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L^p INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. Let $f(z)$ be an entire function of exponential type τ and for any complex number ζ, let $D_\zeta[f(z)] = \tau f(z) + i(1-\zeta)f'(z)$ be the polar derivative of $f(z)$, with respect to ζ. This definition is due to Rahman and Schmeisser [10]. Since $\lim_{\zeta\to\infty} \frac{D_\zeta[f(z)]}{-i\zeta} = f'(z)$, the polar derivative is a generalization of the ordinary derivative. In this paper we obtain L^p inequalities for the polar derivative of entire functions of exponential type satisfying $f(z) \equiv e^{i\tau z}\{\overline{f(\bar{z})}\}$ and for functions satisfying $f(z) \equiv e^{i\tau z} f(-z)$. Our results generalize some of the known results.

1. Introduction and statement of results

An entire function $f(z)$ is said to be of exponential type τ if it is of order less than 1 or it is of order 1 and type less than or equal to τ. We will denote this class of functions by ε_τ. For $f \in \varepsilon_\tau$, define $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. The indicator function $h_f(\theta)$ of f is defined by

$$h_f(\theta) = \lim_{r\to\infty} \sup \frac{\log|f(re^{i\theta})|}{r}.$$

A classical result of Bernstein (see Boas [1, p. 206]) states that if $f \in \varepsilon_\tau$ and if $\|f\| = 1$, then

$$\|f'\| \leqslant \tau. \tag{1}$$

It was proved by Boas [2] that if $h_f(\pi/2) = 0$ and $f(x+iy) \neq 0$ for $y > 0$, then (1.1) can be replaced by

$$|f'(x)| \leqslant \tau/2, \quad -\infty < x < \infty. \tag{2}$$

For $f \in \varepsilon_\tau$, we define with respect to a complex number ζ, the function $D_\zeta[f]$ as

$$D_\zeta[f(z)] = \tau f(z) + i(1-\zeta)f'(z).$$

The above definition is due to Rahman and Schmeisser [10].

Note that $\lim_{\zeta\to\infty} \frac{D_\zeta[f(z)]}{\zeta} = -if'(z)$.

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Let $L^p(\mathbb{R})$, $1 \leqslant p < \infty$ denote the class of measurable functions f, for which $\int_{-\infty}^{\infty} |f(x)|^p\, dx$ is finite. It is well known (see Boas [1, p. 211] that if $f(z)$ is an entire function of exponential type τ belonging to $L^p(\mathbb{R}), 1 \leqslant p < \infty$, then

$$\|f'\|_p = \left(\int_{-\infty}^{\infty} |f'(x)|^p\, dx\right)^{1/p} \leqslant \tau \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p} = \tau \|f\|_p. \tag{3}$$

The above inequality is clearly a generalization of inequality (1.1). For generalizations of inequality (1.3), see Rahman and Schmeisser [11].

As a generalization of the inequality (1.2), Rahman [9, Theorem 2] proved

THEOREM A. *If $f(z)$ is an entire function of exponential type τ belonging to $L^p(\mathbb{R}), 1 \leqslant p < \infty$, $h_f(\pi/2) = 0$ and $f(z) \neq 0$ for $\Im z > 0$, then for $p \geqslant 1$, we have*

$$\|f'\|_p \leqslant C_p^{1/p}\, \tau\, \|f\|_p, \tag{4}$$

where $C_p = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^p\, d\alpha = 2^{-p}\sqrt{\pi}\,\Gamma(\frac{1}{2}p + 1)/\Gamma(\frac{1}{2}p - 1)$.

The above inequality for $p > 0$ was extended by Rahman and Schmeisser [11, Corollary 3]. The inequality analogous to (1.3) for functions of exponential type not vanishing in $\Im z > k, (k \leqslant 0)$ was obtained by Govil and Rahman [6, Theorem 8]. For functions of exponential type τ satisfying $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, Govil and Jain [5, inequality (6)] proved

THEOREM B. *If $f(z)$ is an entire function of exponential type τ belonging to $L^p, (1 \leqslant p < \infty)$ on the real axis, $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, then for $p \geqslant 1$,*

$$\frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p} \leqslant \left(\int_{-\infty}^{\infty} |f'(x)|^p\, dx\right)^{1/p} \leqslant \tau C_p^{1/p} \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}, \tag{5}$$

where $C_p = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha$.

