Mathematical
Inequalities
& Applications
Volume 6, Number 3 (2003), 487–495

INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER DAMPED HALF–LINEAR DIFFERENTIAL EQUATIONS WITH FORCING TERM

WAN–TONG LI

(*communicated by D. Bainov*)

Abstract. By using an inequality due to Hardy, Littlewood and Polya and averaging functions, several interval oscillation criteria are established for the second-order damped half-linear differential equation with forcing term of the form $(r(t)|y'(t)|^{\alpha-1}y'(t))' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = e(t)$ that are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line, where $\alpha > 0$. In particular, several examples that dwell upon the importance of our results are also included.

1. Introduction

In this paper we are concerned with the problem of interval oscillation of second-order damped superlinear differential equation with forcing

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = e(t), \quad t \geqslant t_0, \tag{1}$$

where $r \in C([t_0,\infty);(0,\infty))$, $p, q, e \in C([t_0,\infty);R)$, and $\alpha > 0$.

By a solution of (1), we mean a function $y : [T_y, \infty), T_y \geqslant t_0$, which has the property $r(t)|y'(t)|^{\alpha-1}y'(t) \in C^1[T_y,\infty)$ and satisfies (1). We restrict our attention only to the nontrivial solution $y(t)$ of (1), i.e., to the solution $y(t)$ such that $\sup\{|y(t)| : t \geqslant T\} > 0$ for all $T \geqslant T_y$. A nontrivial solution of (1) is called oscillatory if it has arbitrary large zeros, otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

When $\alpha = 1$ and $p(t) \equiv 0$, Eq. (1) reduces to the equation

$$(r(t)y'(t))' + q(t)y(t) = e(t). \tag{2}$$

Numerous oscillation criteria have been obtained for Eq. (2); see Keener [1], Rainkin [2], Skidmore and Bowers [3], Skidmore and Leighton [4], and Teufel [5]. In these

Mathematics subject classification (2000): 34C10, 34C15.
Key words and phrases: Interval criteria, oscillation, superlinear differential equations, damping, forced term.

Supported by the NNSF of China (10171040), the NSF of Gansu Province of China (ZS011-A25-007-Z), the Foundation for University Key Teacher by the Ministry of Education of China, and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of Ministry of Education of China.

papers, the authors established oscillation criteria for a more general nonlinear equation by employing a technique introduced by Kartsatos [6] where it is additionally assumed that $e(t)$ be the second derivative of an oscillatory function $h(t)$ and their oscillation results require the information of q on the entire half-line $[t_0,\infty)$.

However, from the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e., if there exists a sequence of subintervals $[a_i,b_i]$ of $[t_0,\infty)$, as $a_i\to\infty$, such that for each i there exists a solution of equation

$$(r(t)y'(t))'+q(t)y(t)=0,\quad t\geqslant t_0, \tag{3}$$

that has at least two zeros in $[a_i,b_i]$, then every solution of Eq. (3) is oscillatory, no matter how "bad" Eq. (3) is (or r and q are) on the remaining parts of $[t_0,\infty)$.

El-Sayed [7] applied this idea to oscillation and established an interval criterion for oscillation of a forced second-order linear differential equation (2).

THEOREM A. *Suppose that there exist two positive increasing divergent sequences* $\{a_n^+\},\{a_n^-\}$ *and two sequences* $\{c_n^+\}$, $\{c_n^-\}$ *such that* c_n^+,c_n^- *are positive numbers and*

$$\begin{aligned} V_n^{\pm} &= \int_{a_n^{\pm}}^{a_n^{\pm}+\pi/\sqrt{c_n^{\pm}}} \Big(c_n^{\pm}\,[1-r(t)]\cos^2\Big\{\sqrt{c_n^{\pm}}(t-a_n^{\pm})\Big\} \\ &+ \big[q(t)-c_n^{\pm}\big]\,\sin^2\Big\{\sqrt{c_n^{\pm}}(t-a_n^{\pm})\Big\}\Big)\,dt \\ &\geqslant 0, \end{aligned} \tag{4}$$

for all $n\geqslant n_0$, *where* n_0 *is a fixed positive integer. Suppose further that* $e(t)$ *satisfies*

$$e(t)\begin{cases} \geqslant 0, & t\in\left[a_n^+,a_n^++\frac{\pi}{\sqrt{c_n^+}}\right], \\ \leqslant 0, & t\in\left[a_n^-,a_n^-+\frac{\pi}{\sqrt{c_n^-}}\right], \end{cases} \tag{5}$$

for all $n\geqslant n_0$. *Then Eq. (2) is oscillatory.*

In 1999, Wong [8] proved a more general oscillation result for Eq. (2).

