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BOUNDARY VALUE PROBLEMS FOR FIRST ORDER PARAMETRIZE DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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(*communicated by S. G. Leela*)

Abstract. In this paper, by means of the method of upper and lower solutions and the monotone iterative technique, the existence of maximal and minimal solutions of the boundary value problem for first order parametrized differential equation with piecewise constant arguments is considered.

1. Introduction

The differential equations with parameters have numerous applications in the mathematical models of controlled process. These equations can be successfully used in physics, population dynamics and economics. Recently, the equations were investigated, and some existence results concerning parametrized boundary value problems were obtained (see Refs [1, 7–10]). M. Feckan studied a class of higher order parametrized boundary value problems by the Neilsen fixed point theory [1]. R. M. Brown applied an approach developed of the Neilsen fixed point theory to a class of two order parametrized equations [7]. T. Jankowski and V. Lakshmikantham discussed the first order differential equations with parameters [8, 10]. In this paper we consider the boundary value problem for first order parametrized differential equation with piecewise constant arguments

$$\begin{cases} x'(t) = f(t, x(t), x([t-k]), \lambda), & t \in J = [0, T], \\ x(-i) = x(0) = x_0, & i = 1, 2, \dots, k, \\ G(x(T), \lambda) = 0 \end{cases} \tag{1.1}$$

where $x_0 \in \mathbf{R}$ is a constant, $[\cdot]$ designates the greatest integer function and $k \in N$, $f \in C[J \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}, \mathbf{R}]$, $G \in (\mathbf{R} \times \mathbf{R}, \mathbf{R})$.

Let $\overline{T} = \begin{cases} [T]+1, & T \neq [T], \\ T, & T = [T], \end{cases}$ and Ω denote the class of all function $x : J \cup \{-k, -k+1, \dots, -1\} \to \mathbf{R}$ satisfying that (i) $x(-i) = x(0)$, $i = 1, 2, \dots k$;

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(ii) $x(t)$ is continuous for $t \in J$; (iii) $x'(t)$ exist and is continuous on the intervals $[n, n+1)$ $(n = 0, 1, \ldots, \overline{T}-1)$ and $[\overline{T}-1, T)$.

By a solution of (1.1) we mean a pair $(x, \lambda) \in \Omega \times \mathbf{R}$ for which problem (1.1) is satisfied.

The method of upper and lower solutions coupled with the monotone iterative technique has been widely used in the treatment of nonlinear differential equations in recent years (see Refs [2–6, 8]). When the method is applied to differential equations with piecewise constant arguments,it usually needs a suitable differential inequality as a comparison principle.

Here we establish a differential inequality as a comparison principle. Then, using the monotone iterative technique and the method of upper and lower solutions we obtain the existence theorems of extremal solutions for (1.1). This paper extends the results of [8, 10].

2. Preliminaries

DEFINITION 2.1. A pair $(v, a) \in \Omega \times \mathbf{R}$ is called a lower solution of (1.1) if

$$v'(t) \leqslant f(t, v(t), v([t-k]), a), \quad t \in J,$$
$$v(0) \leqslant x_0, \quad 0 \leqslant G(v(T), a);$$

an upper solution of (1.1) if the above inequalities are reversed.

Assume that v, $w \in \Omega$, a, $b \in \mathbf{R}$ such that $v(t) \leqslant w(t)$ on J and $a \leqslant b$. Let $[v, w] \times [a, b]$ denote the sector $\{(\eta, \lambda) \in \Omega \times \mathbf{R} : v \leqslant \eta \leqslant w,\ a \leqslant \lambda \leqslant b\}$. For any $(\eta_i, \lambda_i) \in \Omega \times \mathbf{R}$ $(i = 1, 2)$, $\eta_1 \leqslant \eta_2$, $\lambda_1 \leqslant \lambda_2$ implies $(\eta_1, \lambda_1) \leqslant (\eta_2, \lambda_2)$.

