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SOME REFINEMENTS OF SLATER, PEČARIĆ AND WANG–WANG'S INEQUALITIES

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Abstract. In this article, we establish some refinements of Slater, Pečarić and Wang-Wang's inequalities.

1. Introduction

Throughout, let $\mathbf{R}$ be the set of some real numbers, $\alpha_i \geqslant 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, and let n be a positive integer. Let

$$G_n = \prod_{i=1}^{n} x_i^{\alpha_i} \quad \text{and} \quad g_n = \prod_{i=1}^{n} x_i^{\frac{1}{n}}, \quad \text{where} \quad x_i > 0, \qquad i = 1, \dots, n,$$

be the weighted and unweighted geometric means and

$$H_n = \left(\sum_{i=1}^{n} \frac{\alpha_i}{x_i} \right)^{-1} \quad \text{and} \quad h_n = \left(\sum_{i=1}^{n} \frac{1}{n x_i} \right)^{-1}, \quad \text{where} \quad x_i > 0, \quad i = 1, \dots, n,$$

be the weighted and unweighted harmonic means of $x_1, \dots, x_n$, respectively.

Similarly, let

$$G_n' = \prod_{i=1}^{n} (1 - x_i)^{\alpha_i} \quad \text{and} \quad g_n' = \prod_{i=1}^{n} (1 - x_i)^{\frac{1}{n}}, \quad \text{where} \quad x_i \in (-\infty, 1), \quad i = 1, \dots, n,$$

be the weighted and unweighted geometric means and

$$H_n' = \left(\sum_{i=1}^{n} \frac{\alpha_i}{1 - x_i} \right)^{-1} \quad \text{and} \quad h_n' = \left(\sum_{i=1}^{n} \frac{1}{n(1 - x_i)} \right)^{-1}, \text{ where } x_i \in (0, 1),\ i = 1, \dots, n,$$

be the weighted and unweighted harmonic means of $1 - x_1, \dots, 1 - x_n$, respectively. In [5], Wang and Wang proved the following Wang-Wang inequality:

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THEOREM A. *If* $x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, \ldots, n$, *then*

$$\frac{h_n}{h_n'} \leqslant \frac{g_n}{g_n'} \tag{1.1}$$

with equality holding if and only if $x_1 = \ldots = x_n$.

In [1], Alzer proved the following generalization of Theorem A:

THEOREM B. *If* $x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, \ldots, n$, *then*

$$\frac{H_n}{H_n'} \leqslant \frac{G_n}{G_n'} \tag{1.2}$$

with equality holding if and only if $x_1 = \ldots = x_n$.

In [4], Slater proved the following companion to Jensen's inequality:

THEOREM C. *Suppose that* f *is convex and increasing (or decreasing) on* (a,b). *Then for* $x_1, \ldots, x_n \in (a,b)$, $p_1, \ldots, p_n \geqslant 0$, $\sum_{i=1}^{n} p_i > 0$ *and* $\sum_{i=1}^{n} p_i f_+'(x_i) \neq 0$, *we have*

$$\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \leqslant f\left(\frac{\sum_{i=1}^{n} p_i x_i f_+'(x_i)}{\sum_{i=1}^{n} p_i f_+'(x_i)}\right). \tag{1.3}$$

The inequality (1.3) remains true if at any occurrence of $f_+'(x)$ *we write instead any value in the interval* $[f_-'(x), f_+'(x)]$, *where* $f_-'(x)$ *and* $f_+'(x)$ *are left and right derivatives of* f *at* x, *respectively.*

If we choose $f(x) = \ln \frac{1-x}{x}$ $\left(x \in \left(0, \frac{1}{2}\right)\right)$ and $x_i \in \left(0, \frac{1}{2}\right)$ $(i = 1, \ldots, n)$ in Theorem C, then the inequality (1.3) yields the Wang-Wang inequality (1.2).

In [4], Slater also proved the following integral analog of (1.3):

THEOREM D. *Suppose that* f *is convex and increasing on* (a,b). *Suppose also that* $X : (E, \xi) \to (a,b)$ *is measurable, that* $f \circ X$, $f_+' \circ X$ *and* $(f_+' \circ X)X$ *are in* $L(\mu)$ *and that* $\int_E (f_+' \circ X) d\mu > 0$. *Then*

$$\int_E (f \circ X) d\mu \leqslant f\left(\frac{\int_E (f_+' \circ X) X d\mu}{\int_E (f_+' \circ X) d\mu}\right). \tag{1.4}$$

