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A NOTE ON WEYL'S INTERLACING INEQUALITY

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(*communicated by G. P. H. Styan*)

Abstract. We shall prove Weyl's interlacing inequality on one dimensional perturbation of Hermitian matrices using fundamental techniques.

The variational principles for eigenvalues of a matrix were discovered in connection with problems of physics. One of the famous work where many of these were used is *The Theory of Sound* by Lord Rayleigh, originally published in 1877, reprinted by Dover in 1945, [4]. The classic book on modern applied mathematics is *Methods of Mathematical Physics* by R. Courant and D. Hilbert [1], is replete with applications of variational principles. For more recent source, see [5].

In this short note we consider two interlacing sets of real numbers and shall prove that they can be realised as the eigenvalues of a Hermitian matrix A, and one of its one dimensional perturbation $B = A + tP$, where t is a nonnegative real number and P is a projection of rank 1. Alternate proofs of this can be seen in [3], [6] and [7] which are based on the theory of calculus of residues in complex analysis. We rather use an elementary technique to prove this. We begin with the following lemma due to Cauchy. For a proof the reader is refered to [2, page 268].

LEMMA 1. *Let* $-\infty < \lambda_1 < \eta_1 < \ldots < \lambda_n < \eta_n < \infty$. *Then*

$$\det\left(\frac{1}{\eta_i - \lambda_j}\right)_{i,j} = \frac{\prod_{i>j}(\eta_i - \eta_j)(\lambda_j - \lambda_i)}{\prod_{i,j}(\eta_i - \lambda_j)}.$$

LEMMA 2. *Let* z_i, x_i *and* t *be complex numbers. Then*

$$\det\left(D(z) - txx^*\right) = z_1 z_2 \ldots z_n - \sum_{j=1}^{n} t \prod_{i \neq j} z_i |x_j|^2,$$

where $D(z)$ *is the diagonal matrix with diagonal entries* $z_1, z_2, \ldots, z_n$ *and* $x = (x_1, x_2, \ldots, x_n)^T$.

Proof. By continuity, on replacing z_i by $z_i + \varepsilon$ if necessary, we can assume that all z_i are non-zero. The desired result is trivially true if $x = 0$. So we assume that

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$x \neq 0$. Note that for $k = x^*x$, the matrix $P = \frac{1}{k}xx^*$ is a projection of rank 1. Then since

$$tD(z)^{-1}xx^*(tD(z)^{-1}xx^*) = (tx^*D(z)^{-1}x)tD(z)^{-1}xx^*,$$

the only non-zero eigenvalue of $tD(z)^{-1}xx^*$ is $tx^*D(z)^{-1}x$. This implies that

$$1 - tx^*D(z)^{-1}x = 1 - t\frac{|x_1|^2}{z_1} - t\frac{|x_2|^2}{z_2} - \cdots - t\frac{|x_n|^2}{z_n}$$

is an eigenvalue of $I - tD(z)^{-1}xx^*$. All other eigenvalues of $I - tD(z)^{-1}xx^*$ are 1. Hence

$$\begin{aligned}\det(D(z) - txx^*) &= \det\left(D(z)(I - tD(z)^{-1}xx^*)\right)\\ &= \det(D(z))\det(I - tD(z)^{-1}xx^*)\\ &= z_1z_2\cdots z_n - \sum_{j=1}^{n} t\prod_{i\neq j} z_i|x_j|^2.\end{aligned}$$

This completes the proof. □

THEOREM 3. *Let* $-\infty < \lambda_1 \leqslant \eta_1 \leqslant \lambda_2 \leqslant \eta_2 \leqslant \ldots \leqslant \lambda_n \leqslant \eta_n < \infty$. *Let* A *be an* $n \times n$ *Hermitian matrix whose eigenvalues are* $\lambda_1, \lambda_2, \ldots, \lambda_n$. *Then there exists* $t \geqslant 0$ *and a projection* P *of rank 1 such that* $B = A + tP$ *has* $\eta_j's$ *as its eigenvalues.*

Proof. We shall prove the result when $-\infty < \lambda_1 < \eta_1 < \lambda_2 < \eta_2 < \ldots < \lambda_n < \eta_n < \infty$ and $t > 0$. The general result then follows on using continuity argument. We can assume that

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

For the existence of a matrix B such that its eigenvalues are $\eta_j's$, we must have

$$t = \operatorname{tr}(B - A) = \sum_{i=1}^{n}(\eta_i - \lambda_i).$$

(Here $\operatorname{tr}(B - A)$ denote the trace of the matrix $(B - A)$). Note that any projection P of rank 1 can be written as

$$P = (x_i\overline{x}_j)_{i,j} \ \text{ with } \ \sum_{i=1}^{n}|x_i|^2 = 1,$$

where the $x_i's$ are complex numbers. Now, for the required result we need to show the existence of suitable $x_i's$. The numbers $\eta_k's$ would be the eigenvalues of $B = A + tP$ if and only if the $\det(\eta_k I - B) = 0$, for $k = 1, 2, \ldots, n$, i.e.,

$$\det\begin{pmatrix} \eta_k - \lambda_1 - t|x_1|^2 & -tx_1\overline{x}_2 & \cdots & -t\overline{x}_n x_1 \\ -t\overline{x}_1 x_2 & \eta_k - \lambda_2 - t|x_2|^2 & \cdots & -t\overline{x}_n x_2 \\ -t\overline{x}_1 x_3 & -t\overline{x}_2 x_3 & \cdots & -t\overline{x}_n x_3 \\ \vdots & \vdots & \ddots & \vdots \\ -t\overline{x}_1 x_n & -t\overline{x}_2 x_n & \cdots & \eta_k - \lambda_n - t|x_n|^2 \end{pmatrix} = 0.$$