Now we develop a comparison result for later use.

LEMMA 2.1. *Suppose that* $m \in \Omega$ *such that*

$$m'(t) \leqslant -Mm(t) - Nm([t-k]), \quad t \in J \tag{2.1}$$

where $M > 0, N \geqslant 0$ *are constants such that*

$$1 - \overline{M}\,\overline{T} \geqslant 0, \tag{2.2}$$

where $\overline{M} = \frac{N}{M}\mathrm{e}^{kM}(\mathrm{e}^{M} - 1)$. *Then* $m(t) \leqslant 0$ *for all* $t \in J$ *if* $m(0) \leqslant 0$.

Proof. Let $p(t) = m(t)\mathrm{e}^{Mt}$, then (2.1) reduces to

$$p'(t) \leqslant -Np([t-k])\mathrm{e}^{M(t-[t-k])}, \tag{2.3}$$

hence

$$p(t) \leqslant p(n-1) - \frac{N}{M}p(n-k-1)[\mathrm{e}^{M(t-n+k+1)} - \mathrm{e}^{Mk}], \tag{2.4}$$

for $t \in [n-1, n)$ $(n = 1, 2, \ldots, \overline{T}-1)$, and for $t \in [\overline{T}-1, T)$,

$$p(t) \leqslant p(\overline{T}-1) - \frac{N}{M}p(\overline{T}-k+1)[\mathrm{e}^{M(t+k+1-\overline{T}} - \mathrm{e}^{Mk}]. \tag{2.5}$$

Notice that by continuity, (2.4) is satisfied on each interval $[n-1,n]$ $(n=1,2,\ldots,\overline{T}-1)$. In particular, we have

$$p(n) \leqslant p(n-1) - \overline{M}p(n-k-1), \quad n=1,2,\ldots,\overline{T}-1. \tag{2.6}$$

If $m(0) \leqslant 0$, this implies, that $p(t) \leqslant 0$ for $t \in [0,1]$ since on this interval

$$p(t) \leqslant p(0) - \overline{M}p(-k) = (1-\overline{M})p(0).$$

In particular, we obtain that

$$p(1) \leqslant (1-\overline{M})p(0). \tag{2.7}$$

Now assume that for $n < \overline{T}-1$, one has $p(t) \leqslant 0$ for every $t \in [n-1,n]$,

$$p(t) \leqslant (1-(n-1)\overline{M})p(0).$$

Let $t \in [n,n+1]$. If $k \geqslant \overline{T}-1$, by (2.4), it is clearly that $p(t) \leqslant (1-n\overline{M})p(0)$. If $k < \overline{T}-1$, by (2.4), we also have

$$p(t) \leqslant (1-(i-1)\overline{M})p(0)$$

for $t \in [i-1,i]$ $(i=1,2,\ldots,k+1)$. By (2.4) and (2.7), for $t \in [k+1,k+2]$ one has

$$\begin{aligned} p(t) &\leqslant p(k+1) - \overline{M}p(1) \leqslant p(k) - \overline{M}p(0) - \overline{M}p(1) \\ &\leqslant p(k-1) - \overline{M}p(-1) - \overline{M}p(0) - \overline{M}p(1) \\ &= p(k-1) - 2\overline{M}p(0) - \overline{M}p(1) \\ &\leqslant \ldots \leqslant p(1) - (k-1)\overline{M}p(0) - \overline{M}p(1) \\ &\leqslant (1-\overline{M})^2p(0) - (k-1)\overline{M}p(0) \leqslant (1-(k+1)\overline{M})p(0). \end{aligned}$$

By (2.4), the hypothesis of induction and a continued application of (2.6), we obtain

$$p(t) \leqslant (1-n\overline{M})p(0) \leqslant 0, \ t \in [n,n+1].$$

Analogously, we can obtain

$$p(t) \leqslant (1-\overline{T}\,\overline{M})p(0) \leqslant 0, \ t \in [\overline{T}-1,T].$$

Thus $m(t) \leqslant 0$ for $t \in J$. Hence the proof of the lemma is completed.