If, in addition we assume that f *is strictly convex, then equality holds in (1.4) if and only if* X *is constant* μ *a.e.*

In [2], Pečarić proved the following simple generalization of (1.3):

THEOREM E. *Suppose that* f *is convex on* (a,b). *If, for* $x_1,\dots,x_n \in (a,b)$ *and* $p_i,\dots,p_n \geqslant 0$ *with* $\sum_{i=1}^{n} p_i > 0$, *we have*

$$\sum_{i=1}^{n} p_i f'_+(x_i) \neq 0, \qquad \frac{\sum_{i=1}^{n} p_i x_i f'_+(x_i)}{\sum_{i=1}^{n} p_i f'_+(x_i)} \in (a,b), \tag{1.5}$$

then the inequality (1.3) holds.

In section 2, we establish some refinements of (1.3)–(1.5). In section 3, we establish a refinement of Wang-Wang inequality.

2. Some refinements of Slater and Pečarić's inequalities

In order to prove our results, we need the following lemmas:

LEMMA 1. ([3, P. 12]) *Let* $f:(a,b)\to\mathbf{R}$ *be a convex function,* $x_0 \in (a,b), m \in [f'_-(x_0), f'_+(x_0)]$. *Then*

$$f(x) \geqslant f(x_0) + m(x - x_0) \tag{2.1}$$

for all $x \in (a,b)$.

LEMMA 2. *Let* $f:(a,b]\to\mathbf{R}$ *be a convex function such that* $f'_-(b)$ *exists. Then*

$$f(x) \geqslant f(b) + f'_-(b)(x-b) \tag{2.2}$$

for all $x \in (a,b]$.

Proof. The inequality (2.2) is trivial if $x=b$. If $x<b$, using the convexity of f, we have, for $x<t<b$ that

$$\frac{f(b)-f(x)}{b-x} \leqslant \frac{f(b)-f(t)}{b-t}.$$

Letting $t \to b$, we have

$$\frac{f(b)-f(x)}{b-x} \leqslant f'_-(b).$$

Hence (2.2) holds.

Now, we are ready to state and prove our results.

THEOREM 1. *Let* $f:(a,b]\to\mathbf{R}$ *be a convex function. Suppose* $x_i \in (a,b]$ $(i=1,\dots,n)$, *let* $y_i \in [f'_-(x_i), f'_+(x_i)]$ *when* $x_i \neq b$, $(i=1,\dots,n)$ *and let* $y_i = f'_-(b)$ *when* $x_i = b$ $(i=1,\dots,n)$. *If* $\gamma \in (a,b]$ *is such that* $\gamma \sum_{i=1}^{n} \alpha_i y_i \geqslant \sum_{i=1}^{n} \alpha_i x_i y_i$ *and the function* F *is defined on* $[0,1]$ *by*

$$F(t) = \sum_{i=1}^{n} \alpha_i f(t\gamma + (1-t)x_i), \tag{2.3}$$

then F *is convex, increasing on* $[0,1]$ *and*

$$\sum_{i=1}^{n} \alpha_i f(x_i) = F(0) \leqslant F(t) \leqslant F(1) = f(\gamma). \tag{2.4}$$

Proof. Since f is convex, it follows from Lemma 1 and Lemma 2 that

$$f(t\gamma + (1-t)x_i) \geqslant f(x_i) + t(\gamma - x_i)y_i$$

for all $t \in [0,1]$ and $x_i \in (a,b]$ $(i = 1,\ldots,n)$. Now,

$$\begin{aligned} F(t) &= \sum_{i=1}^{n} \alpha_i f(t\gamma + (1-t)x_i) \\ &\geqslant \sum_{i=1}^{n} \alpha_i f(x_i) + t\left(\gamma \sum_{i=1}^{n} \alpha_i y_i - \sum_{i=1}^{n} \alpha_i x_i y_i\right) \qquad (2.5) \\ &\geqslant \sum_{i=1}^{n} \alpha_i f(x_i) = F(0), \end{aligned}$$

for all $t \in [0,1]$.

Note that the composition of a convex function and a linear function is convex. Also note that a positive constant multiple of a convex function and a sum of convex functions are convex. Hence F is convex on $[0,1]$. If $0 < x < y \leqslant 1$ then it follows from the convexity of F and (2.5) that

$$\frac{F(y) - F(x)}{y - x} \geqslant \frac{F(x) - F(0)}{x - 0} \geqslant 0, \tag{2.6}$$

which shows that F is increasing on $(0,1]$ and (2.4) holds. This completes the proof.