THEOREM B. *Suppose that for any* $T\geqslant t_0$, *there exist* $T\leqslant s_1<t_1\leqslant s_2<t_2$ *such that*

$$e(t)\begin{cases} \leqslant 0, & t\in[s_1,t_1], \\ \geqslant 0, & t\in[s_2,t_2]. \end{cases} \tag{6}$$

Denote $D(s_i,t_i)=\{u\in C^1[s_i,t_i]:u(t)\not\equiv 0,\ u(s_i)=u(t_i)=0\}$, $i=1,2$. *If there exists* $u\in D(s_i,t_i)$ *such that*

$$Q_i(u)=\int_{s_i}^{t_i}\big[qu^2-r(u')^2\big]\,dt\geqslant 0, \tag{7}$$

for $i = 1, 2$*, then Eq. (2) is oscillatory.*

From the above results we see that Theorems A and B are only concerned with the linear equation (2). For nonlinear equations, Li and Agarwal [12, 13] considered the following equations

$$(r(t)y'(t))' + q(t)f(y(t)) = e(t), \quad t \geqslant t_0,$$

and

$$(r(t)y'(t))' + p(t)y'(t) + q(t)f(y(t)) = e(t), \quad t \geqslant t_0,$$

where the function f satisfies

$$f'(y) \geqslant \mu > 0 \quad \text{for} \quad y \neq 0$$

or

$$\frac{f(y)}{y} \geqslant \nu > 0 \quad \text{for } y \quad \neq 0,$$

and obtained the similar results. But, above mentioned results are not applied to the suprelinear equation (1).

Motivated by the ideas of El-Sayed [7], Li [9, 10, 11], and Li and Agarwal [12, 13, 14], in this paper we obtain, by using an inequality due to Hardy, Littlewood and Polya [15], we obtain several new interval criteria for oscillation, that is, criteria given by the behavior of Eq. (1) (or of r, p, q, and e) only on a sequence of subintervals of $[t_0, \infty)$. These oscillation criteria extend the above mentioned results. Finally, several examples that dwell upon the importance of our results are also included.

2. Main results

In order to prove our results we use the following well-known inequality which is due to Hardy, Littlewood and Polya [15].

LEMMA 1. *If* X *and* Y *are nonnegative, then*

$$X^\gamma + (\gamma - 1)Y^\gamma \geqslant \gamma X Y^{\gamma-1}, \quad \gamma > 1,$$

where equality holds if and only if $X = Y$*.*

In the sequel we say that a function $H := H(t)$ belongs to a function class $D(s_i, t_i) = \{H \in C^1[s_i, t_i] : H(t) \not\equiv 0,\ H(s_i) = H(t_i) = 0\}$, $i = 1, 2$, denoted by $H \in D(s_i, t_i)$, $i = 1, 2$.

THEOREM 1. *Suppose that for any* $T \geqslant t_0$*, there exist* $T \leqslant s_1 < t_1 \leqslant s_2 < t_2$ *such that* $q(t) \geqslant 0$ *on* $[s_1, t_1] \cap [s_2, t_2]$ *and (6) holds. If there exists* $H \in D(s_i, t_i)$ *such that*

$$\int_{s_i}^{t_i} H^2(t)q(t)dt \geqslant \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{r(t)}{|H(t)|^{\alpha-1}} \left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right|^{\alpha+1} dt, \qquad (8)$$

for $i = 1, 2$, then every solution of Eq. (1) is oscillatory.