Therefore by Lemma 2, we get

$$(\eta_k-\lambda_1)(\eta_k-\lambda_2)\cdots(\eta_k-\lambda_n)-t\sum_{j=1}^{n}\prod_{i\neq j}(\eta_k-\lambda_i)|x_j|^2=0.$$

From the strict inequalities between the $\eta_k's$ and $\lambda_k's$ and with $t\neq 0$, we obtain

$$\frac{|x_1|^2}{\eta_k-\lambda_1}+\frac{|x_2|^2}{\eta_k-\lambda_2}+\cdots+\frac{|x_n|^2}{\eta_k-\lambda_n}=\frac{1}{t}, \tag{1}$$

$k=1,2,\ldots,n$. By Lemma 1, the $\det\left(\frac{1}{\eta_i-\lambda_j}\right)_{i,j}\neq 0$. Therefore the above system of linear equations has a unique real solution in $|x_j|^2$. To settle the problem, we claim that (1) has a unique positive solution. This can be seen by rewriting the system (1) trivially in the following way, i.e.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\eta_2-\lambda_1} & \frac{1}{\eta_2-\lambda_2} & \cdots & \frac{1}{\eta_2-\lambda_n} \\ \frac{1}{\eta_3-\lambda_1} & \frac{1}{\eta_3-\lambda_2} & \cdots & \frac{1}{\eta_3-\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\eta_n-\lambda_1} & \frac{1}{\eta_n-\lambda_2} & \cdots & \frac{1}{\eta_n-\lambda_n} \end{pmatrix} \begin{pmatrix} |x_1|^2\frac{1}{\eta_1-\lambda_1} \\ |x_2|^2\frac{1}{\eta_1-\lambda_2} \\ |x_3|^2\frac{1}{\eta_1-\lambda_3} \\ \vdots \\ |x_n|^2\frac{1}{\eta_1-\lambda_n} \end{pmatrix} = t^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

For any invertible matrix U, we write (2) as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\eta_2-\lambda_1} & \frac{1}{\eta_2-\lambda_2} & \cdots & \frac{1}{\eta_2-\lambda_n} \\ \frac{1}{\eta_3-\lambda_1} & \frac{1}{\eta_3-\lambda_2} & \cdots & \frac{1}{\eta_3-\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\eta_n-\lambda_1} & \frac{1}{\eta_n-\lambda_2} & \cdots & \frac{1}{\eta_n-\lambda_n} \end{pmatrix} UU^{-1} \begin{pmatrix} |x_1|^2\frac{1}{\eta_1-\lambda_1} \\ |x_2|^2\frac{1}{\eta_1-\lambda_2} \\ |x_3|^2\frac{1}{\eta_1-\lambda_3} \\ \vdots \\ |x_n|^2\frac{1}{\eta_1-\lambda_n} \end{pmatrix} = t^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Take $U=\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$, the above system becomes

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \frac{1}{\eta_2-\lambda_2} & \cdots & \frac{1}{\eta_2-\lambda_n} \\ 1 & \frac{1}{\eta_3-\lambda_2} & \cdots & \frac{1}{\eta_3-\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{\eta_n-\lambda_2} & \cdots & \frac{1}{\eta_n-\lambda_n} \end{pmatrix} \begin{pmatrix} \left(\sum_{1}^{n}|x_i|^2\frac{1}{\eta_1-\lambda_i}\right) \\ |x_2|^2\frac{\lambda_2-\lambda_1}{\eta_1-\lambda_2} \\ |x_3|^2\frac{\lambda_3-\lambda_1}{\eta_1-\lambda_3} \\ \vdots \\ |x_n|^2\frac{\lambda_n-\lambda_1}{\eta_1-\lambda_n} \end{pmatrix} = t^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

Once again (3) can trivially be seen equivalent to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{\eta_2-\lambda_2} & \cdots & \frac{1}{\eta_2-\lambda_n} \\ 0 & \frac{1}{\eta_3-\lambda_2} & \cdots & \frac{1}{\eta_3-\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{\eta_n-\lambda_2} & \cdots & \frac{1}{\eta_n-\lambda_n} \end{pmatrix} \begin{pmatrix} \left(\sum_1^n |x_i|^2 \frac{1}{\eta_1-\lambda_i}\right) \\ -|x_2|^2 \frac{1}{\eta_1-\lambda_2} \\ -|x_3|^2 \frac{1}{\eta_1-\lambda_3} \\ \vdots \\ -|x_n|^2 \frac{1}{\eta_1-\lambda_n} \end{pmatrix} = t^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{4}$$

The claim follows on using induction hypothesis. This completes the proof. □

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