In this paper, we firstly present the following generalization of Theorem B.

THEOREM 1. *Let $f(z)$ be an entire function of exponential type belonging to $L^p\ (1 \leqslant p < \infty)$ on the real axis. If $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, then for $p \geqslant 1$ and all complex numbers ζ,*

$$\begin{aligned} \frac{\tau}{2}\big||\zeta| - 1\big| \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p} &\leqslant \left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\, dx\right)^{1/p} \\ &\leqslant C_p^{1/p}\tau(|\zeta| + 1) \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p} \end{aligned} \tag{6}$$

where $C_p = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha$ is as defined in Theorem B.

Theorem B is clearly a special case of Theorem 1, because if we divide throughout by $|\zeta|$ in (1.6) and make $|\zeta| \to \infty$, we get (1.5). Theorem B can also be obtained from Theorem 1, by taking $\zeta = 0$ in (1.6) and noting that $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, implies $|f'(x)| = |\tau f(x) + i f'(x)| = \big\{|D_\zeta[f(x)]|\big\}_{\zeta=0}$.

Since $\lim_{p\to\infty} C_p = 1/2$, if we make $p \to \infty$ in (1.6), we get

COROLLARY 1. *Let* $f(z)$ *be an entire function of exponential type* τ *and let* $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. *If* $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, *then for all complex numbers* ζ,

$$\frac{\tau}{2}\big||\zeta|-1\big|\|f\| \leqslant \|D_\zeta[f]\| \leqslant \frac{\tau}{2}(|\zeta|+1)\|f\|. \tag{7}$$

Dividing (1.7) by $|\zeta|$ and making $|\zeta| \to \infty$, we get

COROLLARY 2. *Let* $f(z)$ *be an entire function of exponential type* τ *and let* $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. *If* $f(z)$ *satisfies* $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, *then*

$$\|f'\| = \frac{\tau}{2}\|f\|. \tag{8}$$

Again, the Corollary 2 can also be obtained from Corollary 1, if we take $\zeta = 0$ in (1.7) and note that $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, implies $|f'(x)| = |\tau f(x) + if'(x)| = \{|D_\zeta[f(x)]|\}_{\zeta=0}$.

If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, satisfying $p(z) \equiv z^n\overline{p(1/\overline{z})}$, then $f(z) = p(e^{iz})$ is an entire function of exponential type n, satisfying $f(z) \equiv e^{inz}\{\overline{f(\overline{z})}\}$, and thus if we apply Corollary 2 to the function $f(z) = p(e^{iz})$, we get

COROLLARY 3. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n, *satisfying* $p(z) \equiv z^n\overline{p(1/\overline{z})}$, *then*

$$\max_{|z|=1} |p'(z)| = \frac{n}{2}\max_{|z|=1} |p(z)|. \tag{9}$$

The above result is due to Govil [4], O'Hara and Rodriguez [8], and Saff and Sheil-Small [12]. Also see Milovanović, Mitrinović and Rassias [7, p. 679, Lemma 3.1.10].

For functions satisfying $f(z) \equiv e^{i\tau z}f(-z)$, we are only able to prove

THEOREM 2. *Let* $f(z)$ *be an entire function of exponential type* τ *belonging to* L^p $(1 \leqslant p < \infty)$ *on the real axis. If* $f(z) \equiv e^{i\tau z}f(-z)$, *then for* $p \geqslant 1$ *and all complex numbers* ζ,

$$\left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\,dx\right)^{1/p} \geqslant \frac{\tau}{2}\big||\zeta|-1\big|\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}. \tag{10}$$

If we divide (1.10) by $|\zeta|$ and make $|\zeta| \to \infty$, we get

COROLLARY 4. *Let* $f(z)$ *be an entire function of exponential type* τ *belonging to* L^p $(1 \leqslant p < \infty)$ *on the real axis. If* $f(z) \equiv e^{i\tau z}f(-z)$, *then for* $p \geqslant 1$

$$\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} \geqslant \frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}. \tag{11}$$

The inequality is sharp and becomes equality for $f(z) = e^{i\tau z}/2$.