Proof. Suppose that $y(t)$ is a nonoscillatory solution which is positive, say $y(t) > 0$ when $t \geqslant T_0$ for some T_0 depending on the solution $y(t)$. Denote

$$u(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \geqslant T_0. \tag{9}$$

Then, for every $t \geqslant T_0$, we obtain

$$u'(t) = -q(t) - \frac{p(t)}{r(t)}u(t) - \alpha\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} + \frac{e(t)}{|y(t)|^{\alpha-1}y(t)}. \tag{10}$$

By assumption, we can choose $s_1, t_1 \geqslant T_0$ so that $e(t) \leqslant 0$ on the interval $I = [s_1, t_1]$ with $s_1 < t_1$. On the interval I, $u(t)$ satisfies by (10),

$$q(t) \leqslant -u'(t) - \frac{p(t)}{r(t)}u(t) - \alpha\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}. \tag{11}$$

Let $H \in D(s_1, t_1)$ be given as in hypothesis. Multiplying H^2 through (11) and integrating over I, we have

$$\int_{s_1}^{t_1} H^2(t)q(t)dt \leqslant -\int_{s_1}^{t_1} H^2(t)\left[u'(t) + \frac{p(t)}{r(t)}u(t) + \alpha\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}\right]dt. \tag{12}$$

Integrating (12) by parts and using the fact that $H(s_1) = H(t_1) = 0$, we obtain

$$\begin{aligned}
\int_{s_1}^{t_1} H^2(t)q(t)dt &\leqslant -\int_{s_1}^{t_1} H^2(t)\left[u'(t) + \frac{p(t)}{r(t)}u(t) + \alpha\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}\right]dt \\
&= \int_{s_1}^{t_1} 2H(t)H'(t)u(t)dt - \alpha\int_{s_1}^{t_1} H^2(t)\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}dt - \int_{s_1}^{t_1} H^2(t)\frac{p(t)}{r(t)}u(t)dt \\
&= \int_{s_1}^{t_1}\left(2H'(t) - \frac{p(t)}{r(t)}H(t)\right)H(t)u(t)dt - \alpha\int_{s_1}^{t_1} H^2(t)\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}dt \\
&\leqslant \int_{s_1}^{t_1}\left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right||H(t)|\,|u(t)|\,dt - \alpha\int_{s_1}^{t_1} H^2(t)\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}dt. \qquad (13)
\end{aligned}$$

Taking

$$X = \left[\alpha H^2(t)\right]^{\alpha/\alpha+1}\frac{|u(t)|}{r^{1/\alpha+1}(t)}, \quad \gamma = \frac{\alpha+1}{\alpha},$$

$$Y = \frac{\alpha^{\alpha/\alpha+1}}{(\alpha+1)^{\alpha+1}}\cdot\frac{r^{\alpha/\alpha+1}(t)|H(t)|^{\alpha}}{H^{2\alpha^2/\alpha+1}(t)}\left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right|^{\alpha}.$$

According to Lemma 1, we obtain for $t \in [s_1, t_1]$

$$\begin{aligned}
&\left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right| |H(t)||u(t)| - \alpha H^2(t)\frac{|u(t)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} \\
&\quad \leqslant \frac{r(t)}{(\alpha+1)^{\alpha+1}} \cdot \frac{|H(t)|^{\alpha+1}}{H^{2\alpha}(t)} \left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right|^{\alpha+1} \\
&\quad = \frac{r(t)}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right|^{\alpha+1}}{|H(t)|^{\alpha-1}}.
\end{aligned}$$

Hence, (13) implies

$$\int_{s_1}^{t_1} H^2(t)q(t)dt \leqslant \int_{s_1}^{t_1} \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{r(t)}{|H(t)|^{\alpha-1}} \left|2H'(t) - \frac{p(t)}{r(t)}H(t)\right|^{\alpha+1} dt,$$

which contradicts the assumption (8). This contradiction proves that $y(t)$ is oscillatory.

When $y(t)$ is eventually negative, we see $H \in D(s_2, t_2)$ and $e(t) \geqslant 0$ on $[s_2, t_2]$ to reach a similar contradiction. The proof is complete.

If $\alpha = 1$ and $p(t) \equiv 0$, by Theorem 1, we have the following corollary, which is the main result of Wong [8].