3. Main results

In order to develop the monotone method for (1.1), we require that f and G satisfy hypothesis:

(H_1) (v,a), $(w,b) \in E \times \mathbf{R}$ are lower and upper solutions of (1.1), respectively, such that $(v,a) \leqslant (w,b)$;

(H_2) f is nondecreasing with respect to the last variable;

(H_3) There exist constants $M,N > 0$ such that

$$f(t,\bar{x},\bar{y},\lambda) - f(t,x,y,\lambda) \geqslant -M(\bar{x}-x) - N(\bar{y}-y)$$

whenever $v(t) \leqslant x \leqslant \bar{x} \leqslant w(t)$, $v([t-k]) \leqslant y \leqslant \bar{y} \leqslant w([t-k])$ for $t \in J$, where $\lambda \in [a,b]$, x, $\bar{x}$, y, $\bar{y} \in \mathbf{R}$.

(H_3') There exist constants M, $N > 0$, $Q \geqslant 0$ such that

$$f(t,\bar{x},\bar{y},\bar{\lambda}) - f(t,x,y,\lambda) \geqslant -M(\bar{x}-x) - N(\bar{y}-y) + Q(\bar{\lambda}-\lambda)$$

whenever $v(t) \leqslant x \leqslant \bar{x} \leqslant w(t)$, $v([t-k]) \leqslant y \leqslant \bar{y} \leqslant w([t-s])$ for $t \in J$ and $a \leqslant \lambda \leqslant \bar{\lambda} \leqslant b$, where $\lambda, \bar{\lambda}$, x, $\bar{x}$, y, $\bar{y} \in \mathbf{R}$.

(H_4) There exist $M_1, N_1 > 0$ such that

$$G(\bar{u},\bar{\lambda}) - G(u,\lambda) \geqslant M_1(\bar{u}-u) - N_1(\bar{\lambda}-\lambda)$$

whenever $v(t) \leqslant u \leqslant \bar{u} \leqslant w(t)$ for $t \in J$ and $a \leqslant \lambda \leqslant \bar{\lambda} \leqslant b$, where u, $\bar{u}$, λ, $\bar{\lambda} \in \mathbf{R}$.

(H_4') There exists $N_1 > 0$ such that

$$G(u,\bar{\lambda}) - G(u,\lambda) \geqslant -N_1(\bar{\lambda}-\lambda)$$

whenever $v(t) \leqslant u \leqslant w(t)$ for $t \in J$ and $a \leqslant \lambda \leqslant \bar{\lambda} \leqslant b$, where u, λ, $\bar{\lambda} \in \mathbf{R}$;

(H_5) G is nondecreasing with respect to the first variable.

THEOREM 3.1. *Assume that* $(H_1) - (H_4)$ *and (2.2) hold. Then there exist monotone sequence* $\{(v_n, a_n)\}$ *and* $\{(w_n, b_n)\}$ *that converge uniformly to the minimal and maximal solutions* (ρ, c) *and* (r, d)*, respectively, of (1.1), that is, if* (x, σ) *is any solution of (1.1) in* $[v,w] \times [a,b]$*, then*

$$\begin{aligned}(v,a) &\equiv (v_0,a_0) \leqslant (v_1,a_1) \leqslant \ldots \leqslant (v_n,a_n) \leqslant (\rho,c) \leqslant (x,\sigma) \\ &\leqslant (r,d) \leqslant (w_n,b_n) \leqslant \ldots \leqslant (w_1,b_1) \leqslant (w_0,b_0) \equiv (w,b).\end{aligned}$$

