n+2}\| \leqslant \theta_n\|x_n - x_{n+1}\|, \tag{4.16}$$

where

$$\theta_n = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\gamma^2\left(1 + \frac{1}{1+n}\right)^2} + \rho\beta\gamma\left(1 + \frac{1}{1+n}\right)\sqrt{1 - 2s + \tau^2} + \rho\tau\xi\mu\left(1 + \frac{1}{1+n}\right) + \rho\nu + 2\sigma.$$

Letting

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\gamma^2} + \rho(\beta\gamma\sqrt{1 - 2s + \tau^2} + \tau\xi\mu + \nu) + 2\sigma.$$

We know that $\theta_n \to \theta$ as $n \to \infty$. It follows from (4.2) that $0 \leqslant \theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (4.16) implies that $\{x_n\}$ is a Cauchy sequence in H and we can suppose that $x_n \to x \in H$.

Since T and A are both H-Lipschitz continuous, from (4.1), we get

$$\begin{aligned}\|u_n - u_{n+1}\| &\leqslant \left(1+\frac{1}{1+n}\right) H(T(x_n), T(x_{n+1})) \\ &\leqslant \left(1+\frac{1}{1+n}\right) \gamma \|x_n - x_{n+1}\|\end{aligned}$$

and

$$\begin{aligned}\|v_n - v_{n+1}\| &\leqslant \left(1+\frac{1}{1+n}\right) H(A(x_n), A(x_{n+1})) \\ &\leqslant \left(1+\frac{1}{1+n}\right) \gamma \|x_n - x_{n+1}\|.\end{aligned}$$

This imply that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in H. Let $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Furhter, we have

$$\begin{aligned}d(u, T(x)) &\leqslant \|u_n - u\| + d(u_n, T(x)) \\ &\leqslant \|u_n - u\| + H(T(x_n), T(x)) \\ &\leqslant \|u_n - u\| + \gamma \|x_n - x\| \to 0.\end{aligned}$$

This implies that $u \in T(x)$. Similarly, we can prove that $v \in A(x)$. By Theorem 3.1, we know that there exists a unique $w \in K(x)$ such that

$$\langle w, y-w\rangle \geqslant \langle x, y-w\rangle - \rho\langle N(u,v), \eta(y,v)\rangle + \rho b(x,w) - \rho b(x,y), \ \forall y \in K(x). \tag{4.17}$$

By applying (4.3) and (4.17) and similar argument as proving (4.11), we can prove that

$$\begin{aligned}\|x_{n+1} - w\| \leqslant\ & 2\|m(x_n) - m(x)\| + \|x_n - x - \rho(N(u_n, v_n) - N(u, v_n))\| \\ & + \rho\nu\|x_n - x\| + \rho\sqrt{1 - 2s + \tau^2}\|N(u_n, v_n) - N(u, v_n)\| \\ & + \rho\tau\|N(u, v_n) - N(u, v)\|.\end{aligned} \tag{4.18}$$

From the assumptions, we know that m and $N(\cdot,\cdot)$ are continuous. Therefore (4.18) implies that $x_n \to w$ as $n \to \infty$. Since $x_n \to x$, we must have $x = w$. It follows from (4.17) that (x, u, v) is a solution of the problem (2.1). This completes the proof.

From Theorem 4.1, we can obtain the following corollaries.

COROLLARY 4.1. *Let T, A, K, N, m, and b be the same as in Theorem 4.1. Let $g : H \to H$ be s-strongly monotone and τ-Lipschitz continuous. Suppose that Hypotheses 2.1 and 2.3, and condition (4.2) hold. Then the iterative sequences $\{x_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm 4.2 converge strongly to x, u, and v respectively and (x, u, v) is a solution of the problem (2.2).*

COROLLARY 4.2. *Let T, A, K, N, m, and b be the same as in Theorem 4.1. Suppose that Hypothesis 2.1. If condition (4.2) holds for $k = \mu\xi + \nu$, then the iterative sequences $\{x_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm 4.3 converge strongly to x, u, and v respectively and (x, u, v) is a solution of the problem (2.3).*

COROLLARY 4.3. *Let T, A, K, N, η, m, and b be the same as in Theorem 4.1. Suppose that Hypothesis 2.3 holds. If condition (4.2) holds for $k = \nu\xi$, then the iterative sequences $\{x_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm 4.4 converge strongly to x, u, and v respectively and (x, u, v) is a solution of the problem (2.4).*

REMARK 4.1. Theorem 4.1 improves and generalizes the corresponding results of [2, 3, 5, 9, 12–14].

REFERENCES

[1] R. P. AGARWAL, Y. J. CHO, N. J. HUANG, Sensitivity analysis for strongly nonlinear quasi-variational inclusions, *Appl. Math. Lett.* **13(6)** (2000), 19–24.
[2] S. S. CHANG, N. J. HUANG, Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces, *J. Math. Anal. Appl.* **158** (1991), 194–202.
[3] X. P. DING, Existence and algorithm of solutions for generalized mixed implicit quasi-variational inequalities, *Appl. Math. Comput.* **113** (2000), 67–80.
[4] R. GLOWINSKI, J. L. LIONS, R. TREMOLIERES, *Numerical Analysis of Variational Inequalities,* North-Holland, Amsterdam, 1981.
[5] N. J. HUANG, On the generalized implicit variational inequalities, *J. Math. Anal. Appl.* **216** (1997), 197–210.
[6] N. J. HUANG, M. R. BAI, Y. J. CHO, S. M. KANG, Generalized nonlinear mixed quasi-variational inequalities, *Comput. Math. Appl.* **40(2–3)** (2000), 205–215.
[7] N. J. HUANG, Y. P. LIU, Y. Y. TANG, M. R. BAI, On the generalized set-valued strongly nonlinear implicit variational inequalities, *Comput. Math. Appl.* **37(10)** (1999), 29–36.
[8] S. B. NADLER, Jr. Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969), 475–487.
[9] M. A. NOOR, Nonlinear quai-complemetarity problems, *Appl. Math. Lett.* **2(3)** (1989), 251–254.
[10] D. PASCALI, S. SBURLAN, *Nonlinear Mapping of Monontone Type,* Sijthoff & Noordhoof, The Netherlands, 1978.
[11] S. H. SHIM, S. M. KANG, N. J. HUANG, Y. J. CHO, Perturbed iterative algoriths with errors for completely generalized strongly nonlinear implicit quasivariational inclusions, *J. Inequal. Appl.* **5(4)** (2000), 381–395.
[12] A. H. SIDDIQI, Q. H. ANSARI, Strongly nonlinear quasi-varaitional inequalities, *J. Math. Anal. Appl.* **149(2)** (1990), 444–450.
[13] J. C. YAO, Existence of generalized variational inequalities, *Oper. Res. Lett.* **15** (1994), 35–40.
[14] GEORGE X. Z. YUAN, *KKM Theory and Applications in Nonlinear Analysis,* Marcel Dekker, New York, 1999.

(Received April 31, 2001)

Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P. R. China

Ya-ping Fang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P. R. China