COROLLARY 1. *Suppose that for any $T \geqslant t_0$, there exist $T \leqslant s_1 < t_1 \leqslant s_2 < t_2$ such that $q(t) \geqslant 0$ on $[s_1, t_1] \cap [s_2, t_2]$ and (6) holds. If there exists $H \in D(s_i, t_i)$ such that*

$$\int_{s_i}^{t_i} H^2(t)q(t)dt > \int_{s_i}^{t_i} r(t)[H'(t)]^2 dt, \tag{14}$$

for $i = 1, 2$, then every solution of Eq. (2) is oscillatory.

Next, we will establish a more general oscillation result for Eq. (1).

THEOREM 2. *Suppose that for any $T \geqslant t_0$, there exist $T \leqslant s_1 < t_1 \leqslant s_2 < t_2$ such that $q(t) \geqslant 0$ on $[s_1, t_1] \cap [s_2, t_2]$ and (6) holds. If there exist $H \in D(s_i, t_i)$ and a positive, function $\phi \in C^1([t_0, \infty))$ such that*

$$\begin{aligned}
&\int_{s_i}^{t_i} H^2(t)\phi(t)q(t)dt \\
&> \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}} \left|2H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H(t)\right|^{\alpha+1} dt,
\end{aligned} \tag{15}$$

for $i = 1, 2$, then every solution of Eq. (1) is oscillatory.

Proof. Suppose that $y(t)$ is a nonoscillatory solution which is positive, say $y(t) > 0$ when $t \geqslant T_0$ for some T_0 depending on the solution $y(t)$. Now, we define

$$w(t) = \phi(t)\frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\alpha-1}y(t)}, \quad t \geqslant T_0.$$

Then, for every $t \geqslant T_0$, we obtain

$$\phi(t)q(t) - \frac{\phi(t)e(t)}{|y(t)|^{\alpha-1}y(t)} = -w'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}. \tag{16}$$

By assumption, we can choose $s_1, t_1 \geqslant T_0$ so that $e(t) \leqslant 0$ on the interval $I = [s_1, t_1]$ with $s_1 < t_1$. On the interval I, $w(t)$ satisfies by (16),

$$\phi(t)q(t) + \frac{\phi(t)|e(t)|}{|y(t)|^{\alpha-1}y(t)} = -w'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}. \tag{17}$$

Similar to the proof of Theorem 1, we have

$$\phi(t)q(t) \leqslant -w'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}. \tag{18}$$

Let $H \in D(s_1, t_1)$ be given as in hypothesis. Multiplying H^2 through (18) and integrating over I, we have

$$\int_{s_1}^{t_1} H^2(t)\phi(t)q(t)dt \leqslant -\int_{s_1}^{t_1} H^2(t)w'(t)dt + \int_{s_1}^{t_1} H^2(t)\left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t)dt - \alpha\int_{s_1}^{t_1} H^2(t)\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt. \tag{19}$$

Integrating (19) by parts and using the fact that $H(s_1) = H(t_1) = 0$, we obtain

$$\begin{aligned}\int_{s_1}^{t_1} H^2(t)\phi(t)q(t)dt &\leqslant -\int_{s_1}^{t_1} 2H(t)H'(t)w(t)dt \\ &\quad + \int_{s_1}^{t_1} H^2(t)\left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)w(t)dt - \alpha\int_{s_1}^{t_1} H^2(t)\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt \\ &\leqslant \int_{s_1}^{t_1}\left|2H(t)H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H^2(t)\right| |w(t)|\, dt \\ &\quad -\alpha\int_{s_1}^{t_1} H^2(t)\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}dt. \end{aligned} \tag{20}$$

Taking

$$X = \left[\alpha H^2(t)\right]^{\alpha/\alpha+1}\frac{|u(t)|}{[r(t)\phi(t)]^{1/\alpha+1}}, \quad q = \frac{\alpha+1}{\alpha},$$

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{r(t)\phi(t)}{(\alpha H^2(t))^{\alpha}}\right)^{\alpha/(\alpha+1)}\left|2H(t)H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H^2(t)\right|^{\alpha}.$$

In view of Lemma 1, we obtain for $t \in [s_1, t_1]$

$$\left|2H(t)H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H^2(t)\right| |w(t)| - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(r(t)\phi(t))^{1/\alpha}}$$
$$\leqslant \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}} \left|2H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H(t)\right|^{\alpha+1}.$$

Hence, (20) implies

$$\int_{s_1}^{t_1} H^2(t)\phi(t)q(t)dt$$
$$\leqslant \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}} \left|2H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H(t)\right|^{\alpha+1} dt,$$

which contradicts the assumption (15). This contradiction proves that $y(t)$ is oscillatory.