Proof. Let $(\eta,\lambda) \in [v,w] \times [a,b]$, we consider the linear systems

$$\begin{aligned}p'(t) &= f(t,\eta(t),\eta([t-k]),\lambda) - M(p(t)-\eta(t)) \\ &\quad -N(p([t-k]) - \eta([t-k])), \quad t \in J,\end{aligned} \tag{3.1}$$

$$p(-i) = x_0, \quad i = 1,2,\ldots,k, \tag{3.2}$$

$$0 = G(\eta(T),\lambda) - N_1(\mu-\lambda) + M_1(u(T) - \eta(T)). \tag{3.3}$$

It is not difficulty to see that for given $(\eta,\lambda) \in [v,w] \times [a,b]$ the system (3.1), (3.2) and (3.3) admits a unique solution (p,μ). Thus for $(\eta,\lambda) \in [v,w] \times [a,b]$, it is not difficult to see that we can define the mapping A by

$$A(\eta,\lambda) = (p,\mu)$$

where (p,μ) is the unique solution of (3.1), (3.2) and (3.3).

We shall show that the mapping A satisfies:

(i) $(v,a) \leqslant A(v,a)$, $A(w,b) \leqslant (w,b)$;

(ii) A is monotone nondecreasing on $[v,w] \times [a,b]$, i.e., for any (η_1,λ_1), $(\eta_2,\lambda_2) \in [v,w] \times [a,b]$, $(\eta_1,\lambda_1) \leqslant (\eta_2,\lambda_2)$ implies $A(\eta_1,\lambda_1) \leqslant A(\eta_2,\lambda_2)$.

To prove (i) we set $A(v,a) = (v_1,a_1)$ and $A(w,b) = (w_1,b_1)$ we only prove $(v,a) \leqslant (v_1,a_1)$ and $(w,b) \geqslant (w_1,a_1)$ can be proved similarly. Setting $m_0(t) = v(t) - v_1(t), q_0 = a - a_1$. Then we have from (H_1), (H_3) and (3.1), (3.2)

$$\begin{aligned} m_0'(t) \leqslant & f(t,v(t),v([t-k]),a) - f(t,v(t),v([t-k]),a) + M(v_1(t) - v(t)) \\ & + N(v([t-k]) - v_1([t-k]) = -Mm_0(t) - Nm_0([t-k]), \quad t \in J \end{aligned}$$

and $m_0(-i) = v(-i) - v_1(-i) \leqslant 0$, $i = 1,2,\ldots,k$, by Lemma 2.1, we have $m_0(t) \leqslant 0$ on J. So $v(t) \leqslant v_1(t)$. On the other hand, we have from (3.3) and (H_1)

$$0 = G(v(T),a) - N_1(a_1 - a) + M_1(v_1(T) - v(T)) \geqslant -N(a_1 - a) = Nq_0,$$

so $q_0 \leqslant 0$ and hence $a \leqslant a_1$. That is $(v,a) \leqslant A(v,a)$.

To prove (ii), let $A(\eta_i,\lambda_i) = (p_i,\mu_i)$ $(i = 1,2)$, $m_1(t) = p_1(t) - p_2(t)$ and $q_1 = \mu_1 - \mu_2$. Then, in view of (H_1), (H_2) and (H_3) we obtain,

$$\begin{aligned} m_1'(t) &= f(t,\eta_1,\eta_1([t-k]),\lambda_1) - M(p_1(t) - \eta_1(t)) - N(u_1([t-k]) - \eta_1([t-k])) \\ & \quad - f(t,\eta_2,\eta_2([t-k]),\lambda_2) + M(p_2(t) - \eta_2(t)) + N(u_2([t-k]) - \eta_2([t-k])) \\ & \leqslant -Mm_1(t) - Nm_1([t-k]). \end{aligned}$$

