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COMPARISON INEQUALITIES FOR PARTIAL FINITE DIFFERENCE EQUATIONS

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Abstract. The aim of the present note is to establish some new comparison inequalities related to certain partial finite difference equations by using a fairly elementary analysis.

1. Introduction

The comparison inequalities on finite difference equations are very useful in the development of the theory of finite difference equations, the theory of convergence of the iterative methods for solving fixed point equations, and the convergence theory of finite difference methods for solving problems for ordinary and partial differential equations. In the past few years many new comparison inequalities see [1,3,4,6-10] have been investigated in order to study the behavior of solutions of various types of finite difference equations. The main purpose of this note is to present some basic comparison inequalities which can be used as tools in the analysis of certain nonlinear partial finite difference equations.

2. Main results

In what follows we denote by R the set of real numbers and $M_0 = \{m_0, m_0 + 1, \dots\}, N_0 = \{n_0, n_0 + 1, \dots\}$, where m_0, n_0 are nonnegative integers. For basic elementary theory of finite difference equations, see [2,5].

Our first result deals with the fundamental finite difference inequality which provides the key to the proof of our main results.

THEOREM 1. *Let $f(m,n,r)$ be defined for $m \in M_0$, $n \in N_0$, $0 \leqslant r < \infty$ and nondecreasing with respect to r for fixed $m \in M_0$, $n \in N_0$. Let $u(m,n)$ and $v(m,n)$ be two functions defined for $m \in M_0$, $n \in N_0$ and $u(m_0,n_0) \leqslant v(m_0,n_0)$. Assume further that*

$$u(m+1,n+1) \leqslant f(m,n,u(m,n)), \tag{1}$$

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$$v(m+1,n+1) \geqslant f(m,n,v(m,n)), \tag{2}$$

for $m \in M_0$, $n \in N_0$. *Then*

$$u(m,n) \leqslant v(m,n), \tag{3}$$

for $m \in M_0$, $n \in N_0$.

Proof. Since $u(m_0,n_0) \leqslant v(m_0,n_0)$, from the nondecreasing character of f we obtain

$$\begin{aligned} u(m_0+1,n_0+1) &\leqslant f(m_0,n_0,u(m_0,n_0)) \\ &\leqslant f(m_0,n_0,v(m_0,n_0)) \\ &\leqslant v(m_0+1,n_0+1). \end{aligned}$$

If the inequality (3) is fulfilled for $m = m_0 + i$, $n = n_0 + i$ $(i = 2,3,\ldots,k)$, it follows by the nondecreasing character of f that

$$\begin{aligned} u(m_0+k+1,n_0+k+1) &\leqslant f(m_0+k,n_0+k,u(m_0+k,n_0+k)) \\ &\leqslant f(m_0+k,n_0+k,v(m_0+k,n_0+k)) \\ &\leqslant v(m_0+k+1,n_0+k+1). \end{aligned}$$

Hence by mathematical induction we obtain the result.

Our main results are embodied in the following theorems.

THEOREM 2. *Suppose that the functions* $W_1(m,n,r), W_2(m,n,r)$ *be nonnegative and defined for* $m \in M_0$, $n \in N_0$, $0 \leqslant r < \infty$ *and nondecreasing with respect to* r *for any fixed* $m \in M_0$, $n \in N_0$. *Let* $z(m,n)$ *be a function defined for* $m \in M_0$, $n \in N_0$ *and that*

$$W_2(m,n,z(m,n)) \leqslant z(m+1,n+1) \leqslant W_1(m,n,z(m,n)), \tag{4}$$

for $m \in M_0$, $n \in N_0$. *Let* $u(m,n)$ *and* $v(m,n)$ *be the solutions of the difference equations*

$$u(m+1,n+1) = W_1(m,n,u(m,n)), \quad u(m_0,n_0) = u_0, \tag{5}$$

$$v(m+1,n+1) = W_2(m,n,v(m,n)), \quad v(m_0,n_0) = v_0, \tag{6}$$

and suppose that $v_0 \leqslant z(m_0,n_0) \leqslant u_0$. *Then*

$$v(m,n) \leqslant z(m,n) \leqslant u(m,n), \tag{7}$$

for $m \in M_0$, $n \in N_0$.

Proof. Applying Theorem 1 to the second part of (4) and (5) we obtain the right half of the inequality in (7). A similar argument yields the left half of the inequality (7).

The next theorem is useful in the study of behavior of solutions of the following finite difference equations

$$x(m+1,n+1) = g(m,n,x(m,n)), \quad x(m_0,n_0) = x_0, \tag{8}$$

$$y(m+1,n+1) = h(m,n,y(m,n)), \quad y(m_0,n_0) = y_0, \tag{9}$$

where $x(m,n), y(m,n), g(m,n,r), h(m,n,r)$ are defined for $m \in M_0$, $n \in N_0$, $0 \leqslant r < \infty$.