When $y(t)$ is eventually negative, we see $H \in D(s_2, t_2)$ and $e(t) \geqslant 0$ on $[s_2, t_2]$ to reach a similar contradiction. The proof is complete.

3. Example

In this section we will show the applications of our oscillation criteria by two examples. We will see that the equations in the examples are oscillatory based on the results in Section 2, though the oscillation cannot be demonstrated by the results of [8, 9, 10].

EXAMPLE 1. Consider the following forced half-linear differential equation

$$\left(r(t)|y'(t)|^{\alpha-1}y'(t)\right)' + p(t)|y'(t)|^{\alpha-1}y'(t) + 5\left(\frac{3}{2}\right)^{8/3} |y(t)|^{\alpha-1}y(t) = \sin t, \quad (21)$$

where $t \geqslant 1, \alpha = \frac{1}{3}, r(t) = p(t) = 2 + \cos t$. The zeros of the forcing term $\sin t$ are $n\pi$. Let $H(t) = \sin t$. For any $T \geqslant 1$, choose n sufficient large so that $n\pi = 2k\pi \geqslant T$ and set $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$ in (2.1). It is easy to verify that

$$\int_{s_1}^{t_1} H^2(t)q(t)dt = K\int_{2k\pi}^{(2k+1)\pi} \sin^2 t dt = \frac{K}{2}\pi,$$

where $K = 5\left(\frac{3}{2}\right)^{8/3}$, and

$$\left(\frac{2}{\alpha+1}\right)^{\alpha+1} \int_{s_1}^{t_1} \frac{r(t)}{|H(t)|^{\alpha-1}} \left|H'(t) - \frac{p(t)}{2r(t)}H(t)\right|^{\alpha+1} dt$$
$$= \left(\frac{3}{2}\right)^{4/3} \int_{2k\pi}^{(2k+1)\pi} \frac{(2+\cos t)}{(\sin t)^{-2/3}} \left|\cos t + \frac{1}{2}\sin t\right|^{4/3} dt$$
$$= \frac{1}{5}K\left(\frac{3}{2}\right)^{-4/3} \int_{2k\pi}^{(2k+1)\pi} (2+\cos t)(\sin t)^{2/3} \left|\cos t + \frac{1}{2}\sin t\right|^{4/3} dt$$
$$\leqslant \frac{1}{5}K\int_{2k\pi}^{(2k+1)\pi} (2+\cos t)dt = \frac{2}{5}K\pi < \frac{1}{2}K\pi,$$

which implies that (8) hold for $i=1$.

Similarly, for $s_2=(2k+1)\pi$ and $t_2=2(k+1)\pi$, we can show that (8) holds. It follows from Theorem 1 that every solution of Eq. (21) is oscillatory.

EXAMPLE 2. Consider the following half-linear differential equation

$$\left(t^{-\lambda}|y'(t)|^{\alpha-1}y'(t)\right)' - t^{-\lambda}|y'(t)|^{\alpha-1}y'(t) + Kt^{-\lambda}|y(t)|^{\alpha-1}y(t) = \sin t, \qquad (22)$$

where $t \geqslant 1$ and $\alpha = 1/5, \lambda > 0$ is constant, $K = 5\,(5(3+\lambda)/6)^{6/5}$ and the zeros of the forcing term $\sin t$ are $n\pi$. Let $H(t) = \sin t$. For any $T \geqslant 1$, choose n sufficient large so that $n\pi = 2k\pi \geqslant T$ and set $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$ in (2.8). Taking $\phi(t) = t^{\lambda}$, then

$$\int_{s_1}^{t_1} H^2(t)\phi(t)q(t)dt = K\int_{2k\pi}^{(2k+1)\pi} \sin^2 t dt = \frac{K}{2}\pi,$$