Note that Corollary 4 can also be obtained from Theorem 2 by setting $\zeta = 0$ in (1.10), because, as is easy to verify, $f(z) = e^{i\tau z}f(-z)$ implies that for $-\infty < x < \infty$, one has $|f'(-x)| = |\tau f(x) + if'(x)| = \{|D_\zeta[f(x)]|\}_{\zeta=0}$, and observing the fact that $\left(\int_{-\infty}^{\infty} |f'(-x)|^p\, dx\right)^{1/p} = \left(\int_{-\infty}^{\infty} |f'(x)|^p\, dx\right)^{1/p}$.

If in (1.10), we make $p \to \infty$, we get

COROLLARY 5. *Let* $f(z)$ *be an entire function of exponential type* τ *and let* $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. *If* $f(z)$ *satisfies* $f(z) \equiv e^{i\tau z}f(-z)$, *then for all complex numbers* ζ, *we have*

$$\|D_\zeta[f]\| \geqslant \frac{\tau}{2}\,|(|\zeta| - 1)|\,\|f\|. \tag{12}$$

If we divide both the sides of the above inequality by $|\zeta|$ and make $|\zeta| \to \infty$, we will get

COROLLARY 6. *Let* $f(z)$ *be an entire function of exponential type* τ *and let* $\|f\| = \sup_{-\infty<x<\infty} |f(x)|$. *If* $f(z)$ *satisfies* $f(z) \equiv e^{i\tau z}f(-z)$, *then*

$$\|f'\| \geqslant \frac{\tau}{2}\|f\|. \tag{13}$$

The above result, which can also be obtained from Corollary 4 by making $p \to \infty$, includes as a special case the following result due to Dewan and Govil [3].

COROLLARY 7. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n, *satisfying* $p(z) \equiv z^n p(1/z)$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{14}$$

2. Lemmas

We will need the following lemmas.

LEMMA 1. *Let* $f(z)$ *be an entire function of exponential type* τ, *belonging to* L^p $(1 \leqslant p < \infty)$ *on the real axis. If* $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, *then for* $p \geqslant 1$

$$\left(\int_{-\infty}^{\infty} |f'(x)|^p\, dx\right)^{1/p} \leqslant \tau\, C_p^{1/p} \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}, \tag{1}$$

where $C_p = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha$ *is as defined in Theorem B.*

The above result is due to Govil and Jain [5, inequality (6)].

LEMMA 2. *If $f(z)$ is an entire function of exponential type τ satisfying $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, then for $-\infty < x < \infty$*

$$|f'(x)| \geqslant \frac{\tau}{2}|f(x)|. \tag{2}$$

Proof. Because $f(z) = e^{i\tau z}\{\overline{f(\overline{z})}\}$, therefore

$$|f'(z)| = |e^{i\tau z}\{\overline{f'(\overline{z})}\} + i\tau e^{i\tau z}\{\overline{f(\overline{z})}\}|.$$