THEOREM 3. *Let the functions $W_1(m,n,r)$ and $W_2(m,n,r)$ be as in Theorem 2. Suppose that the functions g and h in (8) and (9) satisfy the condition*

$$W_2(m,n,|x-y|) \leqslant |g(m,n,x)-h(m,n,y)| \leqslant W_1(m,n,|x-y|), \tag{10}$$

for $m \in M_0$, $n \in N_0$. Let $u(m,n)$ and $v(m,n)$ be solutions of the equations (5) and (6) for $m \in M_0$, $n \in N_0$. Let $x(m,n)$ and $y(m,n)$ be solutions of the equations (8) and (9) and assume that $v_0 \leqslant |x_0-y_0| \leqslant u_0$. Then

$$v(m,n) \leqslant |x(m,n)-y(m,n)| \leqslant u(m,n), \tag{11}$$

for $m \in M_0$, $n \in N_0$.

Proof. Let $z(m,n)=|x(m,n)-y(m,n)|$. Then $z(m_0,n_0)=|x(m_0,n_0)-y(m_0,n_0)| \leqslant u(m_0,n_0)$. On account of the nondecreasing nature of $W_1(m,n,r)$ we obtain

$$\begin{aligned} z(m_0+1,n_0+1) &= |x(m_0+1,n_0+1)-y(m_0+1,n_0+1)| \\ &= |g(m_0,n_0,x(m_0,n_0))-h(m_0,n_0,y(m_0,n_0))| \\ &\leqslant W_1(m_0,n_0,|x(m_0,n_0)-y(m_0,n_0)|) \\ &\leqslant W_1(m_0,n_0,u(m_0,n_0)) \\ &= u(m_0+1,n_0+1). \end{aligned}$$

If the inequality $z(m,n) \leqslant u(m,n)$ is fulfilled for $m=m_0+i$, $n=n_0+i$ $(i=2,3,\dots,k)$, then it follows by the nondecreasing nature of $W_1(m,n,r)$ that

$$\begin{aligned} z(m_0+k+1,n_0+k+1) &= |x(m_0+k+1,n_0+k+1)-y(m_0+k+1,n_0+k+1)| \\ &= |g(m_0+k,n_0+k,x(m_0+k,n_0+k)) \\ &\quad - h(m_0+k,n_0+k,y(m_0+k,n_0+k))| \\ &\leqslant W_1(m_0+k,n_0+k,|x(m_0+k,n_0+k)-y(m_0+k,n_0+k)|) \\ &\leqslant W_1(m_0+k,n_0+k,u(m_0+k,n_0+k)) \\ &= u(m_0+k+1,n_0+k+1). \end{aligned}$$

Hence by mathematical induction we obtain $|x(m,n)-y(m,n)| \leqslant u(m,n)$, for $m \in M_0$, $n \in N_0$. The proof of the left half of the inequality in (11) is similar.

3. An application

As an application, we present the following theorem which deals with the dependency of solutions of a certain partial difference equation on initial values.

THEOREM 4. *Let $F(m,n,z)$ be defined for $m \in M_0$, $n \in N_0$, $|z|<\infty$ and satisfy an inequality*

$$|F(m,n,x)-F(m,n,y)| \leqslant W(m,n,|x-y|), \tag{12}$$

for $m \in M_0$, $n \in N_0$, $x, y \in R$, *where* $W(m,n,r)$, $m \in M_0$, $n \in N_0$, $0 \leqslant r < \infty$ *is nonnegative and nondecreasing with respect to* r *for fixed* $m \in M_0$, $n \in N_0$. *Let* $z(m,n,m_0,n_0,z_i)$ $(i = 1,2)$ *be solutions of*

$$z(m+1,n+1) = F(m,n,z(m,n)), \tag{13}$$

with the given initial conditions

$$z(m_0,n_0,m_0,n_0,z_i) = z_i \tag{14}$$

for $i = 1,2$. *Let* $r(m,n)$ *be a solution of the equation*

$$r(m+1,n+1) = W(m,n,r(m,n)), \quad r(m_0,n_0) = r_0, \tag{15}$$

and $|z_1 - z_2| \leqslant r_0$. *Then we obtain a relation*

$$|z(m,n,m_0,n_0,z_1) - z(m,n,m_0,n_0,z_2)| \leqslant r(m,n), \tag{16}$$

for $m \in M_0$, $n \in N_0$.

The proof follows by a suitable application of Theorem 3 and here we omit the details.

In concluding we note that the results given in the above theorems can be extended very easily for functions of more than two independent variables. We leave the formulation of such results to the reader to fill in where needed.

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