REMARK 1. Simply using Lemma 1, we see that Theorem 1 remains true if the interval $(a,b]$ is replaced by (a,b). Let f be an increasing (or decreasing) convex function on (a,b) and $\alpha_i = \frac{p_i}{\sum_{j=1}^{n} p_j}$, where $p_i \geqslant 0$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} p_i > 0$. If $x_i \in (a,b)$, $y_i = f'_+(x_i)$ $(i = 1,\ldots,n)$, $\sum_{i=1}^{n} p_i f'_+(x_i) \neq 0$, $\gamma = \frac{\sum_{i=1}^{n} p_i x_i f'_+(x_i)}{\sum_{i=1}^{n} p_i f'_+(x_i)}$, then, for all $t \in [0,1]$, we have

$$\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} = F(0) \leqslant F(t) \leqslant F(1) = f\left(\frac{\sum_{i=1}^{n} p_i x_i f'_+(x_i)}{\sum_{i=1}^{n} p_i f'_+(x_i)}\right) \tag{2.7}$$

which refines (1.3). Thus Theorem 1 gives a refinement of Theorem C.

REMARK 2. Let $\alpha_i, p_i, x_i, g, y_i$ be defined as in Remark 1 and let f be convex on (a,b), $\sum_{i=1}^{n} p_i f'_+(x_i) \neq 0$ and $\gamma = \frac{\sum_{i=1}^{n} p_i x_i f'_+(x_i)}{\sum_{i=1}^{n} p_i f'_+(x_i)} \in (a,b)$. Then (2.7) still holds. Thus Theorem 1 gives a refinement of Theorem E.

THEOREM 2. *Suppose that $f : (a,b) \to \mathbf{R}$ is convex on (a,b) and $X : (E,\xi) \to (a,b)$ is measurable such that $f \circ X$, $f'_+ \circ X$ and $(f'_+ \circ X)X$ are in $L(\mu)$. If $\eta \in (a,b)$ is such that $\eta \int_E (f'_+ \circ X) d\eta \geqslant \int_E (f'_+ \circ X) d\mu$ and the function G is defined on $[0,1]$ by*

$$G(t) = \int_E f(t\eta + (1-t)X(s)) d\mu(s), \tag{2.8}$$

then G is convex, increasing on $[0,1]$, and

$$\int_E (f \circ X) d\mu = G(0) \leqslant G(t) \leqslant G(1) = f(\eta). \tag{2.9}$$

Proof. That G is convex follows immediately from the convexity of f. Also, it follows from Lemma 1 that

$$f(t\eta + (1-t)X(s)) \geqslant (f \circ X)(s) + t(\eta - X(s))(f'_+ \circ X)(s)$$

for all $t \in [0,1]$ and $s \in E$. Now, integrating with respect to μ, we have

$$\begin{aligned} G(t) &= \int_E f(t\eta + (1-t)X(s)) d\mu(s) \\ &\geqslant \int_E (f \circ X) d\mu + t\left[\eta \int_E (f'_+ \circ X) d\mu - \int_E (f'_+ \circ X) X d\mu\right] \\ &\geqslant \int_E (f \circ X) d\mu = G(0) \end{aligned}$$

for all $t \in [0,1]$. As noted in the proof of Theorem 1, we see that G is increasing on $[0,1]$. This completes the proof.

REMARK 3. In Theorem 2, let f be convex and increasing such that $\int_E (f'_+ \circ X) d\mu > 0$ and $\eta = \frac{\int_E (f'_+ \circ X) X d\mu}{\int_E (f'_+ \circ X) d\mu}$. Then

$$\int_E (f \circ X) d\mu = G(0) \leqslant G(t) \leqslant G(1) = f\left(\frac{\int_E (f'_+ \circ X) d\mu}{\int_E (f'_+ \circ X) d\mu}\right)$$

which refines (1.4). Thus Theorem 2 is a refinement of Theorem D.

3. A refinement of Wang-Wang inequality

THEOREM 3. *Let* $\alpha > 0$, $\beta > 0$, $I = \left(0, \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right)$, *and let* $x_i \in I$ $(i = 1,\ldots,n)$. *If* $\gamma \in I$ *and* $\gamma \leqslant \frac{(\alpha-\beta)H_nH_n'+\beta H_n}{\alpha H_n'+\beta H_n}$. *Define* P *on* $[0,1]$ *by*

$$P(t) = \frac{\left[\prod_{i=1}^{n}(t\gamma + (1-t)x_i)^{\alpha_i}\right]^{\alpha}}{\left[\prod_{i=1}^{n}(1-t\gamma-(1-t)x_i)^{\alpha_i}\right]^{\beta}}. \tag{3.1}$$