Hence, for $-\infty < x < \infty$,

$$\begin{aligned} |f'(x)| &= |f'(x) - i\tau f(x)| \\ &\geqslant \tau|f(x)| - |f'(x)|, \end{aligned} \tag{3}$$

from which (2.2) follows. □

LEMMA 3. *If $f(z)$ is an entire function of exponential type τ satisfying $f(z) \equiv e^{i\tau z}f(-z)$, then for $-\infty < x < \infty$*

$$|f'(-x)| = |\tau f(x) + if'(x)|. \tag{4}$$

Proof. Since $f(z) \equiv e^{i\tau z}f(-z)$, we get for $-\infty < x < \infty$,

$$\begin{aligned} |f'(x)| &= |i\tau e^{i\tau x}f(-x) - e^{i\tau x}f'(-x)| \\ &= |\tau f(-x) + if'(-x)|, \end{aligned} \tag{5}$$

and on replacing x by $-x$, the relation (2.4) follows. □

LEMMA 4. *Let $f(z)$ be an entire function of exponential type τ. If $f(z) \equiv e^{i\tau z}f(-z)$ then for $-\infty < x < \infty$ and $p \geqslant 1$*

$$\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} \geqslant \frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}. \tag{6}$$

The inequality is sharp and becomes equality for $f(z) = e^{i\tau z}/2$.

Proof. Because $f(z) \equiv e^{i\tau z}f(-z)$, we have for $-\infty < x < \infty$,

$$\begin{aligned} |f'(x)| &= |\tau f(-x) + if'(-x)|, \text{ by (2.5)} \\ &\geqslant \tau|f(-x)| - |f'(-x)|, \end{aligned}$$

which is equivalent to

$$\tau|f(-x)| \leqslant |f'(x)| + |f'(-x)|. \tag{7}$$

Inequality (2.7) clearly implies that for $p \geqslant 1$

$$\tau\left(\int_{-\infty}^{\infty} |f(-x)|^p\,dx\right)^{1/p} \leqslant \left(\int_{-\infty}^{\infty} \{\,|f'(x)| + |f'(-x)|\,\}^p\,dx\right)^{1/p},$$

which on applying Minkowski's Inequality and noting that $\left(\int_{-\infty}^{\infty}|f(-x)|^p\,dx\right)^{1/p} = \left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{1/p}$, gives for $p \geqslant 1$

$$\begin{aligned}\tau\left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{1/p} &\leqslant \left(\int_{-\infty}^{\infty}|f'(x)|^p\,dx\right)^{1/p} + \left(\int_{-\infty}^{\infty}|f'(-x)|^p\,dx\right)^{1/p}\\ &= \left(\int_{-\infty}^{\infty}|f'(x)|^p\,dx\right)^{1/p} + \left(\int_{-\infty}^{\infty}|f'(x)|^p\,dx\right)^{1/p}\\ &= 2\left(\int_{-\infty}^{\infty}|f'(x)|^p\right)^{1/p},\end{aligned}$$

from which (2.6) follows. □

3. Proofs of the theorems

Proof of Theorem 1. Since $f(z)$ is an entire function of exponential type τ, satisfying $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, hence for any complex number ζ and $-\infty < x < \infty$, we have

$$\begin{aligned}|D_\zeta[f(x)]| &= |\tau f(x) + i(1-\zeta)f'(x)|\\ &\leqslant |\zeta|\,|f'(x)| + |\tau f(x) + if'(x)|\\ &= |\zeta|\,|f'(x)| + |f'(x) - i\tau f(x)|\\ &= |\zeta|\,|f'(x)| + |f'(x)|, \quad \text{by (2.3)}\\ &= (|\zeta|+1)\,|f'(x)|,\end{aligned}$$

which implies

$$\int_{-\infty}^{\infty}|D_\zeta[f(x)]|^p\,dx \leqslant (|\zeta|+1)^p\int_{-\infty}^{\infty}|f'(x)|^p\,dx. \tag{1}$$

Since $f(z)$ satisfies $f(z) \equiv e^{i\tau z}\{\overline{f(\overline{z})}\}$, if we combine (3.1) with Lemma 1, we get for any ζ and $p \geqslant 1$,

$$\left(\int_{-\infty}^{\infty}|D_\zeta[f(x)]|^p\,dx\right)^{1/p} \leqslant (|\zeta|+1)\tau\, C_p^{1/p}\left(\int_{-\infty}^{\infty}|f(x)|^p\,dx\right)^{1/p},$$

and thus the inequality on the right hand side of (1.6) is established.