Since $m_1(-i) \equiv 0$ for $i = 1,2,\ldots,k$, it follows from Lemma 2.1 that $m_1(t) \leqslant 0$ on J, and $p_1(t) \leqslant p_2(t)$ on J. Further, from (3.3) and (H_1), (H_4)

$$\begin{aligned} 0 &= G(\eta_1(T),\lambda_1) - N_1(\mu_1 - \lambda_1) + M_1(p_1(T) - \eta_1(T)) \\ & \quad - G(\eta_2(T),\lambda_2) + N_1(\mu_2 - \lambda_2) - M_1(p_2(T) - \eta_2(T)) \\ &= G(\eta_1(T),\lambda_1) - G(\eta_2(T),\lambda_2) + N_1(\lambda_1 - \lambda_2) \\ & \quad - M_1(\eta_1(T) - \eta_2(T)) - N_1 q_1 + Q(p_1(T) - p_2(T)) \leqslant -N_1 q_1, \end{aligned}$$

so $q_1 \leqslant 0$ and hence $\mu_1 \leqslant \mu_2$. Therefore (ii) holds.

Let $v_0 \equiv v$, $w_0 \equiv w$, $a_0 = a$, $b_0 = b$, we construct sequences $\{(v_n,a_n)\}$ and $\{(w_n,b_n)\}$ by

$$(v_n,a_n) = A(v_{n-1},a_{n-1}), (w_n,b_n) = A(w_{n-1},b_{n-1}), \ n = 1,2,\ldots.$$

We get that

$$\begin{aligned} v_0 \leqslant v_1 \leqslant \ldots \leqslant v_n \leqslant w_n \leqslant \ldots \leqslant w_1 \leqslant w_0, \\ a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant b_n \leqslant \ldots \leqslant b_1 \leqslant b_0. \end{aligned}$$

Employing standard techniques ([3]), it can be shown that the sequence $\{(v_n,a_n)\}$ and $\{(w_n,b_n)\}$ converge uniformly and monotonically to (ρ,c) and (r,d), respectively. Indeed, (ρ,c), (r,d) are solutions of (1.1) in view of the continuity of f and G, and the definition of the above sequences.

To prove that (ρ,c), (r,d) are extremal solutions of (1.1), let $(\bar{p},\bar{\mu}) \in [v,w] \times [a,b]$ be any solution of (1.1). Suppose that there exists a positive integer n such that $v_n(t) \leqslant \bar{p}(t) \leqslant w_n(t)$ on J and $a_n \leqslant \bar{\mu} \leqslant b_n$. Then, setting $m(t) = v_{n+1}(t) - \bar{u}(t), q = a_{n+1} - \bar{\mu}$, we have

$$\begin{aligned} m'(t) &\leqslant f(t, v_n(t), v_n([t-k]), a_n) - M(v_{n+1}(t) - v_n(t)) \\ &\quad -N(v_{n+1}([t-k]) - v_n([t-k])) - f(t, \bar{p}(t), \bar{p}([t-k]), \bar{\mu}) \\ &\leqslant f(t, v_n(t), v_n([t-k]), \bar{\mu}) - f(t, \bar{p}(t), \bar{p}([t-k]), \bar{\mu}) + M(v_n(t) - \bar{p}(t)) \\ &\quad +N(v_n([t-k]) - \bar{p}([t-k])) - Mm(t) - Nm([t-k]) \\ &\leqslant -Mm(t) - Nm([t-k]). \end{aligned}$$

Since $m(-i) = 0$ for $i = 1, 2, \ldots, k,$ it follow from Lemma 2.1 that $m(t) \leqslant 0$ on J, and $v_{n+1}(t) \leqslant \bar{p}(t)$ on J. Further, we have that, from (3.3) and (H_4),

$$\begin{aligned} 0 &= G(v_n(T), a_n) - N_1(a_{n+1} - a_n) + M_1(v_{n+1}(T) - v_n(T)) - G(\bar{p}(T), \bar{\mu}) \\ &\leqslant M_1(v_n(T) - \bar{p}(T)) - N_1(a_n - \bar{\mu}) - N_1(a_{n+1} - a_n) + M_1(v_{n+1}(T) - v_n(T)) \\ &\leqslant -Nq, \end{aligned}$$