Then P *is decreasing on* $[0,1]$ *and*

$$\frac{\gamma_\alpha}{(1-\gamma)^{\beta}} = P(1) \leqslant P(t) \leqslant P(0) = \frac{G_n^{\alpha}}{G_n'^{\beta}}. \tag{3.2}$$

Proof. Let $f(x) = \beta\ln(1-x) - \alpha\ln x$, $x \in I$. Then

$$f''(x) = \frac{[\sqrt{\alpha}(1-x)-\sqrt{\beta}x][\sqrt{\alpha}(1-x)+\sqrt{\beta}x]}{x^2(1-x)^2} \geqslant 0$$

for $x \in I$. Hence f is convex on I. Now

$$y_i = f'(x_i) = \frac{-\beta}{1-x_i} - \frac{\alpha}{x_i}, \quad (i = 1,\ldots,n),$$
$$\sum_{i=1}^{n}\alpha_i y_i = \frac{-\alpha H_n' - \beta H_n}{H_nH_n'} < 0,$$
$$\sum_{i=1}^{n}\alpha_i x_i y_i = (\beta-\alpha) - \frac{\beta}{H_n'}$$

and

$$\frac{\sum_{i=1}^{n}\alpha_i x_i y_i}{\sum_{i=1}^{n}\alpha_i y_i} = \frac{(\alpha-\beta)H_nH_n'+\beta H_n}{\alpha H_n'+\beta H_n}.$$

Since $\gamma \in I$ and $\gamma \leqslant \frac{(\alpha-\beta)H_nH_n'+\beta H_n}{\alpha H_n'+\beta H_n}$ one has $\gamma\sum_{i=1}^{n}\alpha_i y_i \geqslant \sum_{i=1}^{n}\alpha_i x_i y_i$. It follows from Theorem 1 that $F(t)$ is increasing on $[0,1]$ and

$$\frac{1}{F(t)} = P(t) = \frac{\left[\prod_{i=1}^{n}(t\gamma + (1-t)x_i)^{\alpha_i}\right]^{\alpha}}{\left[\prod_{i=1}^{n}(1-t\gamma-(1-t)x_i)^{\alpha_i}\right]^{\beta}} > 0.$$

Hence $\frac{1}{F(x)}$ is decreasing on $[0,1]$, and

$$\frac{\gamma^{\alpha}}{(1-\gamma)^{\beta}} = P(1) \leqslant P(t) \leqslant P(0) = \frac{G_n^{\alpha}}{G_n'^{\beta}}.$$

This completes the proof.

REMARK 4. In Theorem 3, let $\gamma = \frac{(\alpha-\beta)H_nH_n'+\beta H_n}{\alpha H_n'+\beta H_n}$. Then

$$P(t) = \frac{(\alpha H_n' + \beta H_n)^{\beta-\alpha}\left[\prod_{i=1}^{n}[t(\alpha-\beta)H_nH_n' + t\beta H_n + (1-t)(\alpha H_n' + \beta H_n)x_i]^{\alpha_i}\right]^{\alpha}}{\left[\prod_{i=1}^{n}[(\alpha H_n' + \beta H_n) - t(\alpha-\beta)H_nH_n' - t\beta H_n - (1-t)(\alpha H_n' + \beta H_n)x_i]^{\alpha_i}\right]^{\beta}} \tag{3.3}$$

is decreasing on $[0,1]$, thus

$$\frac{(\alpha H_n' + \beta H_n)^{\beta-\alpha}[(\alpha-\beta)H_nH_n' + \beta H_n]^{\alpha}}{[\alpha H_n' - (\alpha-\beta)H_nH_n']^{\beta}} = P(1) \leqslant P(t) \leqslant P(0) = \frac{G_n^{\alpha}}{G_n'^{\beta}}. \tag{3.4}$$

If we choose $\alpha = \beta = 1$, then $I = \left(0, \frac{1}{2}\right]$, and (3.3) becomes

$$P(t) = \frac{\prod_{i=1}^{n}[tH_n + (1-t)(H_n + H_n')x_i]^{\alpha_i}}{\prod_{i=1}^{n}[(H_n + H_n') - tH_n - (1-t)(H_n + H_n')x_i]^{\alpha_i}} \tag{3.5}$$

and (3.4) implies

$$\frac{H_n}{H_n'} = P(1) \leqslant P(t) \leqslant P(0) = \frac{G_n}{G_n'}. \tag{3.6}$$

Thus Theorem 3 gives a refinement of Theorem B.

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