and

$$\begin{aligned}
&\left(\frac{1}{\alpha+1}\right)^{\alpha+1}\int_{s_1}^{t_1} \frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}}\left|2H'(t) + \left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{r(t)}\right)H(t)\right|^{\alpha+1} dt \\
&= \left(\frac{5}{6}\right)^{6/5}\int_{2k\pi}^{(2k+1)\pi} (\sin t)^{4/5}\left|2\cos t + \frac{\lambda}{t}\sin t + 1\right|^{6/5} dt \\
&\leqslant \left(\frac{5}{6}\right)^{6/5}\int_{2k\pi}^{(2k+1)\pi} \left(3 + \frac{\lambda}{t}\right)^{6/5} dt \\
&= \left(\frac{5}{6}\right)^{6/5}\int_{2k\pi}^{(2k+1)\pi} (3+\lambda)^{6/5}\, dt \\
&= 2\left(\frac{5(3+\lambda)}{6}\right)^{6/5}\pi = \frac{2}{5}K\pi < \frac{1}{2}K\pi,
\end{aligned}$$

which implies that (15) hold for $i=1$.

Similarly, for $s_2=(2k+1)\pi$ and $t_2=2(k+1)\pi$, we can show that (15) holds. It follows from Theorem 2 that every solution of Eq. $(22\ )$ is oscillatory.

REFERENCES

[1] M. S. KEENER, *Solutions of a certain linear nonhomogeneous second order differential equations*, Appl. Anal., **1** (1971), 57–63.

[2] S. M. RAINKIN, *Oscillation theorems for second order nonhomogeneous linear differential equations*, J. Math. Anal. Appl., **53** (1976), 550–553.

[3] A. SKIDMORE AND J. J. BOWERS, *Oscillatory behavior of solutions of* $y'' + p(x)y = f(x)$, J. Math. Anal. Appl., **49** (1975), 317–323.

[4] A. SKIDMORE AND W. LEIGHTON, *On the equation* $y'' + p(x)y = f(x)$, J. Math. Anal. Appl., **43** (1973), 46–55.

[5] H. TEUFEL, *Forced second order nonlinear oscillations*, J. Math. Anal. Appl., **40** (1972), 148–152.

[6] A. G. KARTSATOS, *Maintenance of oscillations under the effect of a periodic forcing term*, Proc. Amer. Math. Soc., **33** (1972), 377–383.
[7] M. A. EL–SAYED, *An oscillation criterion for a forced second order linear differential equations*, Proc. Amer. Math. Soc., **118** (1993), 813–817.
[8] J. S. WONG, *Oscillation criteria for a forced second order linear differential equation*, J. Math. Anal. Appl. , **231** (1999), 235–240.
[9] W. T. LI, *Oscillation of certain second–order nonlinear differential equations*, J. Math. Anal. Appl., **217** (1998), 1–14.
[10] W. T. LI, *Oscillation theorems for second order nonlinear differential equations with integrable coefficients*, Mathl. Comput. Modelling, **37** (7) (2000), 71–79.
[11] W. T. LI, *Positive solutions of second order nonlinear differential equations*, J. Math. Anal. Appl., **221** (1998), 326–337.
[12] W. T. LI AND R. P. AGARWAL, *Interval oscillation criteria related to integral averaging technique for certain nonlinear differential equations*, J. Math. Anal. Appl., **245** (2000), 171–188.
[13] W. T. LI, AND R. P. AGARWAL, *Interval oscillation criteria for second order nonlinear differential equations with forceing term*, Applicable Analysis, **75** (3/4) (2000), 341–347.
[14] W. T. LI, AND R. P. AGARWAL, *Interval oscillation criteria for second order forced nonlinear differential equations with damping*, PanAmerican Mathematical Journal, **11** (3) (2001), 109–117.
[15] G. H. HARDY, LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd Edition, Cambridge University Press, Cambridge, 1988.

(Received November 6, 2002)

Wan–Tong Li
Department of Mathematics
Lanzhou University
Lanzhou, Gansu 730000
People's Republic of China
e-mail: wtli@lzu.edu.cn