so $q \leqslant 0$, and hence $a_{n+1} \leqslant \bar{\mu}$. That is, $(v_{n+1}, a_{n+1}) \leqslant (\bar{u}, \bar{\mu})$. Similarly, we obtain $(\bar{p}, \bar{\mu}) \in (w_{n+1}, b_{n+1})$. Since $(\bar{p}, \bar{\mu}) \in [v, w] \times [a, b]$, by induction we get $(\bar{p}, \bar{\mu}) \in [v_n, w_n] \times [a_n, b_n]$ for every n. Therefore, $(\bar{p}, \bar{\mu}) \in [\rho, r] \times [c, d]$ by taking limit as $n \to \infty$. The proof of the theorem is complete.

THEOREM 3.2. *Assume that* (H_1), (H_3'), (H_4'), (H_5) *and (2.2) hold. Then the conclusion of Theorem 3.1 is valid.*

Proof. Let $(\eta, \lambda) \in [v, w] \times [a, b]$, we consider the linear systems

$$0 = G(\eta(T), \lambda) - p_1(\mu - \lambda), \tag{3.4}$$

$$\begin{aligned} p'(t) &= f(t, \eta(t), \eta([t-s]), \lambda) - M(p(t) - \eta(t)) \\ &\quad -N(p([t-k]) - \eta([t-k])]) + Q(\mu - \lambda), \end{aligned} \tag{3.5}$$

$$p(-i) = x_0, \quad i = 1, 2, \ldots, k. \tag{3.6}$$

It is obviously, that (3.4), (3.5) and (3.6) has a unique solution (p, μ). Thus for any $(\eta, \lambda) \in [v, w] \times [a, b]$, we can define the mapping A by

$$A(\eta, \lambda) = (p, \mu).$$

Thus for $(\eta, \lambda) \in [v, w] \times [a, b]$, we can define the mapping A by

$$A(\eta, \lambda) = (p, \mu)$$

where (p, μ) is the unique solution of (3.4), (3.5) and (3.6).

We shall show that the mapping A satisfies:

(i) $(v, a) \leqslant A(v, a)$, $A(w, b) \leqslant (w, b)$;

(ii) A is monotone nondecreasing on $[v, w] \times [a, b]$, i.e., for any (η_1, λ_1), $(\eta_2, \lambda_2) \in [v, w] \times [a, b]$, $(\eta_1, \lambda_1) \leqslant (\eta_2, \lambda_2)$ implies $A(\eta_1, \lambda_1) \leqslant A(\eta_2, \lambda_2)$.

To prove (i) we set $A(v, a) = (v_1, a_1)$ and $A(w, b) = (w_1, b_1)$ we only prove $(v, a) \leqslant (v_1, a_1)$ and $(w, b) \geqslant (w_1, a_1)$ can be proved similarly. Setting $m_0(t) = v(t) - v_1(t), q_0 = a - a_1$. First we have from (3.4) and (H_1)

$$0 = G(v(T), a) - N_1(a_1 - a) \geqslant -N(a_1 - a) = Nq_0,$$

so $q_0 \leqslant 0$ and hence

$$a \leqslant a_1. \tag{3.7}$$

On the other hand, from (H_1), (H_3'), (3.5), (3.6) and (3.7), we have

$$\begin{aligned} m_0'(t) &\leqslant f(t,v(t),v([t-k]),a) - f(t,v(t),v([t-k]),a) + M(v_1(t)-v(t)) \\ &\quad + N(v([t-k]) - v_1([t-k])) + Q(a_1 - a) \\ &= -Mm_0(t) - Nm_0([t-k]), \quad t \in J, \\ m_0(-i) &= v(-i) - v_1(-i) \leqslant 0, \ \ i = 1,2,\dots,k, \end{aligned}$$

by Lemma 2.1, we have $m_0(t) \leqslant 0$ on J. So $v(t) \leqslant v_1(t)$. That is $(v,a) \leqslant A(v,a)$.