To prove the inequality on the left hand side of (1.6), note that for any complex number ζ,

$$\begin{aligned}|D_\zeta[f(x)]| &= |\zeta f'(x) + i\tau f(x) - f'(x)|\\ &\geqslant \big||\zeta|\,|f'(x)| - |f'(x) - i\tau f(x)|\big|\\ &= \big|\zeta|\,|f'(x)| - |f'(x)|\big|, \quad \text{by (2.3)}\\ &= \big||\zeta| - 1\big|\,|f'(x)|.\end{aligned}$$

Thus for any complex number ζ,

$$|D_\zeta[f(x)]| \geqslant \big||\zeta| - 1\big||f'(x)|. \tag{2}$$

Inequality (3.2), when combined with Lemma 2, gives for $-\infty < x < \infty$ and for any complex number ζ,

$$|D_\zeta[f(x)]| \geqslant \frac{\tau}{2}\big||\zeta| - 1\big|\,|f(x)|, \tag{3}$$

from which the inequality on the left hand side of (1.6) follows. This completes the proof of Theorem 1.

Proof of Theorem 2. Since $f(z)$ is an entire function of exponential type τ and satisfies $f(z) \equiv e^{i\tau z} f(-z)$, we have for any complex number ζ, and $-\infty < x < \infty$

$$\begin{aligned} |D_\zeta[f(x)]| &= |\,\tau f(x) + i(1-\zeta)f'(x)| \\ &\geqslant |\,\zeta|\,|f'(x)| - |\tau f(x) + if'(x)| \\ &= |\,\zeta\,|\,|f'(x)| - |f'(-x)|, \text{ by (2.4).} \end{aligned}$$

The above inequality is clearly equivalent to

$$|\,\zeta\,||f'(x)| \leqslant |D_\zeta[f(x)]| + |f'(-x)|,$$

which implies for $p \geqslant 1$ and for any complex number ζ,

$$|\,\zeta\,|\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} \leqslant \left(\int_{-\infty}^{\infty} \{|D_\zeta[f(x)]| + |f'(-x)|\}^p\,dx\right)^{1/p}. \tag{4}$$

If we apply Minkowski's inequality to the right hand side of (3.4), we easily get for $p \geqslant 1$ and $|\,\zeta| \geqslant 1$,

$$\begin{aligned} \left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\,dx\right)^{1/p} &\geqslant |\,\zeta|\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} - \left(\int_{-\infty}^{\infty} |f'(-x)|^p\,dx\right)^{1/p} \\ &= |\,\zeta\,|\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p} - \left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx.\right)^{1/p}, \end{aligned}$$

which is equivalent to

$$\left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\,dx\right)^{1/p} \geqslant (\,|\,\zeta\,| - 1)\left(\int_{-\infty}^{\infty} |f'(x)|^p\,dx\right)^{1/p}, \tag{5}$$

and this when combined with Lemma 4 gives

$$\left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\,dx\right)^{1/p} \geqslant \frac{\tau}{2}(\,|\,\zeta\,| - 1)\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx.\right)^{1/p}. \tag{6}$$

Similarly, one can prove that for $|\,\zeta\,| \leqslant 1$, we have

$$\left(\int_{-\infty}^{\infty} |D_\zeta[f(x)]|^p\,dx\right)^{1/p} \geqslant \frac{\tau}{2}(1 - |\,\zeta\,|)\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}. \tag{7}$$

If we combine (3.6) and (3.7), we will get

$$\left(\int_{-\infty}^{\infty} |D_{\zeta}[f(x)]|^{p}\, dx\right)^{1/p} \geqslant \frac{\tau}{2}\big|1-|\zeta|\big| \left(\int_{-\infty}^{\infty} |f(x)|^{p}\, dx\right)^{1/p}, \tag{8}$$

which is (1.10), and the proof of the Theorem 2 is thus complete.

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