To prove (ii), let $A(\eta_i,\lambda_i) = (p_i,\mu_i)$ $(i=1,2)$, $m_1(t) = p_1(t) - p_2(t)$ and $q_1 = \mu_1 - \mu_2$. Then, from (3.4) and (H_1), (H_4') and (H_5), we have

$$\begin{aligned} 0 &= G(\eta_1(T),\lambda_1) - N_1(\mu_1 - \lambda_1) + M_1(p_1(T) - \eta_1(T)) \\ &\quad - G(\eta_2(T),\lambda_2) + N_1(\mu_2 - \lambda_2) - M_1(p_2(T) - \eta_2(T)) \\ &= G(\eta_1(T),\lambda_1) - G(\eta_2(T),\lambda_2) + N_1(\lambda_1 - \lambda_2) \\ &\quad - M_1(\eta_1(T) - \eta_2(T)) - N_1 q_1 + Q(p_1(T) - p_2(T)) \\ &\leqslant -N_1 q_1, \end{aligned}$$

so $q_1 \leqslant 0$ and hence

$$\mu_1 \leqslant \mu_2. \tag{3.8}$$

Further, in view of (H_1), (H_3') and (3.8) we obtain,

$$\begin{aligned} m_1'(t) &= f(t,\eta_1,\eta_1([t-k]),\lambda_1) - M(p_1(t) - \eta_1(t)) - N(u_1([t-k]) - \eta_1([t-k])) \\ &\quad - f(t,\eta_2,\eta_2([t-k]),\lambda_2) + M(p_2(t) - \eta_2(t)) + N(u_2([t-k]) - \eta_2([t-k])) \\ &\quad + Q(\mu_1 - \eta_1 + \mu_2 - \eta_2) \\ &\leqslant -Mm_1(t) - Nm_1([t-k]). \end{aligned}$$

Since $m_1(-i) \equiv 0$ for $i = 1,2,\dots,k$, it follows from Lemma 2.1 that $m_1(t) \leqslant 0$ on J, and $p_1(t) \leqslant p_2(t)$ on J. Therefore (ii) holds.

The further proof is analogous to that of Theorem 3.1 and Therefore we omit it. The proof is complete.

REFERENCES

[1] M. FECKAN, *Parametrized singular boundary value problem*, J. Math. Anal. Appl., **188** (1994), 417–425.
[2] S. LEELA, M. N. OGUZTORELI, *Periodic boundary value problem for differential equations with delay and monotone iterative method*, J. Math. Anal. Appl., **122** (1987), 301–307.
[3] G. S. LADDE, V. LAKSHMIKANTHAM, A. S. VATSALA, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985.
[4] H. XU, E. LIZ, *Boundary value problems for functional differential equations*, Nonlinear Analysis, **41** (2000), 971–988.
[5] J. J. NIETO, J. YU, J. YAN, *Monotone iterative methods for functional differential equations*, Nonlinear Analysis, **32** (1998), 741–747.
[6] F. ZHANG, Z. MA, J. YAN, *Periodic boundary value problems and monotone iterative methods for first order impulsive differential equations with delay*, Indian J. Pure and Applied Math., **32** (2001), 1695–1707.
[7] R. F. BROWN, *Topological identification of multiple solutions to parametrized nonlinear equations*, Pacific J. Math., **131** (1988), 51–69.

[8] T. JANKOWSKI AND V. LAKSHMIKANTHAM, *Monotone iterations for differential equations with a parameter*, J. Appl. Math. Stoch. Anal. **10** (1997), 273–278.
[9] S. STANK, *On a class of functional boundary value problems for the equation* $x'' = f(t, x, x', x'', \lambda)$, Ann. Pol. Math., **59** (1994), 225–237.
[10] T. JANKOWSKI, *Monotone iterations for first order differential equations with a parameter*, Acta Math. Hungar., **84** (1999), 65–80.

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